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CLASSES OF MENGER SPACES WITH THE FIXED POINT PROPERTY FOR PROBABILISTIC CONTRACTIONS

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Abstract. We present the most important contributions to the theory of probabilistic contractions of Sehgal type and a new method of obtaining fixed point theorems on Menger spaces under Archimedean triangular norms.

1. Some history

1.1. **Introduction.** The notion of a Probabilistic Metric has been introduced by **K. Menger in 1942** as a function

$$S \times S \ni (p,q) \xrightarrow{\mathcal{F}} F_{pq} \in \mathcal{D}_+$$

where $\mathcal{D}_+$ is the set of all distribution functions F, for which $F(0) = 0$, and the following axioms are imposed:

I. $F_{xy} = \varepsilon_0$ if and only if $x = y$

II. $F_{xy} = F_{yx}\ \forall x, y \in X$.

III_M. $F_{xz}(t+s) \geq T(F_{xy}(t), F_{yz}(s))$

Here $T : [0,1] \times [0,1] \to [0,1]$ is supposed to satisfy the conditions: T is nondecreasing in each variable , it is symmetric and $T(1,1) = 1$, $T(a,1) > 0\ \forall\ a > 0$.

Thus the *the first idea of Menger* was to use distribution functions instead of nonegative real numbers as values of the metric .The second *useful idea* was the formulation of the triangle inequality (III_M) .

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A. Wald in 1943 proposed the following triangle inequality , accepted by Menger himself in subsequent works:

$$(III_w) \quad F_{pq} \geq F_{pr} * F_{rq}$$

which admits a natural interpretation

$$Prob\{dist(p,q) < x\} \geq Prob\{dist(p,r) + dist(r,q) < x\}$$

if the "distances" are considered independent random variables .

B.Schweizer &A.Sklar in 1960 (see also [47]) reconsidered the problem of the triangle ienquality by imposing the associativity for T; thus $([0,1],T)$ is a commutative semigroup with 1 as unity and T(a,b)≤Min{a,b}.Min, Prod and $W = Max\{Sum - 1, 0\}$ are the most important t-norms ; $(S, \mathcal{F}, T)$ is called *PM-space* and T a triangular norm or *t-norm.*

In the same works they introduce two new important notions, *the* (ε, λ)-*topology* ($\mathcal{F}$-topology, strong topology), generated by $\{N_p(\varepsilon,\lambda)\}_{\varepsilon>0,\ \lambda\in(0,1)}$ and *the* (ε, λ)-*uniformity(* $\mathcal{F}$-uniformity),generated by $\{U(\varepsilon,\lambda)\}_{\varepsilon>0,\ \lambda\in(0,1)}$(see 2.1. below).

In the same year Schweizer, Sklar and R.Thorp proved that *if* $\sup_{x<1} T(x,x) = 1$, *then the* (ε, λ) *-uniformity exists and it is metrizable.*

Later on, J.Nagata & B.Morrel and, independently, U.Hohle, proved that the above condition on T is *the weakest one* which ensures the existence of the $\mathcal{F}$-uniformity.

In 1962 -1963, A.N. Šerstnev proposed a new formulation of the triangle inequality , by means of a nondecreasing (semigroup) operation τ on $\mathcal{D}_+$(a *t-function*)

$$III_S \quad F_{pq} \geq \tau(F_{pr}, F_{rq}) \ , \ \tau : \mathcal{D}_+ \times \mathcal{D}_+ \to \mathcal{D}_+ \ , \tau(H_0) = H_0$$

and has formulated explicitly a metric-like function which seemingly agrees with the uniformity , namely

$$d(p,q) = \Sigma \frac{1}{2_n} \cdot \frac{1 - F_{pq}(\frac{1}{n})}{2 - F_{pq}(\frac{1}{n})}$$

Later on I observed that d does not verify yhe triangle inequality , and therefore this is not an adequate example.***Šerstnev*** introduced also the notion of random normed space[1].

Now we present some results concerning the

1.2. **Fixed Point Principles in Menger spaces. In 1966** V.M Sehgal & A.T Bharucha- Reid(see [50] and [51]) have introduced the notion of probabilistic contraction on Menger spaces, namely functions $f : S \to S$ s.t. there is an L , $0 < L < 1$, for which

$$\text{(PC)} \quad F_{fp,fq}(Lx) \geq F_{pq}(x), \forall x > 0,$$

and they proved the following

E1. *Any probabilistic contraction on a complete Menger space (S, F, Min) has a fixed point , which is the limit of the successive approximations, defined by $p_{n+1} = fp_n$, that is a principle of Banach- type holds for the t-norm Min .*

In 1976 , G.L.Cain and R.Kasriel proved the above theorem of Sehgal&Bharucha-Reid by a different method : *If $d_b(p,q) = sup\{x, F_{pq} \leq b\}$ is defined on (S, F, Min), then d_b is a semimetric on S and $\{d_b\}_{b\in(0,1)}$ generates the (ε, λ)-topology ;* and they obtained the Banach's principle from a classical result for $(S, \{d_b\})$.

In 1971 , H.Sherwood gave an example of a complete Menger space $(S, \mathcal{F}, T_m)$ together with a probabilistic contraction with no fixed points(see also[47]).

A fundamental result has been obtained by O. Hadžić **in 1978** (cf [9], [10], [11] and [15])which extended the class of t-norms for which th. BP. holds:

E2. *If T is continuous and T^n are equicontinuous at $x = 1$, then the B.P. holds in every complete Menger space $(S, \mathcal{F}, T)$*

[1] At the West University of Timişoara a Seminar on PM-spaces has been created in 1972. The most part of the contributions to this Seminar are due to D.Barbu,Gh.Bocşan, Gh.Constantin, I.Istrăţescu, D.Mihet, V.Radu and D.Zaharie. A number of 125 papers have been issued in preprints and most of them appeared in well known periodicals. A series of monographs (three volumes until now) has been created. Some of the topics of the Seminar are *Measures of noncompactness , Fixed points , NonArchimedean PM-spaces,Construction of deterministic metrics and Random operators &equations.*

In 1983 I have shown that the condition of continuity of T can be dropped . Also I obtained the following characterization of the *t-norms of Hadžić- type :*

E3. *The following are equivalent , for a continuous t-norm T,*

a) $\{T^n\}$ *is equicontinuous at* $x = 1$

b) $\forall\, a \in [0,1)$, $\exists\, b \in [a,1)$ *such that* $T(b,b) = b$

By using this result one can obtain the following:

E4. *Let T be a continuous t-norm . Then the B.P. holds in every complete Menger space* $(S, \mathcal{F}, T)$ *if, and only if, T is of Hadžić- type.*

E5. *The following are equivalent*

α) *There exists a complete* $(S, \mathcal{F}, T_m)$ *in which the Banach's Principle fails;*

β) *There exists a complete* $(S, \mathcal{F}, Prod)$ *complete, in which the Banach's Principle fails;*

γ) *There exists a complete* $(S, \mathcal{F}, T)$, *where T is not of Hadžić- type t-norm, in which the Banach's Principle fails.*

E6. *E4 is essentially equivalent to the classical Banach Principle.*

Remark. In the general case for equicontinuous T^n (at $x = 1$) one can use a method similar to that proposed by Cain & Kasriel, by using a countable family of pseudo-metrics:

$$b_n \nearrow 1 \ , \ T(b_n, b_n) = b_n$$

$$d_n(p,q) = inf\{t \ , \ F_{pq}(t) \geq b_n\}, n = 1, 2, ...$$

1.3. **Hicks contractions and generalizations. In 1983,** T.L.Hicks has introduced a different condition for contractions

$$(c_h) \quad \exists L < 1 \vdots F_{pq}(t) > 1 - t \ \Rightarrow \ F_{fpfq}(Lt) > 1 - Lt$$

and has shown that

E7. *Every c_h-contraction on a complete Menger space $(S,\mathcal{F},Min)$ has a unique fixed point , which is the limit of the succesive approximations .*

The idea of Hicks' proof is as follows:

- Construct a metric ρ on S, for which
- (S,ρ) is a complete metric space and f is an L strict contraction on (S,ρ).
- Apply the classical B.P.

I essentially extended the above result in two directions(see e.g. [40, 42]).

(a) *E7 remains true for every $T \geq$ W and the proof is similar to that of Hicks ;*

(b) *The result of Hicks is true for every t-norm with the property* $\sup_{t<1} T(t,t) = 1$.

If we observe that the condition (c_h) can be rewritten in the forms :

$$t > 1 - F_{pq}(t) \Rightarrow Lt > 1 - F_{fpfq}(Lt)$$

$$t > h \circ F_{pq}(t) \Rightarrow Lt > h \circ F_{fpfq}(Lt)$$

where $h(u) = 1 - u$, $u \in [0,1]$, we can give more extensions.

Let $\mathcal{M}$ be the family of mappings m:$[0,\infty] \to [0,\infty]$, such that

a) $m(t) = 0 \iff t = 0$;

b) m is continuous ;

c) $m(t+s) \geq m(t) + m(s)$.

Lemma. *Let us suppose that*

(i) $m \in \mathcal{M}$;

(ii) $h : [0,1] \to [0,\infty]$ *is a continuous decreasing function, and* $h(1) = 0$;

(iii) $(S,\mathcal{F},T)$ *is a Menger space,with* $T \geq T_h$.

Then

$$\rho(p,q) = k_{mh}(p,q) := sup\{t\ ,\ m(t) \leq h \circ F_{pq}(t)\}$$

gives a metric which generates the (ε,λ)*-uniformity.*

Theorem 1.3. *Consider a complete Menger space* $(S, \mathcal{F}, T)$ *and a self mapping f of S such that*

$$h \circ F_{pq}(t) < m(t) \Rightarrow h \circ F_{fpfq}(Lt)] < m(Lt)$$

where h is as the Lemma , with $h(0) < \infty$. *Then f has a unique fixed point (which is the limit of successive approximations).*

Remarks.

(1) If $T \geq T_h$, then f is a ρ-strict contraction and the classical BP can be applied . In this case $h(0)$ may be ∞.

(2) The formula

$$\rho(p, q) = sup\{t \ , \ m_1(t) \leq 1 - F_{pq}(m_2(t))\}$$

where $F_{pq}(t) = Prob(d(p, q) < t)$, gives a metric for the convergence in probability , extending the case $m_1(t) = m_2(t) = t$, when one obtains the Ky Fan metric.

(3) The above ideas and methods have been used and extended by many authors(see [4], [5], [6, 7], [11, 13, 14, 15], [25], [48], [54]).

1.4. **Some comparisons.** The following examples, essentially taken from the very interesting paper [48], clarify the independence of the two types of contractions:

1. Let (S, d) be a metric space, and $f : S \to S$ an $L-$ isometry: $d(fp, fq) = Ld(p, q)$. If we set $F_{pq}(x) = \frac{x}{x+d(p,q)}$, then we obtain a Menger space (S, F, Min).
   (a) f is a probabilistic contraction, since $F_{fpfq}(L\varepsilon) = F_{pq}(\varepsilon)$.
   (b) It is easily seen that $\rho(p, q) = 2d(p, q)/(d(p, q) + sqr(d^2(p, q) + 4d(p, q)))$;and $\rho(p, q) \to 1$ for $d(p, q) \to 1$,so f cannot be a strict contraction on (S,ρ).
2. Let S={0,1,2,...} and d(p,q)= max$\{L^p, L^q\}, L \in (0, 1)$. Define

$$F_{pq}(x) = \begin{cases} 0, & x \leq d(p,q) \\ 1 - d(p,q), & d(p,q) \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

If we take $f(p) = p + 1$, then f is a Hicks contraction . Now if $1 \geq x > L_1$, $x > Ld(p,q)$, then $d(fp, fq) = Ld(p,q) < x \leq 1$,and $F_{fp\ fq}(x) = F_{p+1\ q+1}(x) = 1 - d(fp, fq)$. On the other hand $F_{pq}(\frac{1}{L_1}) = 1$, that is $F_{fp\ fq}(x) < F_{pq}(\frac{x}{L_1})$ and so f is not a probabilistic contraction.

3. Generally , if we have a probabilistic contraction f, then $\rho(p,q) < \varepsilon \Rightarrow \varepsilon > 1 - F_{pq}(\varepsilon) \Rightarrow F_{pq}(\varepsilon) > 1 - \varepsilon \Rightarrow F_{fpfq}(\varepsilon) \geq F_{fpfq}(L\varepsilon) \geq F_{pq}(\varepsilon) > 1 - \varepsilon \Rightarrow \rho(fp, fq) < \varepsilon$, thus $\rho(fp, fq) \leq \rho(p,q)$,that is f is nonexpansive.This explains in some sense the counterexamples of type Sherwood.For more details,examples and counterexamples, see [48], [42] and [15].

2. The "fixed point property" for t-norms

2.1. **Probabilistic (semi)metric spaces.** Let $\mathcal{D}_+$ be the family of all distribution functions F (nondecreasing and left continuous on $\mathbf{R}$, with $\inf F = 0$ and $\sup F = 1$) for which $F(0) = 0$. For every $a \geq 0$, ε_a will be the unique element of $\mathcal{D}_+$ for which $\varepsilon_a(a+) - \varepsilon_a(a) = 1$.

Definition 2.1.1 (cf. [46],[47],[7],[42]). Let X be a nonempty set and $\mathcal{F} : X \times X \longrightarrow \mathcal{D}_+$ a given mapping ($\mathcal{F}(x,y)$ will be denoted by F_{xy}). The pair $(X, \mathcal{F})$ is called a *probabilistic semimetric space* (shortly PSM-space) if

I. $F_{xy} = \varepsilon_0$ if and only if $x = y$

II. $F_{xy} = F_{yx}\ \forall x, y \in X$.

If any kind of "triangle inequality" is verified we use the term *probabilistic metric space* (PM-space). The weakest one is that proposed in [46]:

III_{SS} $[F_{xy}(t) = 1, F_{yz}(t) = 1] \Rightarrow F_{xz}(t+s) = 1$

If there exists a triangular norm T [46] such that

III_M $F_{xz}(t+s) \geq T(F_{xy}(t), F_{yz}(s))$

then one uses the term *Menger space*. A more general form for III_M, giving σ-*Menger spaces,* has been formulated by using some operations σ on $[0,\infty)$, instead of the addition (cf. [42], [44]).

In [18] is proposed the inequality

$$III_H \ \forall\varepsilon > 0\ \exists\delta > 0\ s.\,t.\ [F_{xy}(\delta) > 1-\delta, F_{yz}(\delta) > 1-\delta] \Rightarrow F_{xz}(\varepsilon) > 1-\varepsilon$$

which has been generalizated as

$$III_h \ \forall\varepsilon > 0\ \exists\delta > 0\ s.t.\ [h\circ F_{xy}(\delta) < \delta\ ,\ h\circ F_{yz}(\delta) < \delta] \Rightarrow h\circ F_{xz}(\varepsilon) < \varepsilon$$

by using additive generators h (cf. [39], [41], [42]).

For every PSM-space $(X,\mathcal{F})$ we can consider the sets of the form

$$U_{\varepsilon,\lambda} = \{(x,y) \in X\times X, F_{xy}(\varepsilon) > 1-\lambda\},\ \varepsilon > 0, \lambda\in(0,1)$$

which generates a semiuniformity denoted by $\mathcal{U}_{\mathcal{F}}$ and a topology $\mathcal{T}_{\mathcal{F}}$. Namely,

$$\mathcal{O}\in\mathcal{T}_{\mathcal{F}} \text{ iff } \forall x\in\mathcal{O}\ \exists\varepsilon>0,\ \lambda\in(0,1) \text{ s. t. } U_{\varepsilon,\lambda}(x)\subset\mathcal{O}$$

Actually $\mathcal{U}_{\mathcal{F}}$ can also be generated by the family of the sets $V_\delta := U_{\delta,\delta}$.

Proposition 2.1.2. *Let* $(X,\mathcal{F})$ *be a PSM-space and define the two- place mapping*

(1) $k(x,y) = \sup\{t | t \leq 1 - F_{xy}(t)\}$. *Then* k *is a semi-metric (of Ky Fan type) on* X *and*

(2) $k(x,y) < \delta \Leftrightarrow F_{xy}(\delta) > 1-\delta\ ,\forall\delta > 0,$

which shows that k *generates the topology* $\mathcal{T}_{\mathcal{F}}$ *(and the semiuniformity* $\mathcal{U}_{\mathcal{F}}$ *).*

The *proof* is easy to reproduce (cf. [16], [40], [17]).

Examples 2.1.3.

(i) If d is a semi-metric on X and we set $F_{xy} := \varepsilon_{d(x,y)}$ then $(X, \varepsilon_{d(.,.)})$ is a PSM-space and $k(x,y) = \min(d(x,y),1)$.

(ii) Let X be the family of all classes of $\mathbf{R}$-valued random variables on a probability measure space $(\Omega,\mathcal{K},P)$. If we set $\mathcal{F}(x,y) = F_{|x-y|}$, the distribution function of $|x-y|$, then $(X,\mathcal{F},W)$ is a Menger space and k is the Ky Fan metric of the convergence in probability (cf. [19]). Here **R** can be replaced e.g. by any separable metric space (with $|x-y| = dist(x,y)$).

It is to be noted that, generally, k *need not be a metric.* In order to ensure the verification of the triangle inequality for k, T. L. Hicks [17] proposed the following form of the triangle inequality for $(X, \mathcal{F})$:

$$III^1.\ [F_{xy}(t) > 1-t, F_{yz}(s) > 1-s] \Rightarrow F_{xz}(t+s) > 1-(t+s)$$

and he observed that the property III^1 holds for every Menger space $(X, \mathcal{F}, T)$ for which $T \geq W$.

As a matter of fact one has the following

Proposition 2.1.4. *Let T be a t-norm such that the property (III^1) holds for every Menger space (X, F, T). Then $T \geq W$.*

Proof. This will follow from the following well known example. Let $X = \{x, y, z\}$, $F_{xy} = F_{yx}, F_{yz} = F_{zy}, F_{xz} = F_{zx}$, where

$$F_{xy}(t) = \begin{cases} 0 & t \leq 0 \\ a & t \in (0,1] \\ 1 & t > 1 \end{cases}, \quad F_{yz}(t) = \begin{cases} 0 & t \leq 0 \\ b & t \in (0,1] \\ 1 & t > 1 \end{cases},$$

$$F_{zx}(t) = \begin{cases} 0 & t \leq 0 \\ T(a,b) & t \in (0,1] \\ 1 & t > 1 \end{cases}$$

and $F_{xx} = F_{yy} = F_{zz} = \varepsilon_0$. Then $(X, \mathcal{F}, T)$ is a Menger space (for which T is the best t-norm) and $k(x,y) = 1-a$, $k(y,z) = 1-b$, while $k(x,z) = 1-T(a,b)$. Thus we see that $k(x,z) \leq k(x,y) + k(y,z) \Leftrightarrow T(a,b) \geq a+b-1$. □

Remark 2.1.5. Let $(X, \mathcal{F}, T)$ as in the above proof and suppose that $T(a,b) < a+b-1$. Therefore $0 < a, b < 1$ and there exists $p > 1$ such that $((1-a)^{\frac{1}{p}} + (1-b)^{\frac{1}{p}})^p > 1 - T(a,b)$. Thus $(1-a)^{\frac{1}{p}} + (1-b)^{\frac{1}{p}} > (1-T(a,b))^{\frac{1}{p}}$ and we see that k_p, given by $k_p(u,v) = \sup\{t | t^p \leq 1 - F_{uv}(t)\}$, is verifying the triangle inequality. This shows that the more general formulae proposed in [39], [41] and [44] can give metrics in many situations.

It is easy to see that for each $m \in \mathcal{M}$ there exists $t_m > 0$ such that $m : [0, t_m) \longrightarrow [0, \infty)$ is strictly increasing and invertible. If we set, for any PSM-space $(X, \mathcal{F})$,

(1_m) $k_m(x, y) = \sup\{t | t \geq 0, m(t) \leq 1 - F_{xy}(t)\}$

then k_m is a semi-metric. Moreover

(2_m) $k_m(x, y) < \delta \Leftrightarrow F_{xy}(\delta) > 1 - m(\delta)$

from which it follows that k_m generates $\mathcal{T}_{\mathcal{F}}$ and $\mathcal{U}_{\mathcal{F}}$.

This suggests the following definition, which extends (III^1):

Definition 2.1.6. A PSM-space $(X, \mathcal{F})$ for which takes place the following triangle inequality

$$III^m.\ [F_{xy}(t) > 1 - m(t), F_{yz}(s) > 1 - m(s)] \Rightarrow F_{xz}(t + s) > 1 - m(t + s)$$

is called *PM-space of type* $\mathcal{M}$.

In [35] there are presented some fixed point theorems in these classes of PM-spaces.

2.2. **On the fpp for triangular norms.** As we have seen,probabilistic contractions have been introduced by V. M. Sehgal [50]. It is now well known that every probabilistic contraction on a complete Menger space (S, $\mathcal{F}$, Min) has a unique fixed point, which is the limit of succesive approximations. In [53] H. Sherwood constructed complete Menger spaces together with probabilistic contractions which do not have fixed points. O. Hadžić [9] introduced a class of t-norms for which the contraction principle holds [10]. In [38] we proved that a continuous t-norm has the fixed point property iff it is of Hadžić-type.

In the present section, we further investigate the fixed point property of t-norms, by using the structure of continuous t-norms as given in [32]. Essentially we prove that *a t-norm does not have the fpp iff in a neighborhood of 1 it has a behavior similar to that of* $W = Max(Sum - 1, 0)$. Thus the counterexample of H. Sherwood can be generally used for all t-norms which are not of Hadžić-type.

Let I denote the closed unit interval. A t-norm is a two-place function $T : I \times I \to I$ such that T is associative, commutative, nondecreasing in each place and such that $T(a,1) = a$ for each $a \in I$. For a fixed t-norm T, T^m is defined inductively on I by

$$(1) \quad T^1(x) = x, \ T^{m+1}(x) = T(T^m(x), x)$$

We say that T *is of h-type* (and write $T \in \mathcal{H}$) if $\{T^m\}$ is equicontinuous at $x = 1$. The following result is a consequence of [37], [38] and [32]:

Lemma 2.2.1. *(i) If T verifies the condition*

$$(2) : \forall a \in (0,1), \exists b \in [a,1), \ s.t \ T(b,b) = b$$

then $T \in \mathcal{H}$.

(ii) If $T \in \mathcal{H}$. and T is continuous, then (2) holds.

Theorem 2.2.2. *Let T be a continuous t-norm. Then $T \notin \mathcal{H}$ iff*

$$(3) \ \exists a \in [0,1) \text{ such that } T(a,a) = a, T(x,x) < x, \forall x \in (a,1)$$

Proof. By Lemma 2.1, $T \notin \mathcal{H}$ iff (2) is false. If $\exists a_0 \in (0,1)$ such that For each $b \in [a_0, 1)$ one has$T(b,b) < b$, then let

$$a = \lim_{m \to \infty} T^m(a_0).$$

Since $T^{m+1}(a_0) \leq T^m(a_0) \leq a_0$, then $a \in [0,1)$ always exists. Moreover, as $T^{2m+1}(a_0) = T(T^m(a_0), T^m(a_0))$ and T is continuous, then $a = T(a,a)$. Let $b \in (a, a_0)$. If $T(b,b) = b$, then $a = T^m(a) < b = T^m(b) \leq T^m(a_0)$, that is $b \leq a$, a contradiction.

Thus, if (2) is false then (3) holds. The converse is obvious. □

Remark 2.2.3. The number a in (3) is uniquely determined and will be denoted by a_T.

From (3) and [32] we obtain the following

Theorem 2.2.4. *Let T be a continuous t-norm. Then $T \notin H$ iff there exist $a_T \in [0,1)$ and an increasing bijection $h_T : [a_T, 1] \to [0,1]$ such that*

$$(4) : T(\alpha,\beta) = h_T^{-1}[T_*(h_T(\alpha), h_T(\beta))], \forall \alpha, \beta \geq a_T$$

where $T_* = W$ or $T_* = Prod$ (T_* depends only on T).

The following lemma is easy to reproduce:

Lemma 2.2.5. *Let T be a continuous t-norm, $T \notin \mathcal{H}$*

(i) If $(S, \mathcal{F}, W)$ is a Menger space, then $(S, e^{\mathcal{F}-1}, Prod)$ is a Menger space with the same (ε, λ)-uniformity;

(ii) If $(S, \mathcal{F}, T_*)$ is a Menger space, then $(S, h_T^{-1} \circ \mathcal{F}, T)$ is a Menger space with the same (ε, λ)-uniformity;

(iii) If $(S, \mathcal{F}, T)$ is a Menger space, then $(S, h_T \circ \mathcal{F}, T_*)$ is a Menger space with the same (ε, λ)-uniformity. (the notations are as in Theorem 2.4).

One says [38] that T *has the fixed point property* (f.p.p.) iff every probabilistic contraction on a complete Menger space $(S, \mathcal{F}, T)$ has a fixed point (it is obvious that this fixed point is unique and it can be obtained as the limit of the succesive approximations).

If $T \in \mathcal{H}$, then it is well known that T has the f.p.p. and this can be proven by different methods([10], [38], [3]) or is a consequence of the classical Banach principle [37].

For t-norms which *are not of h-type* we have the following

Theorem 2.2.6. *Let T be an arbitrary but fixed t-norm such that $T \notin \mathcal{H}$. Then the following are equivalent*

(i) T does not have the f.p.p.;

(ii) Prod does not have the f.p.p.;

(iii) W does not have the f.p.p.

Proof. Firstly we observe that the constructions in Lemma 2.5 do not change the property of f of being a probabilistic contraction. Therefore the equivalence (ii)⇔(iii) results for Lemma 2.5 (i), for $Prod \geq W$. The fact that (i)$\Rightarrow$(ii) or (iii) is a consequence of Lemma 2.5. (iii). The implication (ii) and (iii)$\Rightarrow$(i) is a consequence of Lemma 2.5. (ii), and the theorem is proved. □

In [53] it is proved by an example that W does not have the fixed point property. Thus we have the following

Corollary 2.2.7 ([38]). *If T is a continuous t-norm such that $T \notin \mathcal{H}$, then T does not have the fixed point property.*

3. Some "iff" conditions for the f.p.p. in the archimedean case

From the above it seems very clear that in order to can hope to obtain some kind of fixed point theorems in the case of Archimedean(or not of Hadzic type) t-norms, one has to impose supplementary conditions either on the probabilistic contractions or on the probabilistic metrics.

Some positive effort has been made in this sense by H. Sherwood [53], R. M. Tardiff [55], V. Radu [42, 43] and E. Părău & V. Radu [34, 35].

Nevertheless we think that the problem has not yet a satisfactory answer, especially for concrete purposes. This is seen from the following simple case of affine mappings on E-spaces:

Example 3.0. Let $L_0(0,1)$ be the space of all classes of random variables on the Lebesgue measure space $((0,1), \mathcal{L}, \lambda)$ and fix the element w defined by the mapping $t \to e^{\frac{1}{t}}$. Let S be any closed (for the convergence in probability) linear subspace of $L_0(0,1)$ which contains w and 1. Now define $f$on S by

$$fp = Lp + (1 - L)w$$

when L is fixed in (0, 1). It is easily seen that

$$f^n p_0 = L^n p_0 + (1 - L^n)w \to w = fw$$

On the other hand, the distribution function of w has the value 0 for $x \leq e$ and $1 - \frac{1}{lnx}$ for $x > e$, such that $\int_1^\infty lnx dF_w(x) = +\infty$. Therefore the conditions of [55] are not verified for $F_{\lambda w \mu w}$, with $\lambda \neq \mu$. At the same time, for every $k > 0$,

$$\int_0^1 w^k(t)dt \geq \int_\epsilon^1 w^k(t)dt \geq \int_\epsilon^1 (1+\frac{k}{t})dt = 1-\epsilon-k\ln\epsilon \xrightarrow{\epsilon\to 0} \infty$$

which shows that we cannot hope to work in any Lebesgue space $L_k(0,1)$:

$$p_0 \in L_k(0,1) \Leftrightarrow f^n p \notin L_k(0,1), n \geq 1$$

The aim of this section is to obtain a *characterization* of the probabilistic contractions, on complete Menger spaces under Archimedean t-norms, which have a fixed point. Our method of proof is very simple and is based upon a *new* family of *metrics* which all generate the strong $\mathcal{F}$-uniformity and seem to be appropriate for studying probabilistic contractions.

3.1. **A family of semi-metrics on PM- spaces.** In the following lemma we introduce a family of nonnegative functions which measure the distance between ϵ_0 and the elements of $\mathcal{D}_+$.Let k be a (fixed) positive real number.

Lemma 3.1.1. *The one-place mapping $\delta_k : \mathcal{D}_+ \to \mathbf{R}_+$, given by*

$$(1) \quad \delta_k(F) := \sup_{x>0}\{x^k[1-F(x)]e^{-x}\},$$

has the following properties:

(i) $\delta_k(F) = 0 <=> F = \epsilon_0$;

(ii) *If* $F_1 \leq F_2$, *then* $\delta_k(F_1) \geq \delta_k(F_2)$;

(iii) $\delta_k(\lambda \circ F) \leq \lambda^k \delta_k(F), \forall \lambda \geq 1$;

(iv) $\delta^{k+1}e^{-\delta} \leq \delta_k(F) \leq \max\{\delta^k, \delta k^k e^{-k}\}$,

where $\delta = \delta(F) := \sup\{t | t \leq 1 - F(t)\}$ *is the écart of Ky Fan.*

(v) $\delta_k(F_n) \to 0 <=> F_n(x) \to 1$, *for each* $x > 0$.

Proof. We will give only the proof of (iv):

a) $\delta_k(F) = \sup\{x^k[1-F(x)]e^{-x}\} \geq \delta^k[1-F(\delta)]e^{-\delta} \geq \delta^{k+1}e^{-\delta}$;

b) If $0 < x \leq \delta$, then $x^k[1-F(x)]e^{-x} \leq \delta^k$. If $\delta < x$, then $1 - F(x) \leq 1 - F(\delta + 0) \leq \delta$.

Therefore $x^k[1-F(x)]e^{-x} \leq \delta x^k e^{-x} \leq \delta k^k e^{-k}$. □

Proposition 3.1.2. *Let $(S,\mathcal{F})$ be a probabilistic metric space and define*

$$(2)\quad e_k(p,q) := \delta_k(F_{pq}) = \sup_{x>0} x^k[1-F_{pq}(x)]e^{-x}, \forall p,q \in S$$

Then

1^o e_k *is a* ***semi-metric*** *for the strong* $\mathcal{F}$*-topology;*

2^o e_k *generates the* $\mathcal{F}$*-uniformity, if the latter exists;*

3^o *If* $(S,\mathcal{F},W)$ *is a Menger space, then*

$$(3)\quad (p,q) \to \theta_k(p,q) := \{e_k(p,q)\}^{\frac{1}{k+1}}$$

gives ***a metric*** *on* S*. Moreover,* $(S,\mathcal{F})$ *is complete if and only if* (S,θ_k) *is complete.*

Proof. 1^o and 2^o follow from Lemma 1.1. and the definitions. In order to prove 3^o, let us recall that $(S,\mathcal{F},W)$ is a Menger space iff the following inequality holds

$$(4)\quad 1-F_{pq}(x) \le 1-F_{pr}(tx)+1-F_{rq}[(1-t)x], \forall p,q,r \in S, \forall x \in \mathbf{R}, \forall t \in [0,1]$$

If we fix $p,q,r \in S$, then we have, for each $x > 0$:

$$x^k[1-F_{pq}(x)]e^{-x} \le x^k[1-F_{pr}(tx)]e^{-tx} + x^k[1-F_{rq}[(1-t)x]e^{-(1-t)x}, \forall t \in (0,1)$$

Then

$$x^k[1-F_{pq}(x)]e^{-x} \le \frac{1}{t^k}e_k(p,r) + \frac{1}{(1-t)^k}e_k(r,q), \forall t \in (0,1)$$

This implies the inequality

$$e_k(p,q) \le \frac{1}{t^k}e_k(p,r) + \frac{1}{(1-t)^k}e_k(r,q), \forall t \in (0,1)$$

and we easily obtain that

$$\{e_k(p,q)\}^{\frac{1}{k+1}} \le \{e_k(p,r)\}^{\frac{1}{k+1}} + \{e_k(r,q)\}^{\frac{1}{k+1}}$$

that is θ_k verifies the triangle inequality.

The last part of the proposition follows from the inequality (iv) of the Lemma 1.1. □

Remark 3.1.3. The above proof shows that if, instead of (4), we have

$$(4') \quad 1 - F_{pq}(x) \leq 1 - F_{pr}(x) + 1 - F_{rq}(x), \forall p, q, r \in S, \forall x \in \mathbf{R},$$

that is $(S, \mathcal{F}, W)$ is nonArchimedean, then e_k **itself is a metric** which generates the $\mathcal{F}$ - uniformity.

3.2. **An iff condition for probabilistic contractions to have a fixed point.** We are in position to give a slight improvement of the results from [42, 43] and [34]:

Theorem 3.2.1. *Let $(S, \mathcal{F}, T)$ be a complete Menger space such that $T \geq W$. If $f : S \to S$ is a probabilistic contraction, that is*

$$(5) \quad F_{fpfq}(x) \geq F_{pq}(\frac{x}{L}), \forall x \in \mathbf{R}$$

*for some $L \in (0,1)$ and all pairs $(p,q) \in S \times S$, then **the following are equivalent***

(5.1) f has a fixed point

(5.2) There exist $p \in S$ and $k \in (0, \infty)$ such that

$$E_k(p) := \sup_{x>0}\{x^k[1 - F_{pfp}(x)]\} < \infty$$

Proof. The implication (5.1) $\Rightarrow$ (5.2) is obvious:

$$p = fp \;\Rightarrow\; F_{pfp}(x) = 1, \;\forall x > 0 \Rightarrow\; E_k(p) = 0$$

Now suppose that $E_k(p) < \infty$ for some $p \in S$ and $k \in (0, \infty)$. From the definition of δ_k we see that $\delta_k(F_{pfp}) \leq E_k(p)$. If we take into account the inequality (5), then we get

$x^k[1 - F_{fpf^2p}(x)]e^{-x} \leq x^k[1 - F_{pfp}(\frac{x}{L})]e^{-x} = L^k\{(\frac{x}{L})^k[1 - F_{pfp}(\frac{x}{L})]\}e^{-x} \leq L^k E_k(p)$,which shows that

$$(6) \quad \theta(fp, f^2p) \leq L^{\frac{k}{k+1}}(E_k(p))^{\frac{1}{k+1}}$$

If we apply (6) for f^n, which verifies (5) with L^n instead of L, then we obtain

$$(8) \quad \sum_{n=0}^{\infty} \theta_k(f^n p, f^{n+1} p) \leq \{\sum_{n=0}^{\infty} (L^{\frac{k}{k+1}})^n\}\{E_k(p)\}^{\frac{1}{k+1}} < \infty$$

This clearly implies that $(f^n p)_{n \geq 0}$ is a Cauchy sequence in the complete metric space (S, θ_k), thus it converges to some point $p_* \in S$. Since (5) implies also the continuity of f ,then p_* is a fixed point which is uniquely determined and globally attractive: $F_{f^n p p_*}(x) \geq F_{pp_*}(\frac{x}{L^n}) \to 1$. □

Remarks 3.2.2. a) Simple examples show that f is generally not contractive relatively to θ_k(or e_k).

b)The suppremum in (5.2) may be infinite for some different values of k or for different points in S. This can be seen from the simple case of the Example3.0. and $fp = Lp + w$. Let $a \in S$ such that

$$\sup x^k [1 - F_{|a|}(x)] < \infty$$

and take $p = \lambda a + \frac{1}{L-1} w$. Our condition (5.2) is verified, for $p - fp = \lambda(1 - L)a$. Clearly f has a fixed point $p_* = \frac{1}{1-L} w$ and it is easily seen that $E_k(p_*) = \infty$.

On the other hand the inequality

$$(10) \quad \int_1^{\infty} ln x dF_{pq}(x) < +\infty$$

does not hold for pairs $p = \lambda p_*$, $q = \mu p_*$ with $\lambda \neq \mu$. Thus we could not apply the results of Tardiff [55], which imposed (10) for **all pairs** (p, q) in $S \times S$.

c) Our condition (5.2) is verified if there exists an element p such that $F_{pfp}(t_p) = 1$ for some $t_p > 0$ (H. Sherwood in [53], Corollary) imposed this condition **for all** F_{pq})

d) The condition (5.2) is verified if F_{pfp} has a finite k moment. Thus Theorem 2.1. slightly extends our results in [34, 42]:

Corollary 3.2.3. *If $T \geq W$ and $(S, \mathcal{F}, T)$ is a complete Menger space, then a given probabilistic contraction f on S has a fixed point if and only if there exist $k > 0$ and*

$p \in S$ such that

$$(11) \quad \int_{\infty}^{0} x^k dF_{pfp}(x) < +\infty$$

Proof. It is well known and easy to see that $\lim_{x \to \infty} x^k(1 - F_{pAp}(x)) = 0$, if (11) holds. □

Remark 3.2.4. A t-norm T is Archimedean if and only if there exists an increasing homeomorphism $h : [0,1] \to [0,1]$ such that

$$(12) \quad T(a,b) = h^{-1}(T_*(h(a), h(b)))$$

where $T_* = W$ or $T_* = Prod$ (see Theorem 2.2.4).

Since $ab \geq a + b - 1$ for all $a, b \in [0,1]$, then we obtain the foolowing.

Theorem 3.2.5. *Let $(S, \mathcal{F}, T)$ be a complete Menger space such that $T \geq T_h$ for some increasing homeomorphism $h : [0,1] \to [0,1]$. Then a probabilistic contraction f of S has a fixed point if and only if there exist $k > 0$ and $p \in S$ such that*

$$(13) \quad \sup_{x>0} x^k[1 - h \circ F_{pfp}(x)] < +\infty$$

The **proof** *follows from the fact that $(S, h \circ \mathcal{F}, W)$ is seen to be a complete Menger space, Q.E.D.*

References

[1] A. T. Bharucha-Reid, Fixed point theorems in probabilistic analysis. Bull. Amer. Math. Soc. 82(1976) 641-657.

[2] Gh.Bocsan & Gh. Constantin, The Kuratowski function and some applications to PM-Spaces. Atti Acad. Naz.Lincei, LV(1973),236-240.

[3] G. L. Cain, Jr. and R. H. Kasriel, Fixed and periodic points of local contractions on PM-space. Math. Systems Theory, vol. 9, No. 4 (1975-76) 289-297.

[4] S-s Chang,Y.J.Cho &S.M.Kang, *Probabilistic Metric Spaces and Nonlinear Operator Theory.* Sichuan Univ. Press, Chengdu,1994

[5] Gh. Constantin,On some classes of contraction mappings in Menger spaces, *Seminarul de Teoria Probabilităţilor şi Aplicaţii*, Nr. 76 (1985)

[6] Gh. Constantin, & I. Istrăţescu, *Elemente de analiză probabilistă si aplicaţii.* Ed. Acad. RSR, 1981.

[7] Gh. Constantin, &I. Istrăţescu, *Elements of Probabilistic Analysis with Applications.* Ed. Acad. and Kluwer Academic Publishers, 1989.

[8] I.Golet,*Structuri 2-Metrice Probabiliste.* Ed.Polit. Timisoara,1999.

[9] O. Hadžić, On the (ε, λ)-topology of probabilistic locally convex spaces. Glasnik Mat. 13(1978) 61-66.

[10] O. Hadžić, A generalization of the contraction principle in PM-spaces. *Review of Research Faculty of Science, Univ. of Novi Sad* 10(1980) 13-21.

[11] O. Hadžić, *Fixed Point Theory in Topological Vector Spaces*. Univ.of Novi Sad,1984

[12] O. Hadžić, Some fixed point theorems in PM-spaces, Review of Research Faculty of Science, Univ. of Novi Sad 15(1985), 23-35

[13] O. Hadžić, Fixed point theorems for multivalued mappings in some classes of fuzzy metric spaces. Fuzzy Sets and Systems, 29(1989) 115-125.

[14] O. Hadžić, On (n, f, g) - locally contractions in PM-spaces, Review of Research Faculty of Science, Univ. of Novi Sad 22(1992), 1, 1-10

[15] O. Hadžić, *Fixed Point Theory in Probabilistic Metric Spaces.* University of Novi Sad, 1995.

[16] T.L. Hicks, Fixed point theory in probabilistic metric space.Review of Research Faculty of Science, Univ. of Novi Sad , 13(1983), 63-72.

[17] T.L. Hicks, Fixed point theory in probabilistic metric space II. Math. Japonica 44(1996), No. 3, 487-493

[18] T. L. Hicks, P. L. Sharma, Probabilistic metric structures: Topological classification, Review of Research, Faculty of Science, Univ. of Novi Sad 14(1984),1 43-50

[19] M. Iosifescu, G. Mihoc & R. Theodorescu, *Teoria Probabilităţilor şi Statistică Matematică*, Ed. Tehnică, Bucureşti, 1966

[20] I.I.Istrăţescu,On some fixed point theorems in generalized Menger spaces. Boll.UMI, (5), 13-A(1976), 96-100.

[21] I.I.Istrăţescu,A fixed point theorem for mappings with a probabilistic contractive iterate. Rev. Roum. Math. Pures Appl.26(1981) 431-435.

[22] V.I.Istrăţescu, *Introducere în Teoria SpatiilorMetrice Probabiliste cu Aplicaţii.* Ed.Tehnica, Bucuresti,1974.

[23] K. Margolis, On some fixed point theorems in generalized complete metric spaces. Bull. Amer. Math. Soc. 74 (1968) 275-282.

[24] B.Margolis, J.B. Diaz: A Fixed Point Theorem of the Alternative for Contractions on a Generalized Complete Metric Space. Bull. Amer. Math. Soc. 74, 305-309, 1968.

[25] D. Miheţ, A fixed point theorem for mappings with contractive iterate in H-spaces, Analele Univ. Timişoara, vol. XXXI, 2 (1993)

[26] D.Miheţ, B-contractions in $\sigma-$Menger spaces, STPA,West Univ.of Timisoara,109(1994).

[27] D.Miheţ, A type of contraction mappings in PM-spaces, STPA,West Univ.of Timisoara,114(1994).

[28] D.Miheţ,On a Theorem of O. Hadžić(seeHa 95,45-48).

[29] D.Miheţ, Inegalitatea triunghiului si puncte fixe in PM-spatii. Doctoral Thesis,West Univ.of Timisoara,1997.

[30] D.Miheţ &V.Radu, A fixed point theorem for mappings with contractive iterate.Analele St. univ. Al. I. Cuza Iasi,1997.

[31] B. Morrel, J. Nagata – Statistical metric spaces as related to topological spaces, Gen. Top. Appl. 9(1978) , 233-237.

[32] P. Mostert, A. Shields, On the structure of semigroups on a compact manifold with boundary. Annals of Math. 65(1957) 117-143.

[33] D.H. Mushtari, A.N. Serstnev, On methods of introducing a topology in random metric spaces. Izv. vysh. Uch. Zav. Math. 6(55)(1966), 99-106.

[34] E. Părău & V. Radu, Some remarks on Tardiff's fixed point Theorem on Menger Spaces. Portugaliae Math. 54(4)(1997)431-440.

[35] E. Părău & V. Radu,On the triangle inquality and probabilistic contractions inPM-spaces. Analele Univ.Bucuresti,1998(to appear)

[36] V. Radu,A Remark on Contractions in Menger Spaces. Seminarul de Teoria Probabilităţilor şi Aplicaţii(=STPA), Universitatea din Timişoara, Nr. 64, 1983.

[37] V. Radu, On the t-norms of Hadžić-type and fixed points in PM-spaces. Review of Research Zb. Radova Faculty of Science, Univ. of Novi Sad 13(1983).

[38] V. Radu, On the contraction principle in Menger spaces. Analele Univ. Timişoara XXII (1-2) (1984) 83-88.

[39] V. Radu – On some contraction - type mappings in Menger spaces, Analele Univ. Timişoara, vol. XXIII, 23(1985) , 61-65

[40] V. Radu, On some fixed point theorems in PM-Spaces. In: Lect. Notes Math. Vol 1233, 125-133, Springer-Verlag, N.Y., 1987.

[41] V. Radu – Deterministic metrics on Menger spaces and applications to fixed point theorems, Babeş-Bolyai University, Fac. of Math. and Ph. Research Seminar, Preprint nr.2(1988), 163-166.

[42] V. Radu, *Lectures on probabilistic analysis.* SURVEYS, LECTURES NOTES AND MONOGRAPHS Series on Probability, Statistics & Applied Mathematics,No 2, Universitatea de Vest din Timişoara, 1994.

[43] V. Radu, A fixed point principle in probabilistic metric spaces under Archimedean triangular norms. Analele Univ. Timişoara 1999 (to appear).

[44] V. Radu & D. Miheţ – A fixed point principle in σ-Menger spaces, Analele Univ. Timişoara, vol. XXXII, fasc. 2,99-110 (1994)

[45] I. Rus – *Principii şi Aplicaţii ale Teoriei Punctului Fix*, Ed. Dacia, 1979 .

[46] B. Schweizer, A. Sklar – Statistical metric spaces, Pacific J. Math 10, 313-334 (1960)

[47] B. Schweizer & A. Sklar, *Probabilistic Metric Spaces.* North-Holland, New York, 1983.

[48] B. Schweizer, H. Sherwood & R. M. Tardiff, Contraction s on PM-spaces: Examples and Counterexamples. Stochastica XII (1) (1988) 5-17.

[49] B. Schweizer, A. Sklar & E. Thorp, The metrization of SM-spaces. Pacific I. Math. 10(1 960) 673-75.

[50] V. M. Sehgal, Some fixed point theorems in functional analysis and probability. Ph.D. dissertation, Wayne State Univ. 1966.

[51] V. M. Sehgal &A. T. Bharucha-Reid, Fixed points of contraction mappings on PM-spaces. Math. Systems Theory 6(1972) 97-100.

[52] A.N. Serstnev, On the probabilistic generalization of metric spaces. Kazan Gos. Univ. Uch. Zap. 124(1964) 3-11.

[53] H. Sherwood, Complete probabilistic metric spaces, Z. Wahr. verw. Geb. 20(1971) 117-128.

[54] M. Stojaković, On some classes of contraction mappings. Math. Japonica 33(1988) 311-318.

[55] R. M. Tardiff, Contraction maps on PM-Spaces. J. Math.Anal.Appl., 155(1990)517-523.

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