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A FIXED POINT APPROACH OF THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract. In an adecquate Banach space the integral operator associated to the initial value problem

$$\begin{cases} u'' + f(t,u,u') = 0, \ t \geq t_0 \\ u(t_0) = U_0 \quad u'(t_0) = U_1 \end{cases} \tag{1}$$

for some $t_0 \geq 1$ (for simplicity) satisfies the requirements of the Schauder-Tychonov theorem if $f(t,u,v)$ is under a Bihari type restriction. The fixed point $u(t)$ of this operator is asymptotic to $at+b$ as $t \to +\infty$, where a, b are real constants.

1. Introduction

Starting with the paper by Bellman [3], functional analysis is frecquently involved in studying the asymptotic behavior of solutions for differential equations. Papers such as those of Massera and Schäffer [5] are now fundamental.

Another important step is made by Corduneanu [4] who uses certain function spaces to analyze those solutions which go to $+\infty$ in the same way as some positive test function g, *i.e.* solutions $x(t)$ such that $|x(t)| = O(g(t))$.

Corduneanu introduces Banach spaces like $(C_g, ||.||_g)$ below:

$$C_g = \{x \in C(\mathbb{R}_+, \mathbb{R}^m) : \lambda_x > 0, |x(t)| \leq \lambda_x g(t), \ t \in \mathbb{R}\}$$

with the norm

$$||x||_g = \sup_{t \geq 0} \frac{|x(t)|}{g(t)}.$$

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Such spaces are used also by Avramescu [1] for solutions $x(t)$ such that $|x(t)| = o(g(t))$.

Following these ideas an adecquate Banach space is introduced herein to study the solutions $u(t)$ of problem (1) which go to some $a_u t + b_u$ as $t \to +\infty$, where a_u, b_u are real constants.

2. The fixed point technique applied to the study of asymptotic behavior

Consider the initial value problem

$$\begin{cases} u'' + f(t, u, u') = 0, t \geq t_0 \\ u(t_0) = U_0 \quad u'(t_0) = U_1 \end{cases}$$

when the following hold true:

(i) The function $f(t, u, v)$ is continuous in $D = \{(t, u, v) : t \in [t_0, +\infty),\ u,\ v \in \mathbb{R}\}$ and $f(t, 0, 0) = 0$ for every $t \geq t_0$.

(ii) There exist three continuous functions h, g_1, $g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq h(t) g_1(\frac{|u_1 - u_2|}{t}) g_2(|v_1 - v_2|) \tag{2}$$

where for $s > 0$ the functions $g_1(s)$, $g_2(s)$ are positive and nondecreasing,

$$A = \int_{t_0}^{\infty} s h(s) ds < +\infty \tag{3}$$

and

$$\sup_{r \geq t_0} \frac{r}{g_1(r) g_2(r)} = +\infty \tag{4}$$

and

$$g_1(0) g_2(0) = 0. \tag{5}$$

On the real linear space $X(t_0) = \{u \in C^1(t_0, +\infty; \mathbb{R}) : \lim_{t \to +\infty} u'(t) = a_u,\ \lim_{t \to +\infty} [u(t) - a_u t] = b_u;\ a_u,\ b_u \in \mathbb{R}\}$ we introduce the norm

$$||u|| = \sup_{t \geq t_0} \{|u'(t)| + |u(t) - a_u t| + \frac{|u(t)|}{t}\}.$$

Proposition 2.1. *The space $(X(t_0), ||.||)$ is complete.*

Proof. Consider $(f_n)_{n\geq 1}$ a Cauchy sequence in $X(t_0)$. Then the sequence of derivatives, $(f'_n)_{n\geq 1}$, is uniformly convergent on $[t_0, +\infty)$ to a continuous function g while $(f_n)_{n\geq 1}$ is pointwise convergent on $[t_0, +\infty)$ to a certain function f. The Weierstrass theorem regarding sequences of derivable functions (see Niculescu [7], Theorem 6.5.4, p. 283-284) shows that f is a C^1–function on $[t_0, +\infty)$ and $f' = g$. Furthermore, $(f_n)_{n\geq 1}$ has local uniform convergence to f since for every $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that

$$\left|\frac{f_n(t)}{t} - \frac{f(t)}{t}\right| < \varepsilon,\ t \geq t_0$$

for every $n \geq N(\varepsilon)$. In this way,

$$\sup_{t\in[t_0,T]} |f_n(t) - f(t)| \leq \varepsilon T,\ n \geq N(\varepsilon)$$

for $T > 0$ fixed. The usual $\varepsilon - N(\varepsilon)$ technique shows that $f \in X(t_0)$ and

$$\lim_{n\to+\infty} a_{f_n} = a_f,\ \lim_{n\to+\infty} b_{f_n} = b_f$$

and $\frac{f_n(t)}{t}$ goes uniformly to $\frac{f(t)}{t}$ on $[t_0, +\infty)$ as $n \to +\infty$ and $f_n(t) - a_{f_n}t$ goes uniformly to $f(t) - a_f t$ on $[t_0, +\infty)$ as $n \to +\infty$.

Finally, f_n goes to f in $X(t_0)$ as $n \to +\infty$. □

The operator $T : X(t_0) \to X(t_0)$ is defined by

$$(Tu)(t) = U_1 t + U_0 - \int_{t_0}^{t} (t - s) f(s, u, u')ds.$$

One has the following estimations:

$$\begin{cases} |(Tu_1 - Tu_2)'(t)| \leq g_1(\|u_1 - u_2\|) g_2(\|u_1 - u_2\|) \int_{t_0}^{\infty} h(s)ds \\ \frac{|(Tu_1 - Tu_2)(t)|}{t} \leq g_1(\|u_1 - u_2\|) g_2(\|u_1 - u_2\|) \int_{t_0}^{\infty} h(s)ds \end{cases}$$

and

$$\left| \int_t^\infty |f(s,u_1,u_1')|\,ds - \int_t^\infty |f(s,u_2,u_2')|\,ds \right| \tag{6}$$
$$\leq \frac{g_1(\|u_1-u_2\|)g_2(\|u_1-u_2\|)}{t} \int_t^\infty sh(s)ds$$

for $u_1, u_2 \in X(t_0)$ and $t \geq t_0$. The values of a_{Tu}, b_{Tu} can be computed from

$$\begin{cases} a_{Tu} = U_1 - \int_{t_0}^{\infty} f(s,u,u')ds \\ b_{Tu} = U_0 + \int_{t_0}^{\infty} sf(s,u,u')ds, \end{cases}$$

since $\lim_{t\to+\infty}\{t\int_t^\infty f(s,u,u')ds\} = 0$ for every $u \in X(t_0)$. Using $u(t) = 0$, this follows easily from (6) since $f(t,0,0) = 0$ for $t \geq t_0$. We need also the formula $(Tu)(t) - a_{Tu}t = U_0 + \int_{t_0}^{t} sf(s,u,u')ds + t\int_t^\infty f(s,u,u')ds$.

All of this shows that $TX_0 \subseteq X_0$ and

$$\|Tu_1 - Tu_2\| \leq 3Ag_1(\|u_1-u_2\|)g_2(\|u_1-u_2\|),\ u_1,u_2 \in X(t_0). \tag{7}$$

A compactness criterion on $X(t_0)$ is the one below.

Proposition 2.2. *Let $M \subset X(t_0)$ satisfy the next properties:*

(i) *For every $\varepsilon > 0$ there exists $L > 0$ such that*

$$|u'(t)| \leq L,\ \ |u(t) - a_u t| \leq L$$

for every $t \geq t_0$ and $u \in M$.

(ii) *For every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that*

$$|u'(t_1) - u'(t_2)| < \varepsilon,\ \ |u(t_1) - u(t_2) - a_u(t_1 - t_2)| < \varepsilon$$

for every t_1, $t_2 \geq t_0$, with $|t_1 - t_2| < \delta(\varepsilon)$, and $u \in M$.

(iii) *For every $\varepsilon > 0$ there exists $Q(\varepsilon) > 0$ such that*

$$|u'(t) - a_u| < \varepsilon,\ \ |u(t) - a_u t - b_u| < \varepsilon$$

for every $t \geq Q(\varepsilon)$ and $u \in M$.

Then, M is relatively compact in $X(t_0)$.

Proof. A simple consequence of the compactness criterion on $C_n^f = \{u \in C(t_0, +\infty; \mathbb{R}^n) : \lim_{t\to+\infty} u(t) = l_u,\ l_u \in \mathbb{R}^n\}$. See Avramescu [2]. ☐

We introduce the straight line $x_0(t) = U_1 t + U_0$. Thus, $T(0) = x_0$. According to (4) $\sup_{r\geq t_0} \frac{r}{g_1(\|x_0\|+r)g_2(\|x_0\|+r)} = +\infty$ and from (3) there exists $b \geq \|x_0\|$ such that $3\int_{t_0}^{\infty} sh(s)ds \leq \frac{b}{g_1(\|x_0\|+b)g_2(\|x_0\|+b)}$.

The set $D_0 = \{u \in X(t_0) : \|u - x_0\| \leq b\}$ is closed and convex.

Theorem 2.3. *The requirements below are satisfied:*

(a) $TD_0 \subseteq D_0$.

(b) *If H is bounded in $X(t_0)$ then TH is relatively compact in $X(t_0)$.*

(c) *The operator T is continuous in $X(t_0)$.*

Proof. For (a) one has the estimation

$$\|Tu - x_0\| \leq 3g_1(\|x_0\| + b)g_2(\|x_0\| + b)\int_{t_0}^{\infty} sh(s)ds \leq b.$$

For (b) one has to test the properties (i), (ii) and (iii) from Proposition 2.2. For (i), if $M = \sup_{h\in H} \|h\|$ then

$$\|Th\| \leq L = \|x_0\| + 3Ag_1(M)g_2(M),$$

according to (7) since $T(0) = x_0$. For (ii), if $t_1 \geq t_2 \geq t_0$ then one has the following estimations:

$$|(Tu)'(t_1) - (Tu)'(t_2)| \leq g_1(M)g_2(M)\int_{t_2}^{t_1} h(s)ds$$

and

$$|(Tu)(t_1) - (Tu)(t_2) - a_{Tu}(t_1 - t_2)|$$
$$\leq g_1(M)g_2(M)\{\int_{t_0}^{\infty} h(s)ds + \int_{t_2}^{t_1} h(s)ds\}(t_1 - t_2).$$

For (iii), again, one has the estimations below:

$$|(Tu)'(t) - a_{Tu}| \leq g_1(M)g_2(M)\int_t^\infty h(s)ds$$

and

$$|(Tu)(t) - a_{Tu}t - b_{Tu}| \leq g_1(M)g_2(M)\int_t^\infty sh(s)ds + |R(u)|(t),$$

where $R(u)(t) = t\int_t^\infty f(s,u,u')ds$. According to (6), $\lim_{t\to+\infty} R(u)(t) = 0$ uniformly with respect to $u \in H$ since $R(0) = 0$ and

$$|R(u)(t)| \leq g_1(M)g_2(M)\int_t^\infty sh(s)ds,\ u \in H.$$

The requirement (c) is justified by (5). If u_n goes to u in $X(t_0)$ as $n \to +\infty$ then

$$||Tu_n - Tu|| \leq 3Ag_1(||u_n - u||)g_2(||u_n - u||) \to 0$$

as $n \to +\infty$. □

According to the Schauder-Tychonov theorem (see Rus [9], Theorem 7.42, p. 58-59) the operator T has a fixed point $u(t)$ in $X(t_0)$. This is exactly the desired solution of problem (1).

Note. Whenever (2) is replaced by the Lipschitz type restriction

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq h(t)\left(\frac{|u_1 - u_2|}{t} + |v_1 - v_2|\right)$$

and (3) is valid the operator $T : X(t_0) \to X(t_0)$ becomes a contraction under some Bielecki type norm. See Mustafa [6].

In what regards the term $\frac{|u_1-u_2|}{t}$ in (2) it appears to be a natural requirement. See Rogovchenko [8], Theorems 1-3.

Since (4) implies that

$$\int_{t_0}^\infty \frac{ds}{g_1(s)g_2(s)} \geq \sup_{r\geq t_0}\frac{r}{g_1(r)g_2(r)} = +\infty,$$

which is the standard Bihari condition, (4) itself can be properly refer to by a Bihari type condition.

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