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Superdense unbounded divergence of a class of interpolatory product quadrature formulas

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Dedicated to Professor Gheorghe Coman on the occasion of his 80th anniversary

Abstract. The aim of this paper is to highlight the superdense unbounded divergence of a class of product quadrature formulas of interpolatory type on Jacobi nodes, associated to the Banach space of all real continuous functions defined on $[-1,1]$, and to a Banach space of measurable and essentially bounded functions $g : [-1,1] \to \mathbb{R}$. Some aspects regarding the convergence of these formulas are pointed out, too.

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1. Introduction

This paper deals with a class of interpolatory product quadrature formulas, regarding their divergence and the convergence rate, as follows. Let C be the Banach space of all continuous functions $f : [-1,1] \to \mathbb{R}$, endowed with the supremum norm $\|\cdot\|$. Denoting by μ the Lebesgue measure on the interval $[-1,1]$, let $(L_p, \|\cdot\|_p)$, $1 \le p \le \infty$, be the Banach space of all measurable functions (equivalence classes of functions, with respect to the equality μ-a.e.) $g : [-1,1] \to \mathbb{R}$, normed by

$$\|g\|_p = \left(\int_{-1}^{1} |g(x)|^p dx\right)^{1/p}, \text{ if } 1 \le p < \infty, \text{ and } \|g\|_\infty = \text{esssup}|g|.$$

According to [7], [8], if $p \in [1,\infty]$ and $\rho \in L_q$ (with $p^{-1}+q^{-1}=1$) are given such that $\rho(x) > 0$ μ-a.e. on $[-1,1]$, the notation $(L_p^{(1/\rho)}, \|\cdot\|_p^{(1/\rho)})$ stands for the Banach space of all measurable functions g for which $g/\rho \in L_p$ and $\|g\|_p^{(1/\rho)} = \|g/\rho\|_p$.

Further, let consider an arbitrary triangular node matrix

$$\mathcal{M} = \{x_{kn} :\ n \ge 1,\ 1 \le k \le n\}$$

so that the n-th row of $\mathcal{M}$, $n \geq 1$, contains n distinct nodes of $[-1,1]$, then let us denote, as usual, by $\mathcal{L}_n f \in \mathcal{P}_{n-1}$ (the space of all polynomials of degree at most $n-1$) and λ_n the Lagrange interpolation polynomial and the Lebesgue function associated to the n-th row of $\mathcal{M}$, respectively, i.e.,

$$(\mathcal{L}_n f)(x) = \sum_{k=1}^{n} f(x_{kn}) l_{kn}(x),\ f \in C,\ \lambda_n(x) = \sum_{k=1}^{n} |l_{kn}(x)|,$$

where l_{kn} are the fundamental Lagrange interpolation polynomials, [2], [10].
The equalities

$$\int_{-1}^{1} g(x) f(x) dx = \int_{-1}^{1} g(x)(\mathcal{L}_n f)(x) dx + R_n(f;g),\ f \in C,\ g \in L_p^{(1/\rho)},\ n \geq 1 \quad (1.1)$$

involving

$$R_n(P,g) = 0,\ \forall\ f \in \mathcal{P}_{n-1} \text{ and } g \in L_p^{(1/\rho)},\ n \geq 1 \quad (1.2)$$

describe *product quadrature formulas of interpolatory type*, associated to the spaces C and $L_p^{(1/\rho)}$.

If $p = 1$, these product quadrature formulas were intensively studied, in their convergence aspects, for various functions $g \in L_1$, $\rho \in L_\infty$ (including $\rho(x) = (1-x)^a(1+x)^b$, $a, b \geq 0$) and node matrices $\mathcal{M}$, [1], [3], [4], [7], [8]. We notice, also, the divergence result obtained by I.H. Sloan and W.E. Smith, for arbitrary node matrices $\mathcal{M}$ and $\rho(x) = 1$, $-1 \leq x \leq 1$, [8, Th. 6]. A recent result, [5], refers to more general product quadrature formulas of interpolatory type, involving polynomial projection operators $\mathcal{L}_n : C \to \mathcal{P}_{n-1}$ (namely $\mathcal{L}_n f \in \mathcal{P}_{n-1}$, $\forall\ f \in C$, and $\mathcal{L}_n f = f$ if and only if $f \in \mathcal{P}_{n-1}$) instead of Lagrange projections in (1.1) and highlights the phenomenon of double condensation of singularities for the corresponding product quadrature formulas (1.1), meaning unbounded divergence on superdense sets belonging to the spaces C and $L_1^{(1/\rho)}$, for arbitrary node matrices $\mathcal{M}$ and $\rho \in L_\infty$, with $\rho(x) > 0$ μ-a.e. on $[-1,1]$.

The aim of this paper is to point out the superdense unbounded divergence of the product quadrature formulas described by (1.1) and (1.2) for $p = \infty$, $\rho(x) = (1-x)^a(1+x)^b$, with $a, b > -1$, and $\mathcal{M} = \mathcal{M}^{(\alpha,\beta)}$, $\alpha > -1$, $\beta > -1$, where $\mathcal{M}^{(\alpha,\beta)}$ is the Jacobi node matrix (namely, its n-th row contains the roots $x_n^{(\alpha,\beta)}$, $1 \leq k \leq n$, of the Jacobi polynomial $P_n^{(\alpha,\beta)}$, $n \geq 1$). Moreover, some aspects regarding the convergence of these formulas (for functions $f \in C$ satisfying a Dini-Lipschitz condition and arbitrary $g \in L_\infty^{(1/\rho)}$) will be presented in the last section.

In this paper, the notation M_k, $k \geq 1$, stands for some positive constants which do not depend on n. Also, we denote by $\omega(f,\cdot)$ the modulus of continuity associated to a function $f \in C$.

2. Unbounded divergence on superdense sets

Suppose that $\rho(x) = (1-x)^a(1+x)^b$, $a, b > -1$ and $\mathcal{M} = \mathcal{M}^{(\alpha,\beta)}$, $\alpha > -1$, $\beta > -1$. Let U_n, $n \geq 1$, be the continuous linear operators defined as

$$\begin{cases} U_n : C \to (L_\infty^{(1/\rho)})^*; \; f \mapsto U_n f \\ (U_n f)(g) = \displaystyle\int_{-1}^{1} g(x)(\mathcal{L}_n f)(x)dx; \; f \in C, \; g \in L_\infty^{(1/\rho)}, \end{cases} \tag{2.1}$$

where Y^* is the Banach space of all continuous linear functionals defined on the normed space Y.

Using standard arguments and classic results of Functional Analysis, we obtain (see also [8]):

$$\|U_n\| = \sup\{\|U_n f\| : \; f \in C, \; \|f\| \leq 1\}$$

and

$$\begin{aligned} \|U_n f\| &= \sup\left\{\left|\int_{-1}^{1} g(x)(\mathcal{L}_n f)(x)dx\right| : \; g/\rho \in L_\infty, \; \|g/\rho\|_\infty \leq 1\right\} \\ &= \sup\left\{\left|\int_{-1}^{1} \rho(x)g(x)(\mathcal{L}_n f)(x)dx\right| : \; g \in L_\infty, \; \|g\|_\infty \leq 1\right\}, \end{aligned}$$

so we get

$$\|U_n\| = \sup\{\|\rho\mathcal{L}_n f\|_1 : \; f \in C, \; \|f\| \leq 1\}, \; n \geq 1. \tag{2.2}$$

Now, we can state:

Theorem 2.1. *Suppose that $\alpha \geq 2a + 3/2$ or $\beta \geq 2b + 3/2$. Then, a superdense set X_0 in the Banach space $L_\infty^{(1/\rho)}$ exists such that for every g in X_0, the set of C consisting of all functions for which the product quadrature formulas described by* (1.1) *and* (1.2) *are unbounded divergent, namely*

$$Y_0(g) = \left\{f \in C : \; \limsup_{n\to\infty} \left|\int_{-1}^{1} g(x)(\mathcal{L}_n f)(x)dx\right| = \infty\right\},$$

is superdense in the Banach space C.

Proof. First, we show that the set $\{\|U_n\| : \; n \geq 1\}$ is unbounded. Similarly to [9], let consider the function $f_n \in C$, $n \geq 1$, defined by

$$f_n(x) = \begin{cases} (-1)^k, & \text{if} \quad x = x_{kn}^{(\alpha,\beta)}, \; 0 \leq k \leq n+1 \\ \text{linear}, & \text{if} \quad x \in [x_{kn}^{(\alpha,\beta)}, x_{k,n-1}^{(\alpha,\beta)}], \; 1 \leq k \leq n+1, \end{cases}$$

where $x_{0n}^{(\alpha,\beta)} = 1$ and $x_{n+1,n}^{(\alpha,\beta)} = -1$.

It follows from (2.2):

$$\|U_n\| \geq \|\rho\mathcal{L}_n f_n\|_1 = \int_{-1}^{1} (1-x)^a(1+x)^b |(\mathcal{L}_n f_n)(x)|dx. \tag{2.3}$$

Next, let us suppose that $\alpha \geq 2a + 3/2 > -1/2$ and set $q_0 = 1 - \dfrac{4(a+1)}{2\alpha+1}$ (so, $0 \leq q_0 < 1$). Using the estimation of [9, formula (3.3), with $p = 1$ and $q = q_0$], we get:

$$\begin{cases} \|U_n\| \geq M_1 \log n, & \text{if} \quad q_0 = 0 \\ \|U_n\| \geq M_2 n^{q_0(\alpha+1/2)}, & \text{if} \quad q_0 > 0. \end{cases} \tag{2.4}$$

The relations (2.3) and (2.4) prove the unboundedness of the set $\{\|U_n\| :\ n \geq 1\}$, if $\alpha \geq 2a + 3/2$; similarly, the same assertion is true for $\beta \geq 2b + 3/2 > -\dfrac{1}{2}$.

Now, we apply the principle of condensation of singularities [3, Theorem 5.2], with $X = L_\infty^{(1/\rho)}$, $T = C$, $Y = \mathbb{R}$, $J = \mathbb{N}^*$ and $A_n(g; f) = (U_n f)(g)$. It is easily seen that the hypotheses 1° and 2° of this principle are fulfilled. In order to show the validity of the hypothesis 3°, denote by $\mathcal{U} = \{U_n :\ n \geq 1\}$ the family of the operators defined by (2.1). Using the principle of condensation of singularities, [3, Th. 5.4], with respect to the family $\mathcal{U}$ and taking into account the unboundedness of the set $\{\|U_n\| :\ n \geq 1\}$, we infer that the set of the singularities of $\mathcal{U}$, namely

$$\mathcal{S}(U) = \{f \in C :\ \sup\{\|U_n f\| :\ n \geq 1\} = \infty\}, \tag{2.5}$$

is superdense in C. Now, take $T_0 = \mathcal{S}(U)$ from (2.5) and remark that

$$\sup\{\|A_n f\| :\ n \geq 1\} = \sup\{\|U_n f\| :\ n \geq 1\} = \infty,$$

for every $f \in T_0$, therefore the hypothesis 3° of [3, Theorem 5.2] holds, too. Finally, denote by $Y_0(g)$ the set of singularities of the family $\mathcal{A}(g) = \{A_n(g, \cdot) :\ n \geq 1\}$, which completes the proof. □

3. Dini-Lipschitz convergence

Let us estimate the quadrature errors $R_n(f; g)$ of (1.1), see also [7], [8]. Denoting by $I : C \to (L_\infty^{(1/\rho)})^*$, the operator given by $(If)(g) = \int_{-1}^{1} g(x) f(x) dx$ and taking into account the interpolatory condition (1.2), we get:

$$|R_n(f; g)| = |(U_n - I)(f - p)(g)| \leq \|U_1 - I\| \cdot \|f - p\| \cdot \|g\|_\infty^{(1/\rho)}. \tag{3.1}$$

Further, we obtain, for every $f \in C$:

$$\|\rho \mathcal{L}_n f\|_1 = \int_{-1}^{1} \rho(x) |(\mathcal{L}_n f)(x)| dx \leq \left(\int_{-1}^{1} \rho(x) \lambda_n(x) dx \right) \|f\|,$$

so, (2.2) leads to:

$$\|U_n\| \leq \|\rho \lambda_n\|_1. \tag{3.2}$$

Similarly, we get

$$\|I\| \leq \|\rho\|_1. \tag{3.3}$$

Now, combining the relations (3.1), (3.2) and (3.3), the estimation

$$\|R_n(f; g)\| \leq M_3(\|\rho\|_1 + \|\rho \lambda_n\|_1) \cdot \|g/\rho\|_\infty \cdot \omega\left(f; \frac{1}{n}\right) \tag{3.4}$$

holds for sufficient large $n \geq 1$.

The following step is to estimate $\|\rho\lambda_n\|_1$. We have:

$$\begin{cases} \|\rho\lambda_n\|_1 = \int_{-1}^{1} (1-x)^a (1+x)^b \lambda_n(x)dx = I_n^{(1)} + I_n^{(2)}, \text{ with} \\ I_n^{(1)} = \int_{-1}^{0} (1-x)^a (1+x)^b \lambda_n(x)dx \text{ and} \\ I_n^{(2)} = \int_{0}^{1} (1-x)^a (1+x)^b \lambda_n(x)dx. \end{cases} \tag{3.5}$$

Using the estimation

$$\lambda_n(x) - 1 \sim |P_n^{(\alpha,\beta)}|\sqrt{n}[1 + (1-x)^{(2\alpha+1)/4} \log n], \ 0 \le x \le 1, \ [6],$$

we obtain

$$I_n^{(2)} \sim \int_0^1 (1-x)^a dx + \sqrt{n} \int_0^1 (1-x)^a |P_n^{(\alpha,\beta)}(x)| dx$$
$$+\sqrt{n}(\log n) \int_0^1 (1-x)^{a+\alpha/2+1/4} |P_n^{(\alpha,\beta)}(x)| dx. \tag{3.6}$$

Next, the estimation [10, formula (7.34.1)]

$$\int_0^1 (1-x)^\mu |P_n^{(\alpha,\beta)}(x)| dx \sim \begin{cases} n^{\alpha-2\mu-2}, & \alpha > 2\mu + 3/2 \\ n^{-1/2} \log n, & \alpha = 2\mu + 3/2 \\ n^{-1/2}, & \alpha < 2\mu + \frac{3}{2} \end{cases} ; \ \alpha, \beta, \mu > -1,$$

gives for $\mu = a$ and $\mu = a + \alpha/2 + 1/4$, respectively:

$$\int_0^1 (1-x)^a |P_n^{(\alpha,\beta)}(x)| dx \sim \begin{cases} n^{\alpha-2a-2}, & \alpha > 2a + 3/2 \\ n^{-1/2} \log n, & \alpha = 2a + 3/2 \\ n^{-1/2}, & \alpha < 2a + 3/2 \end{cases} \tag{3.7}$$

$$\int_0^1 (1-x)^{a+\alpha/2+1/4} |P_n^{(\alpha,\beta)}| dx \sim n^{-1/2}. \tag{3.8}$$

Finally, a combination of (3.6), (3.7) and (3.8) yields:

$$I_n^{(2)} \sim 1 + \log n + \begin{cases} n^{\alpha-2a-3/2}, & \alpha > 2a + 3/2 \\ \log n, & \alpha = 2a + 3/2 \\ 1, & \alpha < 2a + 3/2. \end{cases} \tag{3.9}$$

A similar estimation holds for $I_n^{(1)}$ of (3.5), namely:

$$I_n^{(1)} \sim 1 + \log n + \begin{cases} n^{\beta-2b-3/2}, & \beta > 2b + 3/2 \\ \log n, & \beta = 2b + 3/2 \\ 1, & \beta < 2b + 3/2. \end{cases} \tag{3.10}$$

Now, we prove the following statement.

Theorem 3.1. *If $\rho(x) = (1-x)^a(1+x)^b$, $-1 < \alpha \leq 2a + 3/2$ and $-1 < \beta \leq 2b + 3/2$, then the product quadrature formulas given by* (1.1) *and* (1.2) *are convergent for each $g \in L_\infty^{(1/\rho)}$ and for each $f \in C$ satisfying a Dini-Lipschitz condition*

$$\lim_{\delta \searrow 0} \omega(f;\delta) \log \delta = 0.$$

Proof. The relations (3.5), (3.9) and (3.10) lead to the estimation $\|\rho\lambda_n\|_1 \sim \log n$. which combined with (3.4) gives:

$$|R_n(f;g)| \leq M_4 \cdot \|g/\rho\|_\infty \cdot \omega\left(f;\frac{1}{n}\right) \log n,$$

for sufficient large $n \geq 1$, which completes the proof. □

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