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ON THE RELATION BETWEEN ABSOLUTELY SUMMING OPERATORS AND NUCLEAR OPERATORS

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Abstract. It is known that every absolutely summing operator acting between $C(\Omega)$, where Ω is an arbitrary compact set, and a space, F, with the Radon-Nikodym property is nuclear.

The purpose of this paper is to show that composing a weakly compact operator with an absolutely summing one we obtain a nuclear operator even the space, F, has not the Radon-Nikodym property.

We give, also, a proof for the "factorisation" theorem and we put an interesting problem.

1. Preliminaries

1.1. **Notations.** Let E, F be Banach spaces over the field Γ. Γ is the set of real, or complex, numbers.

1) $L(E,F) := \{T : E \to F : T \text{ is linear and bounded}\}$.

2) $E^* := L(E,\Gamma)$.

3) $U_E := \{x \in E : \|x\| \leq 1\}$.

4) Let $e^* \in E^*$ and $e \in E$, $\langle e, e^* \rangle := e^*(e)$.

5) Let $e^* \in E^*$ and $f \in F$. We denote by $e^* \otimes f$ the following operator: $e^* \otimes f : E \to F, (e^* \otimes f)(e) = \langle e, e^* \rangle \cdot f$.

1.2. **Definition [5].** Let E be a Banach space. A subset $A \subset E$ is said to be **weakly compact** if it is compact in the weak topology, $\sigma(E, E^*)$.

1.3. **Definition [5].** Let E, F be Banach spaces and $T \in L(E,F)$. T is said to be **weakly compact** if TU_E is relatively weakly compact.

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1.4. **Definition [3].** Let E, F be Banach spaces. An operator $T \in L(E,F)$ is called **absolutely summing** ($T \in ABS(E,F)$) if there is a constant $c \geq 0$ such that:

$$\sum_{i=1}^{n} \|Tx_i\| \leq c \cdot \sup\left\{\sum_{i=1}^{n} |\langle x_i, x^* \rangle| : x^* \in U_{E^*}\right\},$$

for every finite family of elements $x_1, x_2, ..., x_n \in E$.

For every $T \in ABS(E,F)$ we define: $\pi(T) := \inf c$.

1.5. **Definition [3].** Let E, F be Banach spaces. An operator $T \in L(E,F)$ is said to be **nuclear** if there is a representation:

$$T = \sum_{i=1}^{\infty} e_i^* \otimes f_i,$$

where $e_i^* \in E^*$ and $f_i \in F$, for every natural i.

We write $T \in N(E,F)$.

1.6. **Definition [1].** Let $(\Omega, \sum, \mu)$be a finite real measure space and E a Banach space.

We say that E has the **Radon -Nikodym property** with respect to $(\Omega, \sum, \mu)$ if, for each μ–continuously vector measure $\vartheta : \sum \to E$ of bounded variation, there is $g \in L_1(\mu, E)$ such that $\vartheta(A) = \int_A g d\mu$ for every $A \in \sum$:

We say that E has the **Radon -Nikodym property** (E has the **R.N.p**) if E has the Radon -Nikodym property with respect to every finite real measure space.

1.7. **Examples of spaces with the R.N.p.[1].** 1) Every reflexive space.(Phillips' theorem)

2) Let F be a Banach space. If $E = F^*$ and, in addition, E is separable then E has the R.N.p.

3) Let I be an arbitrary set, $I \neq \emptyset$. Then $l_1(I)$ has the R.N.p.

4) Let $1 < p < \infty$ and X be a space with the R.N.p. Then $L_p(X, \mu)$ has the R.N.p.

1.8. **Examples of spaces without the R.N.p.[1].** 1) $(c_0, \|\cdot\|_\infty)$, where $c_0 := \{x = \{x_n\}_n : x_n \in \Gamma \text{ and } x_n \to 0\}$, $\|x\|_\infty := \sup\limits_n |x_n|$.

2) $L_1(\mu)$, where μ is a finite and non-purely atomic measure.

1.9. **Definition [7].** The Banach space $\widetilde{F}$ is said to have **the extension property** if every $T \in L(E, \widetilde{F})$, where E is an arbitrary Banach space, can be extended to any Banach space $\widetilde{E}$ containing E as a suspace, where the extension $\widetilde{T} : \widetilde{E} \to \widetilde{F}$ is linear and bounded.

1.10. **Example [7].** The Banach space $(l_\infty(\Gamma), \|\cdot\|_\infty)$ has the metric extension theory.

1.11. **Theorem (the "Domination" theorem)[3].** Let $T \in L(E,F)$. $T \in ABS(E,F)$ if and only if there is a regular normalized measure μ on U_{E^*} such that:

$$\|Tx\| \leq \pi(T) \cdot \int\limits_{U_{E^*}} |\langle x, x^* \rangle| \, d\mu(x^*),$$

for every $x \in E$.

1.12. **Corolar [3].** Let J be the inclusion from $C(\Omega)$ into $L_1(\mu)$, where Ω is a compact set and μ is a measure with the properties from the "domination" theorem. Then:

$$J \in ABS(C(\Omega), L_1(\mu)).$$

1.13. **Theorem (the "Factorization" theorem) [3].** Let E, F be Banach spaces, F having the extension property, and $T \in ABS(E,F)$. Then there exist the operators:

1) $A \in L(E, C(U_{E^*}))$,

2) $Y \in L(L_1(\mu), F)$, where μ is a regular, positive, normalised, Borel measure on U_{E^*}, likewise in the "domination" theorem,

such that: $T = Y \circ J \circ A$, where J is the inclusion from $C(U_{E^*})$ into $L_1(\mu)$.

Proof. (authors'adaptation)

Construction

1) Let $A : E \to C(U_{E^*})$ be defined, for every $x \in E$, by: $Ax := J_x$, where $J_x : E^* \to \Gamma, J_x(x^*) = \langle x, x^* \rangle$, for every $x^* \in E^*$.

From the definition it follows that $A \in L(E, C(U_{E^*}))$, $\|Ax\| = \|J_x\| = \|x\|$, the corolar of the Hahn-Banach teorem, and further $\|A\| = 1$.

2) We consider now the inclusion,J, from $C(U_{E^*})$ into $L_1(\mu)$. From the corolar 1.12 we obtain that $J \in ABS(C(U_{E^*}), L_1(\mu))$.

3) Let $\widetilde{Y} : Im(J \circ A) \to F$ be defined by $\widetilde{Y}((J \circ A)x) := Tx$.

We prove now that $\widetilde{Y} \in L(Im(J \circ A), F)$.

a) The linearity is obvious

b) $\left\|\widetilde{Y}((J \circ A)x)\right\| = \|Tx\| \leq \pi(T) \cdot \int\limits_{U_{E^*}} |\langle x, x^* \rangle| \, d\mu(x^*) =$

$= \pi(T) \cdot \int\limits_{U_{E^*}} |J_x(x^*)| \, d\mu(x^*) = \pi(T) \cdot \|J_x\| = \pi(T) \cdot \|(J \circ A)x\|.$

So $\widetilde{Y}$ is bounded on $Im(J \circ A)$.

F has the extension property so $\widetilde{Y}$ can be extended to Y defined on $L_1(\mu)$.

In conclusion we obtain the announced factorization of T. □

1.14. **Remark [3].** If F has not the extension property the factorization of an operator

$T \in ABS(E, F)$ is as follows:

$T = Y \circ J \circ A$, where $A \in L(E, C(U_{E^*}))$, J is the inclusion from $C(U_{E^*})$ into $\overline{Im(J \circ A)E} \subset L_1(\mu)$ and $Y \in L\left(\overline{Im(J \circ A)E}, F\right)$.

1.15. **Theorem (Davies, Figiel, Johnson, Pelczynski) [5].** Let E, F be Banach spaces. Every weak compact operator $S : E \to F$ can be factorised through a reflexive Banach space.

1.16. **Theorem [3].** Let Ω be a compact set and F a space with the R.N.p. Then every $T \in ABS(C(\Omega), F)$ is nuclear.

2. Result

2.1. Theorem. Let E, F, G be Banach spaces, F having, in addition, the extension property.

If $S : F \to G$ is weak compact and $T \in ABS(E, F)$ then $S \circ T$ is nuclear.

Proof. From the factorisation theorem it follows that $T = Y \circ J \circ A$, likewise the factorization theorem, and from teorem 1.15 it follows that $S = U \circ V$, where $V \in L(F, R)$, R being a reflexive space, and $U \in L(R, G)$.

Further $S \circ T = U \circ V \circ Y \circ J \circ A$.

From the following facts $V \circ Y \circ J \in ABS\left(C\left(U_{E^{*}}, R\right)\right)$ and R is a space with the R.N.p. we obtain that $V \circ Y \circ J$ is nuclear.

In conclusion $S \circ T = U \circ V \circ Y \circ J \circ A$ is nuclear. □

3. Open Problem

Let E, F be Banach spaces, F having, in addition, the R.N.p. Any $T \in ABS(E, F)$ admits a factorisation, likewise in the "factorization" theorem. So:

$T = Y \circ J \circ A$, where $A : E \to C\left(U_{E^{*}}\right)$, $J : C\left(U_{E^{*}}\right) \to \overline{Im\left(J \circ A\right)} \subset L_1\left(\mu\right)$, $Y : \overline{Im\left(J \circ A\right)} \to F$.

From the facts that $Y \circ J \in ABS\left(C\left(U_{E^{*}}\right), F\right)$ and F is a space with the R.N.p it follows that $Y \circ J$ is nuclear.

In conclusion $T = Y \circ J \circ A$ must be nuclear.

But it is false because we can give a contraexample.

If we consider the identity from l_1 to l_2, $I : l_1 \to l_2$, this operator is ABS and l_2 , being a Hilbert space, is a space with the R.N.p., it follows that $I : l_1 \to l_2$ must be nuclear. But it is false because $I : l_1 \to l_2$ isn't even compact.

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