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Sufficient conditions for univalence obtained by using the Ruscheweyh-Bernardi differential-integral operator

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Abstract. In this paper we introduce the Ruscheweyh-Bernardi differential-integral operator $T^m : A \to A$ defined by

$$T^m[f](z) = (1-\lambda)R^m[f](z) + \lambda B^m[f](z),\ z \in U,$$

where R^m is the Ruscheweyh differential operator (Definition 1.3) and B^m is the Bernardi integral operator (Definition 1.1). By using the operator T^m, the class of univalent functions denoted by $T^m(\lambda, \beta)$, $0 \le \lambda \le 1$, $0 \le \beta < 1$, is defined and several differential subordinations are studied.

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1. Introduction and preliminaries

The theory of differential subordinations was introduced by S.S. Miller and P.T. Mocanu in two articles in 1978 [9] and 1981 [10]. This theory subsequently became very popular and its development was broad and fast. Important contributions to this theory can be found in older papers like [5] and newer publications like [13], [18], [4], [14] and [15].

We use the well-known definitions and notations:

Denote by U the unit disc of the complex plane

$$U = \{z \in \mathbb{C} :\ |z| < 1\}.$$

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Let $\mathcal{H}(U)$ be the space of holomorphic functions in U and let

$$A_n = \{f \in \mathcal{H}(U) :\ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots,\ z \in U\},$$

with $A_1 = A$.

Let $S = \{f \in A :\ f \text{ is univalent in } U\}$ be the class of holomorphic and univalent functions in the open unit disc U with the conditions $f(0) = 0$ and $f'(0) = 1$, that is the holomorphic and univalent functions with the following power series development

$$f(z) = z + a_2z^2 + \dots,\ z \in U.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we denote by

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) :\ f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \dots,\ z \in U\}.$$

Denote by

$K = \left\{f \in A :\ \operatorname{Re}\left(\frac{zf''(z)}{zf'(z)} + 1\right) > 0,\ z \in U\right\}$ the class of normalized convex functions in U and let

$S^* = \left\{f \in A :\ \operatorname{Re}\frac{zf'(z)}{f(z)} > 0,\ z \in U\right\}$ denote the class of starlike functions in U.

The core of the theory of differential subordination is found in the monograph published in 2000 by S.S. Miller and P.T. Mocanu [11].

Definition of subordination ([11, p. 4])

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there is a function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$ such that $f(z) = g(w(z))$ for $z \in U$. If g is univalent, then $f \prec g$ or $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition of second-order differential subordination ([11, p. 7])

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the second-order differential subordination

(i) $\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z),\ z \in U$

then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solutions of the differential subordination, or more simply, a dominant if $p \prec q$ for all p satisfying (i).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (i) is said to be the best dominant of (i). (Note that the best dominant is unique up to a rotation of U).

If we require the more restrictive condition $q \in \mathcal{H}[a, n]$ then p is called an (a, n)-solution, q an (a, n)-dominant and $\widetilde{q}$ the best (a, n)-dominant.

Definition of Briot-Bouquet differential subordination [11, p.80] Let $r, l \in \mathbb{C}$, $r \neq 0$ and let h be a univalent function in U, with $h(0) = a$, and let $p \in \mathcal{H}[a, n]$ satisfy

(ii) $p(z) + \dfrac{zp'(z)}{rp(z) + l} \prec h(z),\ z \in U.$

The first-order differential subordination is called the Briot-Bouquet differential subordination.

In 1969 Bernardi [3] introduced the operator

(iii) $F(z) = \frac{\gamma+1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt$, for $\gamma = 1, 2, 3, \ldots$,
which generalizes the Libera operator.

Studying subordination properties by using differential and integral operators is a classic topic still of interest at this time, interesting results being currently obtained in forms of criteria for univalence of functions. A recent approach in using operators is to mix a differential and an integral operator as it the case in the very recent papers [1], [16] and [19]. This idea is also used in the present paper for introducing a new differential-integral operator mixing Ruscheweyh differential operator and Bernardi integral operator and by using it, a new class of univalent functions. Some criteria for univalence are derived from proving theorems containing subordination results related to this newly introduced operator.

To prove our main results, we need the following:

Definition 1.1. [17] For $f \in A$, $m \in \mathbb{N}$, $\gamma \in \mathbb{N}^* = \{1, 2, \ldots\}$, the integral operator $B^m : A \to A$ is defined by

$$B^0[f](z) = f(z)$$

$$B^1[f](z) = \frac{\gamma+1}{z^\gamma}\int_0^z B^0[f](t)\cdot t^{\gamma-1}dt = \frac{\gamma+1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt$$

$$\vdots$$

$$B^m[f](z) = \frac{\gamma+1}{z^\gamma}\int_0^z B^{m-1}[f](t)\cdot t^{\gamma-1}dt. \tag{1.1}$$

Remark 1.2. a) For $m = 1$, $\gamma \in \mathbb{N}^*$, we obtain Bernardi integral operator (iii) defined in [3].

b) For $m = 1$, $\gamma = 1$, we obtain Libera integral operator defined in [7].

c) For $m = 1$, $\gamma = 0$ we obtain Alexander integral operator defined in [2].

d) If $f \in A$ and $f(z) = z + a_2z^2 + a_3z^3 + \ldots = z + \sum_{k=2}^{\infty} a_kz^k$, then

$$B^m[f](z) = z + \sum_{k=2}^{\infty}\frac{(\gamma+1)^m}{(\gamma+k)^m}a_kz^k. \tag{1.2}$$

e) For $f \in A$, $m \in \mathbb{N}$, $\gamma \in \mathbb{N}^*$. we obtain

$$z(B^m[f](z))' = (\gamma+1)B^{m-1}[f](z) - \gamma B^m[f](z),\ z \in U. \tag{1.3}$$

Definition 1.3. [20] For $f \in A$, $m \in \mathbb{N}$, the differential operator $R^m : A \to A$ is defined by

$$R^0[f](z) = f(z)$$

$$R^1[f](z) = z(R^0[f](z))' = zf'(z)$$

$$\vdots$$

$$(m+1)R^{m+1}[f](z) = z(R^m[f](z))' + mR^m[f](z),\ z \in U. \tag{1.4}$$

Remark 1.4. If $f \in A$, $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$R^m[f](z) = z + \sum_{k=2}^{\infty} C_{m+k-1}^{m} a_k z^k = z + \sum_{k=2}^{\infty} \begin{bmatrix} m+k-1 \\ m \end{bmatrix} a_k z^k. \tag{1.5}$$

Lemma A. [13, Th. 10.2.1] *Let $r, l \in \mathbb{C}$, $r \neq 0$, and let h be a convex function that satisfies*

$$\operatorname{Re}\left[r \cdot h(z) + l\right] > 0,\ z \in U.$$

If $p \in \mathcal{H}[h(0), n]$, then

$$p(z) + \frac{zp'(z)}{r \cdot p(z) + l} \prec h(z)$$

implies

$$p(z) \prec h(z),\ z \in U.$$

Lemma B. (Hallenbeck and Ruscheweyh [11, Th. 3.1.b, p. 71]) *Let h be a convex function in U, with $h(0) = a$, $\mu \neq 0$ and $\operatorname{Re}\mu \geq 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\mu} zp'(z) \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z),\ z \in U,$$

where

$$q(z) = \frac{\frac{\mu}{n}}{z^{\frac{\mu}{n}}} \int_0^z h(t) \cdot t^{\frac{\mu}{n}-1} dt,\ z \in U.$$

The function q is convex and is the best dominant.

2. Main results

In this paper we define a differential-integral operator $T^m : A \to A$, we define a class of holomorphic univalent functions and study several Briot-Bouquet differential subordinations obtained by using this operator.

Definition 2.1. Let $m \in \mathbb{N}$, $0 \leq \lambda \leq 1$. Denote by $T^m : A \to A$,

$$T^m[f](z) = (1-\lambda) R^m[f](z) + \lambda B^m[f](z),\ z \in U, \tag{2.1}$$

where R^m is given by (1.4) and B^m is given by (1.1).

Remark 2.2. a) If $f \in A$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, and using (1.2) and (1.5), we have

$$\begin{aligned} T^m[f](z) &= (1-\lambda)\left(z + \sum_{k=2}^{\infty} \begin{bmatrix} m+k-1 \\ m \end{bmatrix} a_k z^k \right) + \lambda \left(z + \sum_{k=2}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m} a_k z^k \right) \\ &= z + \sum_{k=2}^{\infty} \left\{ \begin{bmatrix} m+k-1 \\ m \end{bmatrix} (1-\lambda) + \lambda \frac{(\gamma+1)^m}{(\gamma+k)^m} \right\} a_k z^k. \end{aligned} \tag{2.2}$$

b) For $\lambda = 1$, the differential-integral operator T^m coincides with Bernardi integral operator (Definition 1.1).

c) For $\lambda = 0$, the differential-integral operator T^m coincides with R^m, Ruschweyh differential operator (Definition 1.3).

Definition 2.3. If $0 \le \beta < 1$, $0 \le \lambda \le 1$, $m \in \mathbb{N}$, we let $B^m(\lambda, \beta)$ stand for the class of functions $f \in A$, which satisfy the inequality

$$\operatorname{Re}(T^m[f](z))' > \beta, \ z \in U, \tag{2.3}$$

where the differential-integral operator $T^m[f]$ is given by (2.1).

Remark 2.4. a) For $m = 0$, $\beta = 0$, $0 \le \lambda \le 1$, the operator $T^m[f]$ becomes

$$\begin{aligned} T_0[f](z) &= (1-\lambda)R^0[f](z) + \lambda B^0[f](z) \\ &= (1-\lambda)f(z) + \lambda f(z) = f(z), \ z \in U, \end{aligned}$$

then $B^m(\lambda, \beta)$ becomes

$$B^0(\lambda, 0) = R = \{f \in A : \ \operatorname{Re} f'(z) > 0, \ z \in U\},$$

called the class of functions with bounded rotation.

This class of functions was studied by J.W. Alexander [2] and he proved that $R \subset S$. J. Krzyz [6] and P.T. Mocanu [12] have proved that $R \not\subset S^*$. A more systematic study of class R was done by Mac Gregor [8].

b) For $m = 0$, $0 \le \beta < 1$, $0 \le \lambda \le 1$, we have

$$B^0(\lambda, \beta) = M(\beta) = \{f \in A : \ \operatorname{Re} f'(z) > \beta\} \subset R.$$

Theorem 2.5. *The set $B^m(\lambda, \beta)$ is convex.*

Proof. Let the functions

$$f_j(z) = z + \sum_{k=2}^{\infty} \alpha_{kj} z^j, \ j = 1, 2, \ z \in U,$$

where

$$\alpha_{kj} = a_{kj} \left\{ (1-\lambda) \begin{bmatrix} m+k-1 \\ m \end{bmatrix} + \lambda \frac{(\gamma+1)^m}{(\gamma+k)^m} \right\},$$

be in the class $B^m(\lambda, \beta)$. It is sufficient to show that the function

$$h(z) = \mu_1 f_1(z) + \mu_2 f_2(z), \ z \in U,$$

with $\mu_1, \mu_2 \ge 0$ and $\mu_1 + \mu_2 = 1$ is in $B^m(\lambda, \beta)$.
Since $h(z) = \mu_1 f_1(z) + \mu_2 f_2(z)$, $z \in U$, we have

$$T^m[h](z) = z + \sum_{k=2}^{\infty} \left\{ (1-\lambda) \begin{bmatrix} m+k-1 \\ m \end{bmatrix} + \lambda \frac{(\gamma+1)^m}{(\gamma+k)^m} \right\} (\mu_1 a_{k1} + \mu_2 a_{k2}) z^k. \tag{2.4}$$

Differentiating (2.4), we have

$$(T^m[h](z))' = 1 + \sum_{k=2}^{\infty} \left\{ (1-\lambda) \begin{bmatrix} m+k-1 \\ m \end{bmatrix} + \lambda \frac{(\gamma+1)^m}{(\gamma+k)^m} \right\} (\mu_1 a_{k1} + \mu_2 a_{k2}) k z^{k-1}.$$

Hence

$$\begin{aligned}\mathrm{Re}\,(T^m[h](z))' &= 1 + \mathrm{Re}\sum_{k=2}^{\infty}\left\{(1-\lambda)\begin{bmatrix} m+k-1 \\ m \end{bmatrix} + \lambda\frac{(\gamma+1)^m}{(\gamma+k)^m}\right\}\mu_1 a_{k1} k z^{k-1} \\ &+ \mathrm{Re}\sum_{k=2}^{\infty}\left\{(1-\lambda)\begin{bmatrix} m+k-1 \\ m \end{bmatrix} + \lambda\frac{(\gamma+1)^m}{(\gamma+k)^m}\right\}\mu_2 a_{k2} k z^{k-1}. \quad (2.5)\end{aligned}$$

Since $f_1, f_2 \in B^m(\lambda, \beta)$, we have

$$\mu_1 \mathrm{Re}\sum_{k=2}^{\infty}\left\{(1-\lambda)\begin{bmatrix} m+k-1 \\ m \end{bmatrix} + \lambda\frac{(\gamma+1)^m}{(\gamma+k)^m}\right\} a_{k1} k z^{k-1} > \mu_1(\beta-1) \quad (2.6)$$

$$\mu_2 \mathrm{Re}\sum_{k=2}^{\infty}\left\{(1-\lambda)\begin{bmatrix} m+k-1 \\ m \end{bmatrix} + \lambda\frac{(\gamma+1)^m}{(\gamma+k)^m}\right\} a_{k2} k z^{k-1} > \mu_2(\beta-1) \quad (2.7)$$

Using (2.6) and (2.7), we obtain

$$\mathrm{Re}\,(T^m[h](z))' > 1 + \mu_1(\beta-1) + \mu_2(\beta-1)$$

and since $\mu_1 + \mu_2 = 1$, we deduce

$$\mathrm{Re}\,(T^m[h](z))' > \beta,$$

i.e. $B^m(\lambda, \beta)$ is convex.

Theorem 2.6. *Let $0 \le \beta < 1$, $0 \le \lambda \le 1$, $m \in \mathbb{N}$, $f \in A$.*
If $f \in B^n(\lambda, \beta)$, then we have

$$\mathrm{Re}\,\frac{T^m[f](z)}{z} > 2\beta - 1 + 2(1-\beta)\ln 2 = \delta.$$

Proof. We prove that $\delta \in [0, 1)$, $\delta = 2\beta(1 - \ln 2) + 2\ln 2 - 1$. For $\ln 2 \approx 0,69$,

$$\begin{aligned}&\delta - 2\beta(1 - 0,69) + 2 \cdot 0,69 - 1 \\ &\quad = 2\beta \cdot 0,31 + 0,38 = \beta \cdot 0,62 + 0,38.\end{aligned}$$

Hence $0 \le \beta < 1$. We have

$$\begin{gathered}0 \le \beta \cdot 0,62 < 0,62, \\ 0,38 \le \beta \cdot 0,62 + 0,38 < 0,62 + 0,38, \\ 0,38 \le \beta \cdot 0,62 + 0,38 < 1, \\ 0,38 \le \delta < 1,\ \ \delta \in [0,38, 1).\end{gathered}$$

Let the convex function

$$h(z) = \frac{1 + (2\beta - 1)z}{1 + z},\ \ 0 \le \beta < 1,\ \ z \in U. \quad (2.8)$$

For $z \in U$, we have $\mathrm{Re}\, h(z) > \beta$ and $h(0) = 1$.
From the hypothesis we have that $f \in B^m(\lambda, \beta)$, then from Definition 2.3 we have

$$\mathrm{Re}\,(T^m[f](z))' > \beta,\ \ z \in U. \quad (2.9)$$

Let

$$p(z) = \frac{T^m[f](z)}{z},\ \ z \in U. \quad (2.10)$$

Using (2.3) in (2.10) we have

$$p(z) = \frac{z + \sum_{k=2}^{\infty} a_k z^k \left\{ \begin{bmatrix} m+k-1 \\ m \end{bmatrix} (1-\lambda) + \lambda \frac{(\gamma+1)^m}{(k+\gamma)^m} \right\}}{z}$$
$$= 1 + \sum_{k=2}^{\infty} a_k z^{k-1} \left\{ \begin{bmatrix} m+k-1 \\ m \end{bmatrix} (1-\lambda) + \lambda \frac{(\gamma+1)^m}{(k+\gamma)^m} \right\},$$
$$p(0) = 1 \text{ and } p \in \mathcal{H}[1,1].$$

From (2.10), we have

$$T^m[f](z) = zp(z), \ z \in U. \tag{2.11}$$

Differentiating (2.11), we obtain

$$(T^m[f](z))' = p(z) + zp'(z). \tag{2.12}$$

Using (2.12) in (2.9), we have

$$\operatorname{Re}[p(z) + zp'(z)] > \beta, \ z \in U. \tag{2.13}$$

Relation (2.13) can be written as a subordination of the form

$$p(z) + zp'(z) \prec h(z) = \frac{1 + (2\beta - 1)z}{1+z}, \ z \in U.$$

Using Lemma B, for $\mu = 1$, $n = 1$, we have

$$p(z) \prec q(z),$$

where

$$q(z) = \frac{1}{z} \int_0^z \frac{1 + (2\beta - 1)t}{t} dt = 2\beta - 1 + 2(1-\beta) \frac{\ln(1+z)}{z}$$

i.e.,

$$\frac{T^m[f](z)}{z} \prec 2\beta - 1 + 2(1-\beta) \frac{\ln(1+z)}{z} = q(z), \ z \in U.$$

The function q is convex and is the best dominant.
Since q is convex function and

$$p(z) \prec q(z) = 2\beta - 1 + 2(1-\beta) \frac{\ln(1+z)}{z}, \ z \in U,$$

we have

$$\operatorname{Re} p(z) > \operatorname{Re} q(1) = 2\beta - 1 + 2(1-\beta) \ln 2 = \delta. \tag{2.14}$$

Using (2.10), the relation (2.14) becomes

$$\operatorname{Re} \frac{T^m[f](z)}{z} > \delta = 2\beta - 1 + 2(1-\beta) \ln 2.$$

From Theorem 2.6 we deduce the following corollary:

Corollary 2.7. *Let $0 \le \lambda \le 1$, $f \in A$, $m \in \mathbb{N}$, $\delta = 2\beta - 1 + 2(1-\beta) \ln 2$.*
If $f \in B^m(\lambda, \delta)$, then

$$\operatorname{Re} \frac{T^m[f](z)}{z} > \delta = 2\beta - 1 + 2(1-\beta) \ln 2.$$

Proof. From the proof of Theorem 2.6, we can see that

$$\frac{T^m[f](z)}{z} \prec q(z) = 2\beta - 1 + 2(1-\beta)\frac{\ln(1+z)}{z},\ z \in U.$$

Since q is convex function, we have that

$$\operatorname{Re}\frac{T^m[f](z)}{z} > \operatorname{Re} q(1) = \delta = 2\beta - 1 + 2(1-\beta)\ln 2.$$

Theorem 2.8. *Let h be a convex function, with $h(0) = 1$ and*

$$\operatorname{Re} h(z) > 0,\ z \in U.$$

If $f \in A$, $0 \le \lambda \le 1$, $m \in \mathbb{N}$ and satisfies the differential subordination

$$(T^m[f](z))' + \frac{z(T^m[f](z))''}{(T^m[f](z))'} \prec h(z) \tag{2.15}$$

then

$$(T^m[f](z))' \prec h(z),\ z \in U.$$

Proof. We let

$$p(z) = (T^m[f](z))',\ z \in U. \tag{2.16}$$

Using (2.2) in (2.16), we have

$$\begin{aligned} p(z) &= \left(z + \sum_{k=2}^{\infty} \left\{ \begin{bmatrix} m+k-1 \\ m \end{bmatrix} (1-\lambda) + \lambda \frac{(\gamma+1)^m}{(\gamma+k)^m} \right\} a_k z^k \right)' \\ &= 1 + \sum_{k=2}^{\infty} \left\{ \begin{bmatrix} m+k-1 \\ m \end{bmatrix} (1-\lambda) + \lambda \frac{(\gamma+1)^m}{(\gamma+k)^m} \right\} a_k k z^{k-1} \\ &= 1 + p_1 z + p_2 z^2 + \dots \end{aligned} \tag{2.17}$$

and $p(0) = 1$, $p \in \mathcal{H}[1,1]$.
Differentiating (2.16), we get

$$\frac{p'(z)}{p(z)} = \frac{(T^m[f](z))''}{(T^m[f](z))'},\quad \frac{zp'(z)}{p(z)} = \frac{z(T^m[f](z))''}{(T^m[f](z))'} \tag{2.18}$$

and

$$p(z) + \frac{zp'(z)}{p(z)} = (T^m[f](z))' + \frac{z(T^m[f](z))''}{(T^m[f](z))'}. \tag{2.19}$$

Using (2.19), the differential subordination (2.15) becomes

$$p(z) + \frac{zp'(z)}{p(z)} \prec h(z),\ z \in U.$$

Using Lemma A, for $r = 1$, $l = 0$, we obtain

$$p(z) \prec h(z),\ z \in U,$$

i.e.

$$(T^m[f](z))' \prec h(z),\ z \in U.$$

From Theorem 2.8 we deduce the following sufficient conditions for univalent function.

Criterion 2.9. *Let*

$$h(z) = \frac{1+(2\beta-1)z}{1+z},\ 0 \le \beta < 1,$$

be convex function with $h(0)=1$ *and* $\operatorname{Re} h(z) > \beta$, $z \in U$.
If $f \in A$, $0 \le \lambda < 1$, $m \in \mathbb{N}$ *and satisfies the differential subordination*

$$(T^m[f](z))' + \frac{z(T^m[f](z))''}{(T^m[f](z))'} \prec \frac{1+(2\beta-1)z}{1+z},$$

then

$$(T^m[f](z))' \prec \frac{1+(2\beta-1)z}{1+z},\ z \in U, \tag{2.20}$$

where $T^m[f]$ *is defined in (2.1). Hence* f *is an univalent function.*

Proof. Since h is convex, with $h(1)=\beta$, $0 \le \beta < 1$, $\operatorname{Re} h(z) > \beta$, relation (2.20) is equivalent to

$$\operatorname{Re}(T^m[f](z))' > \operatorname{Re} h(1) = \beta.$$

From Definition 2.3, we have $f \in B^m(\lambda, \beta)$, hence f is an univalent function.

Criterion 2.10. *Let*

$$h(z) = \frac{1-z}{1+z},$$

with $h(0)=1$, $\operatorname{Re} h(z) > 0$, $z \in U$.
If $f \in A$, $0 \le \lambda < 1$, $m \in \mathbb{N}$ *and satisfies the differential subordination*

$$(T^m[f](z))' + \frac{z(T^m[f](z))''}{(T^m[f](z))'} \prec \frac{1-z}{1+z}$$

then

$$(T^m[f](z))' \prec \frac{1-z}{1+z},\ z \in U, \tag{2.21}$$

where $T^m[f]$ *is defined in (2.1). Hence* f *is an univalent function.*

Proof. Since h is convex, with $h(1)=0$, $\operatorname{Re} h(z) > 0$, $z \in U$, relation (2.21) becomes

$$\operatorname{Re}(T^m[f](z))' > \operatorname{Re} h(1) > 0,\ z \in U.$$

From Definition 2.3, we have $f \in B^m(\lambda, 0)$, hence f is an univalent function.

Theorem 2.11. *Let* h *be a convex function,* $h(0)=1$, *with*

$$\operatorname{Re} h(z) > 0,\ z \in U.$$

If $f \in A$, $0 \le \lambda \le 1$, $m \in \mathbb{N}$ *and satisfies the differential subordination*

$$\frac{T^m[f](z)}{z} + \frac{z(T^m[f](z))'}{T^m[f](z)} - 1 \prec h(z),\ z \in U, \tag{2.22}$$

then

$$\frac{T^m[f](z)}{z} \prec h(z),\ z \in U.$$

Proof. We let

$$p(z)=\frac{T^m[f](z)}{z},\ z\in U. \tag{2.23}$$

Using (2.2) in (2.23), we get

$$\begin{aligned} p(z)&=\frac{z+\sum_{k=2}^{\infty}\left\{\begin{bmatrix} m+k-1\\ m\end{bmatrix}(1-\lambda)+\lambda\frac{(\gamma+1)^m}{(\gamma+k)^m}\right\}a_kz^k}{z}\\ &=1+\sum_{k=2}^{\infty}\left\{\begin{bmatrix} m+k-1\\ m\end{bmatrix}(1-\lambda)+\lambda\frac{(\gamma+1)^m}{(\gamma+k)^m}\right\}a_kz^{k-1}\\ &=1+p_1z+p_2z^2+\dots \end{aligned}$$

and $p(0)=1$, $p\in\mathcal{H}[1,1]$.
From (2.23), we have

$$zp(z)=T^m[f](z),\ z\in U. \tag{2.24}$$

Differentiating (2.24), we get

$$\frac{1}{z}+\frac{p'(z)}{p(z)}=\frac{(T^m[f](z))'}{T[f](z)},$$

$$1+\frac{zp'(z)}{p(z)}=\frac{z(T^m[f](z))'}{T^m[f](z)},$$

and

$$p(z)+\frac{zp'(z)}{p(z)}=\frac{T^m[f](z)}{z}+\frac{z(T^m[f](z))'}{T^m[f](z)}-1. \tag{2.25}$$

Using (2.25), the differential subordination (2.22) becomes

$$p(z)+\frac{zp'(z)}{p(z)}\prec h(z),\ z\in U.$$

Using Lemma A, for $r=1$, $l=0$, we get

$$p(z)\prec h(z),$$

i.e.

$$\frac{T^m[f](z)}{z}\prec h(z),\ z\in U.$$

Example 2.12. Let

$$f(z)=z+\frac{6}{31}z^2,\ m=2,\ k=2,\ \gamma=1,\ \lambda=\frac{1}{2},\ \beta=\frac{1}{3},$$

$$T^2[f](z)=z+\frac{1}{3}z^2,\ h(z)=\frac{1-\frac{1}{3}z}{1+z},\ h(0)=1,\ \operatorname{Re}h(z)>\frac{1}{3},$$

$$(T^2[f](z))'=1+\frac{2}{3}z,\ (T^2[f](z))''=\frac{2}{3}.$$

Using Theorem 2.8, we get:

$$1+\frac{2}{3}z+\frac{2z}{3+2z}\prec\frac{1-\frac{1}{3}z}{1+z},$$

implies

$$1+\frac{2}{3}z\prec\frac{1-\frac{1}{3}z}{1+z},\ z\in U.$$

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