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A FIXED POINT THEOREM FOR HICKS-TYPE CONTRACTIONS IN PROBABILISTIC METRIC SPACES

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Abstract. A fixed point theorem for mappings with contractive iterate at a point on bounded uniform spaces is proved. As particular cases some fixed point theorems of Hicks - type are obtained.

1. A fixed point theorem

Definition 1.1. Let X be a nonempty set and $\mathcal{B}$ be the class of all functions $\beta : X \times X \to [0,\infty)$ with the properties

$\beta 1)$ $\beta(x,y) = 0 \Leftrightarrow x = y$

$\beta 2)$ $\beta(x,y) = \beta(y,x)(\forall)x,y \in X$

$\beta 3)$ $(\forall)\varepsilon > 0\ (\exists)\delta > 0 : (\beta(x,y) < \delta, \beta(y,z) < \delta) \Rightarrow \beta(x,z) < \varepsilon$

$\beta 4)$ $(\exists)M > 0 : \beta(x,y) \leq M, (\forall)x,y \in X.$

A B - *space* is a pair $(X,\beta)$with $\beta \in \mathcal{B}$.

Proposition 1.2 ([1]). *If β satisfies $\beta 1) - \beta 3)$ then the family*

$$U_\beta = \{S_{\beta,\varepsilon}\}_{\varepsilon>0}, where S_{\beta,\varepsilon} = \{(x,y) \in X^2 | \beta(x,y) < \varepsilon\}$$

is a base for a metrizable uniformity on X (which we will call the U_β - uniformity).

The proof is easy to be reproduced.

Definition 1.3. If M is a positive number, Φ_M means the class of all functions $\varphi : [0,\infty) \to [0,\infty)$ for which there exists $\alpha > M$ such that $\lim_{n\to\infty} \varphi^n(\alpha) = 0$. ($\varphi^n = \varphi \circ \varphi \circ \cdots \circ \varphi$).

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Definition 1.4. Let (X,β) be a $\mathcal{B}$ space and $\varphi:[0,\infty)\to[0,\infty)$ be a function. We say that the mapping $f:X\to X$ has *a* $(O-\varphi)$ *-contractive iterate at* $x\in X$ if for each $y\in O_f(x)$ there exists
$n=n(y)\in\mathbf{N}$ such that

$$(\varepsilon>0, z\in O_f(y), \beta(y,z)<\varepsilon)\Rightarrow\beta(f^n y, f^n z)<\varphi(\varepsilon),$$

where $O_f(y)$ denoted the set $\{f^n y | n\in\mathbf{N}\}$.

Theorem 1.5. *Let* (X,β) *be a* U_β *- complete* $\mathcal{B}$ *-space. If* $\varphi\in\Phi_M$ *and* $f:X\to X$ *is a continous mapping which has a* $(O-\varphi)$ *- contractive iterate at* $x\in X$*, then* f *has a fixed point which can be obtained by the successive approximation method, starting from an arbitrary point of* $O_f(x)$.

Proof. We will show that for every $y\in O_f(x)$ the sequence $\{f^n y\}_{n\in\mathbf{N}}$ is a *Cauchy sequence.*

Lemma 1.6. *Let* (X,β) *be a* $\mathcal{B}$ *space,* $\varphi\in\Phi_M$ *and* $f:X\to X$ *be a mapping with* $(O-\varphi)$ *-contractive iterate at* $x\in O_f(x)$*. Then, for every* $\varepsilon>0$ *there exists* $m_0=m(y,\varepsilon)\in\mathbf{N}$ *such that*

$$\beta(f^{m_0}y, f^{m_0+m}y)<\varepsilon\ (\forall)m\in\mathbf{N}.$$

Proof of Lemma. First we show that for every $s\in\mathbf{N}$ there exists $u=u(s,y)\in\mathbf{N}$ such that, for every $\alpha>M$,

$$(1)\quad \beta(f^{u(s)}y, f^{u(s)+m}y)<\varphi^s(\alpha),\ (\forall)\ m\in\mathbf{N}.$$

(1) is true for $s=0$, because β satisfies $\beta4$) and $M<\alpha$. Next, for every $j\geq 0$, let us define recursively the numbers $v(i)$ and $u(j)$ by $v(0)=u(0)$, $v(j+1)=n(f^{u(j)}x)$ and $u(j+1)=u(j)+v(j+1)$.

Then we have the following implication :
$\beta(f^{u(s)}y, f^{u(s)+m}y)<\varphi^s(\alpha)\Rightarrow\beta(f^{v(s+1)}(f^{u(s)}y), f^{v(s+1)}(f^{u(s)+m}y))<\varphi^{s+1}(\alpha)$, that is

$$\beta(f^{u(s)}y, f^{u(s)+m}y)<\varphi^s(\alpha)\Rightarrow\beta(f^{u(s+1)}y, f^{u(s+1)+m)}y)<\varphi^{s+1}(\alpha).$$

So, the relation (1) is proved by induction. □

Now let $\varepsilon > 0$ be given. Since $\varphi \in \Phi_M$, then there exists $\alpha > M$ such that $\varphi^n(\alpha) \to 0$. For this α let us consider $s_0 \in \mathbf{N}$ for which $\varphi^{s_0}(\alpha) < \varepsilon$. If we take in (1) $m_0 = u(s_0)$ then we obtain

$$\beta(f^{m_0}y, f^{m_0+m}y) < \varepsilon \ (\forall) \ m \in \mathbf{N}$$

and the lemma is proved.

Now it is easy to show that $\{f^n(y)\}$ is a Cauchy sequence:
For given $\varepsilon > 0$ we consider $\delta(\varepsilon)$ from $\beta 3)$.
By *Lemma 1.6* there exists $m_0 \in \mathbf{N}$ such that

$$\beta(f^{m_0}y, f^{m_0+p}y) < \delta, (\forall) p \in \mathbf{N},$$

hence

$$\beta(f^{m_0}y, f^n y) < \delta, \beta(f^{m_0}y, f^{n+m}y) < \delta, \ (\forall) \ n \geq m_0, (\forall) m$$

From $\beta 3)$ it follows that there exists $m_0 = m_0(y, \varepsilon) \in \mathbf{N}$ such that

$$\beta(f^n y, f^{n+m}y) < \varepsilon, \ (\forall) n \geq m_0, \ (\forall) m \in \mathbf{N},$$

i.e.$\{f^n(y)\}_{n \in \mathbf{N}}$ is a Cauchy sequence.

Since (X, β) is U_β - complete, then there exists $u_* \in X$, $u_* = \lim_{n \to \infty} f^n y$. From the continuity of f it follows that $f(u_*) = u_*$. □

Corollary 1.7. *If (X, β) is a complete $\mathcal{B}$-space and $f : X \to X$ is a continous mapping with the property that for every $y \in X$ there exists $n = n(y) \in \mathbf{N}$ such that, for every $z \in O_f(y)$,*

$$(\varepsilon > 0, \beta(y, z) < \epsilon) \Rightarrow \beta(f^n y, f^n z) < \varphi(\varepsilon)$$

then f has a fixed point which can be obtained by the successive approximation method, starting from an arbitrary point of X.

2. Applications to PM - spaces

Definition 2.1 ([10]). A PSM space $(S, \mathcal{F})$ is called a H - *space* if the following triangle inequality takes place :

(PM_{3H}) $(\forall)\varepsilon > 0$ $(\exists)\delta > 0 : (F_{pq}(\delta) > 1-\delta, F_{qr}(\delta) > 1-\delta) \Rightarrow F_{pr}(\varepsilon) > 1-\varepsilon.$

Two important examples are

Example 2.2.

a) Every σ - Menger space $(S, \mathcal{F}, T)$ with $\sup_{a<1} T(a,a) = 1$ and $\inf_{b>0}\sigma(b,b) = 0$ is a H - space.

b) If $(S, \mathcal{F}, \tau)$ is a Serstnev space and the t - function τ is continuous at $(\varepsilon_0, \varepsilon_0)$, then $(S, \mathcal{F})$ is a H - space (for the basic notions used here see [6], [14] or [15]).

Lemma 2.3. *Let r be a strictly decreasing and continuous function on $[0, \infty)$ such that $r(0) = 1$ and there exists $\alpha > 0$ such that $r(\alpha) = 0$. If $(S, \mathcal{F})$ is a PSM space and K_r is the mapping defined on $S \times S$ by*

$$K_r(p,q) = \sup\{t > 0 | F_{pq}(t) \le r(t)\}$$

then

a) K_r satisfies $\beta 4)$, $\beta 1)$ and $\beta 2)$.

b) $K_r(p,q) < t \Leftrightarrow F_{pq}(p,q) > r(t)$

c) If $(S, \mathcal{F})$ is a H - space then K_r satisfies $\beta 3)$ and the uniformity U_{K_r} is the $\mathcal{F}$ - uniformity.

Proof. Let $g(t) = F_{pq}(t) - r(t)$. Then g is a left conntinous and strictly increasing function on $[0, \infty)$. If we denote $A = \{t > 0 | g(t) \ge 0\}$ then because $g(t) > 0$ if $t > \alpha$, we have $A \subset [0, \alpha]$ and we can choose $M = \alpha$. So K_r satisfies $\beta 4)$.

Let now $p = q$. Then from PM1), $F_{pq}(t) = 1$ $(\forall)t > 0$, so $r(0) = 1$ from which it follows $K_r(p,q) = 0$. Conversely, if we suppose that $K_r(p,q) = 0$, then $F_{pq}(t) > r(t)$ $(\forall)t > 0$, so $F_{pq}(0+) = 1$ and, from PM 1) again, $p = q$.

$\beta 2)$ follows immediately from $PM2) : F_{pq} = F_{qp}$ $(\forall)p, q \in S$.

b) Let p, q be fixed and $m = \sup A = K_r(p,q)$.

If $(x_n) \subset A, x_n \nearrow m$, then, by the left continuity of g it follows $g(m) \leq 0$ and by the monotonocity of g we deduce that $g(t) > 0 \Rightarrow t > m$. So we proved the implication $F_{pq}(t) > r(t) \Rightarrow K_r(p,q) < t$. The converse implication immediately follows from the definition of $K_r(p,q)$.

c) Let us suppose that $(S, \mathcal{F})$ is a H - space. We prove that $(\forall)\varepsilon > 0(\exists)\delta > 0 : (F_{pq}(\delta) > r(\delta), F_{qr}(\delta) > r(\delta)) \Rightarrow F_{pr}(\varepsilon) > r(\varepsilon)$ and then $\beta 3)$ follows from b).

Let $\varepsilon > 0$ be fixed. We choose $\varepsilon_1 = \min(\varepsilon, 1 - r(\varepsilon))$ and let δ_1 be the ε_1 - correspondent from $PM3_H)$. There exists $\delta < \delta_1$ such that $r(\delta) > 1 - \delta_1$. Then, $F_{pq}(\delta) > r(\delta), F_{qr}(\delta) > r(\delta)) \Rightarrow (F_{pq}(\delta_1) > 1 - \delta_1$ $F_{qr}(\delta_1) > 1 - \delta_1) \Rightarrow F_{pr}(\varepsilon_1) > 1 - \varepsilon_1 \Rightarrow F_{pr}(\varepsilon) > r(\varepsilon)$.

The fact that the U_{K_r} uniformity is the $\mathcal{F}$ uniformity immediately follows from b).

The lemma is proved. □

The mapping d_{m_1,m_2} bellow has been introduced by V. Radu ([12]). Let $\mathcal{M}$ be the family of all functions $m : [0,\infty) \to [0,\infty)$ such that

m1) $m(t+s) \geq m(t) + m(s)$ $(\forall)t, s \geq 0$

m2) $m(t) = 0 \Leftrightarrow t = 0$

m3) m is continous.

Let $(S, \mathcal{F})$ be a PSM space. If $f : [0,1] \to \mathbf{R}$ is a continuous and strictly decreasing function with $f(1) = 0$ and $m_1, m_2 \in \mathcal{M}$, then d_{m_1,m_2} is the mapping defined on S^2 by

$$d_{m_1,m_2}(p,q) = \sup\{t \geq 0 | m_1(t) \leq f \circ F_{pq}(m_2(t)\}.$$

Lemma 2.4. *a)* $(\exists)$ $M > 0 : d_{m_1,m_2}(p,q) \leq M,$ $(\forall)$ $p, q \in S$.

b) d_{m_1,m_2} *satisfies* $\beta 1), \beta 2)$.

c) $d_{m_1,m_2}(p,q) < t \Leftrightarrow f \circ F_{pq}(m_2(t)) < m_1(t)$

d) If $(S, \mathcal{F})$ *is a* H *- space, then* $\beta = d_{m_1,m_2}$ *satisfies* $\beta 3)$ *and the uniformity* U_β *is the* $\mathcal{F}$ *- uniformity.*

Proof. For a) let us observe that $\lim_{t\to\infty} m_1(t) = \infty$ and so there exists $t_0 > 0$ such that $m_1(t_0) > f(0)$. From this it follows that $d_{m_1,m_2}(p,q) < t_0$ $(\forall) p, q \in S$, because $f \circ F_{pq}(m_2(t_0) \leq f(0) < m_1(t_0)$.
b), c) and d) follows from [12, Theorem 1]. □

From *Lemma 2.3* and *Lemma2.4* it follows that if $(S, \mathcal{F})$ is a H - space, then (S, β) is a $\mathcal{B}$ space (in the first case $\beta = K_r$ and $M = \alpha$ and in the second one $\beta = d_{m_1,m_2}$ and $M = t_0$), and the $\mathcal{U}_\beta$ - uniformity is the $\mathcal{F}$ - uniformity. Thus we can transpose the results from the previous paragraph to contractions on H - space.

Theorem 2.5. *Let* $(S, \mathcal{F})$ *be a complete* H *- space and* $f : S \to S$ *be a continuous mapping with the property that there exists* $p \in S$ *such that*
$(\forall) p \in O_f(p) (\exists) n = n(q) \in \mathbf{N} : (t > 0, r \in O_f(q), F_{qr}(t) > 1 - t) \Rightarrow F_{f_q^n, f_r^n}(\varphi(t) > 1 - \varphi(t)$, *where* $\varphi \in \Phi_1$. *Then* f *has a fixed point.*

The proof follows from *Lemma 2.3* ($r(t) = 1 - t$) and *Theorem 1.5.*
For $\varphi(t) = kt$ one obtains *Theorem 1.2* from [11].

Theorem 2.6. *Let* $(S, \mathcal{F})$ *be a complete* H *- space and* $f : S \to S$ *be a continuous mapping with the property that for every* $p \in S$ *there exist* $n = n(p) \in \mathbf{N}$ *such that, for every* $q \in O_f(p)$,
$(t > 0, F_{pq}(t) > 1 - t) \Rightarrow F_{f_p^n f_q^n}(\varphi(t)) > 1 - \varphi(t)$, *where* $\varphi \in \Phi_1$.
Then f *has a fixed point.*

The proof follows from *Lemma2.3* with $r(t) = 1 - t$ and *Corollary 1.7*

Corollary 2.7 ([7, Theorem 1.a)]). *If $(S, \mathcal{F}, T)$ is a complete Menger space under the t - norm T satisfying $\sup_{a<1} T(a,a) = 1$ and $f : S \to S$ is a continuous mapping with the property that for every $x \in S$ there exists $n(x) \in \mathbf{N}$ such that, for every $v \in O_f(x)$,*

$$(r > 0, F_{xv}(r) > 1 - r) \Rightarrow F_{f^{n(x)}x f^{n(x)}v}(g(r)) > 1 - g(r)$$

where $g : [0,\infty) \to [0,\infty)$ is a mapping which satisfies $\lim_{n\to\infty} g^n(r) = 0$ for every $r > 0$ and $g(u) < u$ $(\forall) u > 0$, then f has a fixed point.

The proof follows from *Theorem 2.6* and *Example 2.2.*

If T=Min and g satisfies stronger conditions of Browder type, then one obtains *Theorem 3.3* from [3].

Theorem 2.8. *Let $(S, \mathcal{F})$ be a complete H - space and $f : S \to S$ be a continuous mapping with the property that there exists $p \in S$ such that*
$(\forall) q \in O_f(p)$ $(\exists) n = n(q) \in \mathbf{N} : (t > 0, r \in O_f(q), f \circ F_{pq}(m_2(t)) < m_1(t)) \Rightarrow$
$f_\circ F_{f_p^n f_q^n}(m_2(\varphi(t)) < m_1(\varphi(t))$
(m_1, m_2 are like in Lemma 2.4 and $\varphi \in \Phi_{t_0}$), then f has a fixed point which is the limit of the succesive approximations, starting from an arbitrary point of $O_f(p)$.

The proof follows from *Lemma 2.4* and *Theorem 2.6.*

If $(S, \mathcal{F})$ is a H - space of the type $(S, \mathcal{F}, T)$ - Menger space under the t - norm T with $\sup_{a<1} T(a,a) = 1$, then from the above theorem we obtain *Theorem 1* from [8].

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