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THE MARKOV PROPERTY FOR THE SOLUTION OF THE STOCHASTIC NAVIER-STOKES EQUATION

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Abstract. We consider the stochastic Navier-Stokes equation of Navier-Stokes type containing a noise part given by a stochastic integral with respect to a Wiener process. The purpose of this paper is to prove that the solution of this nonlinear equation is a Markov process. We take into consideration the properties of the Galerkin approximations.

1. Introduction

The stochastic Navier-Stokes equation has important physical and technical applications. It describes the behavior of a viscous velocity field of an incompressible liquid. The equation on the domain of flow $G \subset \mathbb{R}^n$ ($n \geq 2$ a natural number) is given by

$$\begin{cases} \dfrac{\partial U}{\partial t} - \nu \Delta U = -(U, \nabla)U + f - \nabla p + \mathcal{C}(U)\dfrac{\partial w}{\partial t} \\ \text{div } U = 0, \quad U(0,x) = U_0(x), \quad U(t,x)\mid_{\partial G} = 0, \; t > 0, \; x \in G, \end{cases} \tag{1}$$

where U is the velocity field, ν is the viscosity, Δ is the Laplacian, ∇ is the gradient, f is an external force, p is the pressure, and U_0 is the initial condition. Realistic models for flows should contain a random noise part, because external perturbations and the internal Browninan motion influence the velocity field. For this reason equation (1) contains a random noise part $\mathcal{C}(U)\dfrac{\partial w}{\partial t}$. Here the noise is defined as the distributional derivative of a Wiener process $\Big(w(t)\Big)_{t\in[0,T]}$, whose intensity depends on the state U.

Throughout this paper we consider strong solutions ("strong" in the sense of stochastic analysis) of a stochastic equation of Navier-Stokes type (we will call it a

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stochastic Navier-Stokes equation) and define the equation in the generalized sense as an evolution equation, assuming that the stochastic processes are defined on a given complete probability space and the Wiener process is given in advance.

An important property in the study of the solutions of stochastic differential equations is the Markov property. This property is used for example in dynamic programming approaches (see [4]) to formulate Bellman's principle, in the theory of random dynamical systems (see [1]) to determine invariant measures, in investigations of the long-time behaviour of the processes (see [8]). In the case of stochastic processes which are also Markov processes we can describe its properties by studying the properties of the corresponding Markov semigroup.

In this paper we prove that the solution of the stochastic Navier-Stokes equation is a Markov process. This property was proved by B. Schmalfuß [6] for the stochastic Navier-Stokes equation, but only for the case of additive noise. Our hypothesis are more general.

The structure of the paper is as follows: In Section 2 we give the assumptions for the Navier-Stokes equation and mention some results concerning the convergence of the Galerkin approximations to the solution of the considered equation. We also prove that the solution depends continuously on the initial data. Section 3 contains the main result of our paper. We prove that the solution of the stochastic Navier-Stokes equation is a Markov process. In Section 4 we give some auxiliary results from stochastic analysis.

Frequently Used Notations

$\rightharpoonup$	weak convergence (in the sense of functional analysis)
I_A	indicator function for the set A
EX	mathematical expectation of the random variable X
$\mathcal{L}_V^2(\Omega)$	space of all $\mathcal{F}$-measurable random variables $u : \Omega \to V$ with $E\|u\|_V^2 < \infty$
$\mathcal{L}_V^2(\Omega \times [0,T])$	space of all $\mathcal{F} \times B([0,T])$-measurable processes $u : \Omega \times [0,T] \to V$ that are adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and $E\int_0^T \|u(t)\|_V^2\,dt < \infty$

2. Assumptions and formulation of the problem

First we state the assumptions about the stochastic evolution equation that will be considered.

(i): $(\Omega, \mathcal{F}, P)$ is a complete probability space and $(\mathcal{F}_t)_{t\in[0,T]}$ is a right continuous filtration such that $\mathcal{F}_0$ contains all $\mathcal{F}$-null sets. $(w(t))_{t\in[0,T]}$ is a real valued standard $\mathcal{F}_t$-Wiener process.

(ii): (V, H, V^*) is an evolution triple (see [10], p. 416), where $(V, \|\cdot\|_V)$ and $(H, \|\cdot\|)$ are separable Hilbert spaces, and the embedding operator $V \hookrightarrow H$ is assumed to be compact. We denote by $(\cdot,\cdot)$ the scalar product in H.

(iii): $\mathcal{A} : V \to V^*$ is a linear operator such that $\langle \mathcal{A}v, v\rangle \geq \nu\|v\|_V^2$ for all $v \in V$ and $\langle \mathcal{A}u, v\rangle = \langle \mathcal{A}v, u\rangle$ for all $u, v \in V$, where $\nu > 0$ is a constant and $\langle\cdot,\cdot\rangle$ denotes the dual pairing.

(iv): $\mathcal{B} : V \times V \to V^*$ is a bilinear operator such that

$$\langle \mathcal{B}(u,v), v\rangle = 0 \quad \text{for all } u, v \in V$$

and there exists a positive constant $b > 0$ such that

$$|\langle \mathcal{B}(u,v), z\rangle|^2 \leq b\|z\|_V^2\|u\|\|u\|_V\|v\|\|v\|_V$$

for all $u, v, z \in V$.

(v): $\mathcal{C} : [0,T] \times H \to H$ is a mapping such that

(a): $\|\mathcal{C}(t,u) - \mathcal{C}(t,v)\|^2 \le \lambda \|u-v\|^2$ for all $t \in [0,T]$, $u, v \in H$, where λ is a positive constant;

(b): $\mathcal{C}(t,0) = 0$ for all $t \in [0,T]$;

(c): $\mathcal{C}(\cdot, v) \in \mathcal{L}^2_H[0,T]$ for all $v \in H$.

(vi): $\Phi : [0,T] \times H \to H$ is a mapping such that

(a): $\|\Phi(t,u) - \Phi(t,v)\|^2 \le \mu \|u-v\|^2$ for all $t \in [0,T]$, $u, v \in H$, where μ is a positive constant;

(b): $\Phi(t,0) = 0$ for all $t \in [0,T]$;

(c): $\Phi(\cdot, v) \in \mathcal{L}^2_H[0,T]$ for all $v \in H$.

(vii): x_0 is a H-valued $\mathcal{F}_0$-measurable random variable such that $E\|x_0\|^4 < \infty$.

Definition 2.1. We call a process $\Big(U(t)\Big)_{t\in[0,T]}$ from the space $\mathcal{L}^2_V(\Omega \times [0,T])$ with $E\|U(t)\|^2 < \infty$ for all $t \in [0,T]$ a **solution of the stochastic Navier-Stokes equation** if it satisfies the equation:

$$
\begin{aligned}
(U(t), v) + \int_0^t \langle \mathcal{A}U(s), v\rangle ds &= (x_0, v) + \int_0^t \langle \mathcal{B}(U(s), U(s)), v\rangle ds \qquad (2) \\
&+ \int_0^t (\Phi(s, U(s)), v) ds + \int_0^t (\mathcal{C}(s, U(s)), v) dw(s)
\end{aligned}
$$

for all $v \in V$, $t \in [0,T]$ and a.e. $\omega \in \Omega$, where the stochastic integral is understood in the Ito sense.

Remark 2.2. If we set $n = 2$, $V = \{u \in \overset{\circ}{W}{}^1_2 (G) : \operatorname{div} u = 0\}$, $\quad H = \bar{V}^{L^2(G)}$ and

$$
\langle \mathcal{A}u, v\rangle = \int_G \sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx, \quad \langle \mathcal{B}(u,v), z\rangle = -\int_G \sum_{i,j=1}^n u_i \frac{\partial v_j}{\partial x_i} z_j dx, \quad \Phi(t,u) = f(t)
$$

for $u, v, z \in V, t \in [0,T]$, then equation (2) can be transformed into (1).

Let $h_1, h_2, \ldots, h_n, \cdots \in H$ be the eigenvectors of the operator $\mathcal{A}$, for which we consider the domain of definition $\operatorname{Dom}(\mathcal{A}) = \{v \in V \mid \mathcal{A}v \in H\}$. These eigenvectors form an orthonormal base in H and they are orthogonal in V. For each $n \in \mathbb{N}$ we consider $H_n := \mathrm{sp}\{h_1, h_2, \dots, h_n\}$ be equipped with the norm induced from H. We write $(H_n, \|\cdot\|_V)$ when we consider H_n equipped with the norm induced from V. We define by $\Pi_n : H \to H_n$ the orthogonal projection of H on H_n

$$\Pi_n h := \sum_{i=1}^{n} (h, h_i) h_i.$$

Let $\mathcal{A}_n : H_n \to H_n$, $\mathcal{B}_n : H_n \times H_n \to H_n$, $\Phi_n, \mathcal{C}_n : [0,T] \times H_n \to H_n$ be defined respectively by

$$\mathcal{A}_n u = \sum_{i=1}^{n} \langle \mathcal{A}u, h_i \rangle h_i, \quad \mathcal{B}_n(u,v) = \sum_{i=1}^{n} \langle \mathcal{B}(u,v), h_i \rangle h_i,$$

$$\mathcal{C}_n(t,u) = \Pi_n \mathcal{C}(t,u), \quad \Phi_n(t,u) = \Pi_n \Phi(t,u), \quad x_{0n} = \Pi_n x_0$$

for all $t \in [0,T]$, $u, v \in H_n$.

The existence of the solution of the Navier-Stokes equation (2) is proved by approximating it by means of the Galerkin method, i.e., by a sequence of solutions of finite dimensional equations $(P_n), n \geq 1$.

For each $n = 1, 2, 3, \dots$ we consider the sequence of finite dimensional evolution equations

$$(P_n) \qquad (U_n(t), v) + \int_0^t (\mathcal{A}_n U_n(s), v) ds = (x_{0n}, v) + \int_0^t (\mathcal{B}_n(U_n(s), U_n(s)), v) ds$$
$$+ \int_0^t (\Phi_n(s, U_n(s)), v) ds + \int_0^t (\mathcal{C}_n(s, U_n(s)), v) dw(s),$$

for all $v \in H_n$, $t \in [0,T]$ and a.e. $\omega \in \Omega$.

We use an analogous method as in [9]. Let $\left(\chi_M\right)$ be a family of Lipschitz continuous mappings such that

$$\chi_M(x) = \begin{cases} 1\,, & \text{if } 0 \leq x \leq M, \\ 0\,, & \text{if } x \geq M+1, \\ M+1-x, & \text{if } x \in (M, M+1). \end{cases}$$

For each fixed $n \in \mathbb{N}$ we consider the solution U_n of equation (P_n) approximated by $\left(U_n^M\right)$ $(M = 1, 2, \ldots)$ which is the solution of the equation

$$(P_n^M) \qquad \begin{aligned} (U_n^M(t), v) &+ \int_0^t (\mathcal{A}_n U_n^M(s), v) ds = (x_{0n}, v) \\ &+ \int_0^t (\chi_M(\|U_n^M(t)\|^2)\mathcal{B}_n(U_n^M(s), U_n^M(s)), v) ds \\ &+ \int_0^t (\Phi_n(s, U_n^M(s)), v) ds + \int_0^t (\mathcal{C}_n(s, U_n^M(s)), v) dw(s), \end{aligned}$$

for all $v \in H_n$, $t \in [0,T]$, and a.e. $\omega \in \Omega$. For this equation we apply the theory of finite dimensional Ito equations with Lipschitz continuous nonlinearities (see [5], Theorem 3.9, p. 289). Hence there exists $U_n^M \in \mathcal{L}^2_{(H_n, \|\cdot\|_V)}(\Omega \times [0,T])$ almost surely unique solution of (P_n^M) which has continuous trajectories in H.

Theorem 2.1. *For each $n \in \mathbb{N}$, equation (P_n) has a solution $U_n \in \mathcal{L}^2_V(\Omega \times [0,T])$, which is unique almost surely and has almost surely continuous trajectories in H. For each $n \in \mathbb{N}$ it holds*

$$P\left(\lim_{M \to \infty} \sup_{t \in [0,T]} \|U_n^M(t) - U_n(t)\|^2 = 0 \right) = 1.$$

Theorem 2.2. *The Navier-Stokes equation (2) has a solution $U \in \mathcal{L}^2_V(\Omega \times [0,T])$, which is almost surely unique and has almost surely continuous trajectories in H. The following convergence holds*

$$\lim_{n \to \infty} E\|U_n(t) - U(t)\|^2 = 0 \quad \textit{for all } t \in [0,T].$$

Lemma 2.3. *There exists a positive constant c such that*

$$E \sup_{t \in [0,T]} \|U(t)\|^4 + E\Big(\int_0^T \|U(s)\|_V^2 ds \Big)^2 \leq cE\|x_0\|^4.$$

The proofs of these results can be found in [2].

Before we investigate the Markov property for the solution of the stochastic Navier-Stokes equation, we prove that the solution U of (2) depends continuously on the initial data x_0.

Theorem 2.4. *Let (x_0^N) be a sequence in H and let $x_0 \in H$ be such that*

$$\lim_{N\to\infty} \|x_0^N - x_0\|^2 = 0.$$

Then for each $t \in [0,T]$ it holds

$$\lim_{N\to\infty} E\|U_N(t) - U(t)\|^2 = 0,$$

where U_N is the solution of (2) satisfying the initial condition $U_N(0) = x_0^N$.

Proof. For all $t \in [0,T]$ and a.e. $\omega \in \Omega$ let

$$e(t) = \exp\left\{ -\frac{b}{\nu}\int_0^t \|U(s)\|_V^2 ds - (\lambda + 2\sqrt{\mu})t \right\}.$$

It follows by the Ito formula that

$$\begin{aligned}
& e(t)\|U(t) - U_N(t)\|^2 + 2\int_0^t e(s)\langle \mathcal{A}U(s) - \mathcal{A}U_N(s), U(s) - U_N(s)\rangle ds \\
= \; & \|x_0 - x_0^N\|^2 + 2\int_0^t e(s)\langle \mathcal{B}(U(s),U(s)) - \mathcal{B}(U_N(s),U_N(s)), U(s) - U_N(s)\rangle ds \\
- \; & \frac{b}{\nu}\int_0^t e(s)\|U(s)\|_V^2 \|U(s) - U_N(s)\|^2 ds - (\lambda + 2\sqrt{\mu})\int_0^t e(s)\|U(s) - U_N(s)\|^2 ds \\
+ \; & 2\int_0^t e(s)(\Phi(s,U(s)) - \Phi(s,U_N(s)), U(s) - U_N(s))ds \\
+ \; & \int_0^t e(s)\|\mathcal{C}(s,U(s)) - \mathcal{C}(s,U_N(s))\|^2 ds \\
+ \; & 2\int_0^t e(s)(\mathcal{C}(s,U(s)) - \mathcal{C}(s,U_N(s)), U(s) - U_N(s))dw(s).
\end{aligned}$$

In view of the properties of $\mathcal{B}$ we can write

$$
\begin{aligned}
&2\langle \mathcal{B}(U(s),U(s)) - \mathcal{B}(U_N(s),U_N(s)), U(s) - U_N(s)\rangle \\
&\qquad = 2\langle \mathcal{B}(U(s) - U_N(s), U(s)), U(s) - U_N(s)\rangle \\
&\qquad \leq \frac{b}{\nu}\|U(s)\|_V^2 \|U(s) - U_N(s)\|^2 + \nu\|U(s) - U_N(s)\|_V^2.
\end{aligned}
$$

Now we use the properties of $\mathcal{A}$, Φ, $\mathcal{C}$, and those of the stochastic integral to obtain

$$
\begin{aligned}
& E \sup_{s\in[0,t]} e(s)\|U(s) - U_N(s)\|^2 \leq \|x_0 - x_0^N\|^2 \\
+ \quad & 4E \sup_{s\in[0,t]} \left| \int_0^s e(r)(\mathcal{C}(r,U(r)) - \mathcal{C}(r,U_N(r)), U(r) - U_N(r))dw(r) \right| \\
\leq \quad & k_1 E \int_0^t \sup_{r\in[0,s]} \left\{ e(r)\|U(r) - U_N(r)\|^2 \right\} ds \\
+ \quad & \frac{1}{2} E \sup_{s\in[0,t]} e(s)\|U(s) - U_N(s))\|^2,
\end{aligned}
$$

where k_1 is a positive constant and $t \in [0,T]$. By Gronwall's Lemma we get

$$
E \sup_{s\in[0,t]} e(s)\|U(s) - U_N(s)\|^2 \leq 4e^{2k_1 T}\|x_0 - x_0^N\|^2
$$

for all $t \in [0,T]$.

We take $t := \mathcal{T}_M^U$, where $\mathcal{T}_M^U$ is the following *stopping time*

$$
\mathcal{T}_M^U = \begin{cases} T\,, \quad \text{if} \quad \displaystyle\int_0^T \|U(s)\|_V^2 ds < M \\ \inf\left\{ t \in [0,T] : \displaystyle\int_0^t \|U(s)\|_V^2 ds \geq M \right\}, \quad \text{otherwise.} \end{cases}
$$

Using the hypothesis and the above inequality it follows that for each fixed $M \in \mathbb{N}$ we have

$$
\lim_{n\to\infty} E\|U(\mathcal{T}_M^U) - U_N(\mathcal{T}_M^U)\|^2 = 0.
$$

Applying Proposition 4.1 for $\mathcal{T} := t$, $\mathcal{T}_M := \mathcal{T}_M^U$, $Q_n(\mathcal{T}) := \|U(\mathcal{T}) - U_N(\mathcal{T})\|^2$ we obtain

$$\lim_{n\to\infty} E\|U_N(t) - U(t)\|^2 = 0.$$

□

3. The Markov property

Let us introduce the following σ-algebras

$$\sigma_{[U(s)]} := \sigma\{U(s)\}, \quad \sigma_{[U(r):r\leq s]} := \sigma\{U(r) : r \leq s\}$$

and the event

$$\sigma_{[U(s)=y]} := \{\omega : U(s) = y\}.$$

For the solution U of the Navier-Stokes equation (2) we define the **transition function**

$$\bar{P}(s,x,t,A) := P(U(t) \in A | \sigma_{[U(s)=x]})$$

with $s,t \in [0,T], s < t, x \in H, A \in B(H)$.

In the following theorem we prove that **the solution of the Navier-Stokes equation is a Markov process**. This means that the state $U(s)$ at time s must contain all probabilistic information relevant to the evolution of the process for times $t > s$.

Theorem 3.1. *(i) For fixed $s,t \in [0,T], s < t, A \in B(H)$ the mapping*

$$y \in H \mapsto \bar{P}(s,y,t,A) \in \mathbb{R}$$

is measurable.

(ii) The following equalities hold

$$P(U(t) \in A|\mathcal{F}_s) = P(U(t) \in A|\sigma_{[U(s)]})$$

and

$$P\Big(U(t) \in A\Big|\sigma_{[U(r):r\leq s]}\Big) = P\Big(U(t) \in A\Big|\sigma_{[U(s)]}\Big)$$

for all $s,t \in [0,T], s < t, y \in H, A \in B(H)$.

Proof. **(i)** Let $s, t \in [0, T], s < t, y \in H$. We denote by $\left(\tilde{U}(t, s, y)\right)_{t \in [s,T]}$ the solution of the Navier-Stokes equation starting in s with the initial value y, i.e. $\tilde{U}(s, s, y) = y$ for a.e. $\omega \in \Omega$.

Let $A \in B(H)$. Without loss of generality we can consider the set A to be closed. Let (a_n) be a sequence of continuous and uniformly bounded functions $a_n : H \to \mathbb{R},\ n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} ||a_n(y) - I_A(y)|| = 0 \quad \text{for all } y \in H. \tag{3}$$

By the uniqueness of the solution of the Navier-Stokes equation and from the definition of the transition function we have

$$\bar{P}(s, y, t, A) = E\Big(I_A(U(t))\Big|\sigma_{[U(s)=y]}\Big) = E\Big(I_A(\tilde{U}(t, s, y))\Big).$$

We consider an arbitrary sequence (y_n) in H such that $\lim\limits_{n\to\infty} ||y_n - y|| = 0$. By using Theorem 2.4 (instead of starting in 0 we start in s) it follows that

$$\lim_{n\to\infty} E||\tilde{U}(t, s, y_n) - \tilde{U}(t, s, y)||^2 = 0. \tag{4}$$

Therefore $\left(\tilde{U}(t, s, y_n)\right)$ converges in probability to $\tilde{U}(t, s, y)$. Using (4) and the Lebesgue Theorem it follows that for all $k \in \mathbb{N}$

$$\lim_{n\to\infty} Ea_k\Big(\tilde{U}(t, s, y_n)\Big) = Ea_k\Big(\tilde{U}(t, s, y)\Big).$$

We conclude that for each $k \in \mathbb{N}$ the mapping

$$y \in H \mapsto Ea_k\Big(\tilde{U}(t, s, y)\Big) \in \mathbb{R}$$

is continuous. Hence it is measurable. By the Lebesgue Theorem and (3) we deduce that for all $y \in H$

$$\lim_{k\to\infty} Ea_k\Big(\tilde{U}(t, s, y)\Big) = EI_A\Big(\tilde{U}(t, s, y)\Big).$$

Consequently, $\bar{P}(s, \cdot, t, A) = EI_A\left(\tilde{U}(t, s, \cdot)\right)$ is measurable, because it is the pointwise limit of measurable functions.

(ii) First we prove that for each fixed $s,t \in [0,T], s < t, y \in H$ the random variable $\tilde{U}(t,s,y)$ (considered as a H-valued random variable) is independent of $\mathcal{F}_s$. By Theorem 2.1 we have

$$\lim_{M\to\infty} \|\tilde{U}_n^M(t,s,y) - \tilde{U}_n(t,s,y)\| = 0 \quad \text{for each } n \in \mathbb{N} \text{ and a.e. } \omega \in \Omega, \tag{5}$$

and by Theorem 2.2 it follows that there exists a subsequence (n') of (n) such that

$$\lim_{n'\to\infty} \|\tilde{U}_{n'}(t,s,y) - \tilde{U}(t,s,y)\| = 0 \quad \text{for a.e. } \omega \in \Omega, \tag{6}$$

where $\left(\tilde{U}_n^M(t,s,y)\right)_{t\in[s,T]}$ and $\left(\tilde{U}_n(t,s,y)\right)_{t\in[s,T]}$ are the solutions of (P_n^M) and (P_n), respectively, if we start in s with the initial value y. Since for fixed n, M the random variable $\tilde{U}_n^M(t,s,y)$ is approximated by Picard-iteration and each Picard-approximation is independent of $\mathcal{F}_s$ (as a H-valued random variable), it follows by Proposition 4.2 that $\tilde{U}_n(t,s,y)$ is independent of $\mathcal{F}_s$. Using (5), (6), and Proposition 4.2 we conclude that $\tilde{U}(t,s,y)$ is independent of $\mathcal{F}_s$.

Let $A \in B(H)$. Now we apply Proposition 4.3 for $\widehat{\mathcal{F}} := \mathcal{F}_s, f(y,\omega) := I_A\left(\tilde{U}(t,s,y)\right), \xi(\omega) := U(s)$. Hence

$$E\left(I_A\left(\tilde{U}(t,s,U(s))\right)\Big|\mathcal{F}_s\right) = E\left(I_A\left(\tilde{U}(t,s,U(s))\right)\Big|\sigma_{[U(s)]}\right). \tag{7}$$

Since the solution of the Navier-Stokes equation is (almost surely) unique it follows that

$$\tilde{U}(t,s,U(s)) = U(t) \quad \text{for all } t \in [s,T] \text{ and a.e. } \omega \in \Omega.$$

Then relation (7) becomes

$$E\left(I_A\left(U(t)\right)\Big|\mathcal{F}_s\right) = E\left(I_A\left(U(t)\right)\Big|\sigma_{[U(s)]}\right).$$

Consequently,

$$P\left(U(t) \in A\Big|\mathcal{F}_s\right) = P\left(U(t) \in A\Big|\sigma_{[U(s)]}\right). \tag{8}$$

We know

$$\sigma_{[U(s)]} \subseteq \sigma_{[U(r):r\leq s]} \subseteq \mathcal{F}_s.$$

Taking into account the properties of the conditional expectation and (8) we deduce that

$$\begin{aligned} P\Big(U(t) \in A \Big| \sigma_{[U(r):r\leq s]}\Big) &= \Big(E\Big(U(t) \in A \Big| \mathcal{F}_s\Big) \Big| \sigma_{[U(r):r\leq s]}\Big) \\ &= E\Big(E\Big(U(t) \in A \Big| \sigma_{[U(s)]}\Big) \Big| \sigma_{[U(r):r\leq s]}\Big) \\ &= P\Big(U(t) \in A \Big| \sigma_{[U(s)]}\Big). \end{aligned}$$

□

Using results from [3] (Chapter 3, Section 9, p. 59) we deduce the following corollary.

Corollary 3.2. *(i) For fixed $s, t \in [0,T], s < t, y \in H$ the mapping*

$$A \in B(H) \mapsto \bar{P}(s, y, t, \cdot) \in \mathbb{R}$$

is a probability measure.

(ii) The Chapman-Kolmogorov equation

$$\bar{P}(s, y, t, A) = \int_H \bar{P}(r, x, t, A)\bar{P}(s, y, r, dx)$$

holds for any $r, s, t \in [0,T], s < r < t, y \in H, A \in B(H)$.

Remark 3.3. We have the **autonomous version** of the stochastic Navier-Stokes equation if for $t \in [0,T]$, $h \in H$ we have $\mathcal{C}(t,h) = \mathcal{C}(h)$ and $\Phi(t,h) = \Phi(h)$. In this case $\Big(U_\Phi(t)\Big)_{t\in[0,T]}$ is a **homogeneous Markov process**, i.e., we have

$$\bar{P}(0, y, t-s, A) = \bar{P}(s, y, t, A) \tag{9}$$

for all $s, t \in [0,T], s < t, y \in H, A \in B(H)$.

Now we prove the above property. Let $s, t \in [0,T], s < t, y \in H$. The solution U_Φ of the Navier-Stokes equation, which starts in s with the initial value y satisfies

$$\begin{aligned} (U_\Phi(t), v) + \int_s^t \langle \mathcal{A}U_\Phi(r), v\rangle dr &= (y, v) + \int_s^t \langle \mathcal{B}(U_\Phi(r), U_\Phi(r)), v\rangle dr \\ &+ \int_s^t (\Phi(U_\Phi(r)), v) dr + \int_s^t (\mathcal{C}(U_\Phi(r)), v) dw(r) \end{aligned}$$

for all $v \in V$ and a.e. $\omega \in \Omega$.

We take $\tilde{U}(r) = U_\Phi(s+r), \tilde{w}(r) := w(s+r) - w(s)$ for $r \in [0, t-s]$. Then for $\tilde{U}(t-s)$ we have

$$\begin{aligned}(\tilde{U}(t-s), v) + \int_0^{t-s} \langle \mathcal{A}\tilde{U}(r), v\rangle dr &= (y, v) + \int_0^{t-s} \langle \mathcal{B}(\tilde{U}(r), \tilde{U}(r)), v\rangle dr \\ &+ \int_0^{t-s} (\Phi(\tilde{U}(r)), v)dr + \int_0^{t-s} (\mathcal{C}(\tilde{U}(r)), v)d\tilde{w}(r)\end{aligned}$$

for all $v \in V$ and a.e. $\omega \in \Omega$. Since $\left(\tilde{w}(r)\right)_{r\in[0,t-s]}$ and $\left(w(r)\right)_{r\in[s,t]}$ have the same distribution and because of the uniqueness of the solution of the Navier-Stokes equation, it follows that $\tilde{U}(t-s)$ and $U_\Phi(t)$ have the same distribution. Hence (9) holds.

In the case of a homogeneous Markov process we denote

$$\bar{p}(y, t, A) := \bar{P}(0, y, t, A)$$

for all $t \in [0, T], y \in H, A \in B(H)$. The Chapman-Kolmogorov equation (see Corollary 3.2) can be rewritten as

$$\bar{p}(y, s+t, A) = \int_H \bar{p}(x, t, A)\bar{p}(y, s, dx) \tag{10}$$

for each $s, t \in [0, T], y \in H, A \in B(H)$.

We consider the following set of probabilty measures on Ω

$$\mathcal{S} := \left\{\mu \Big| \int_H ||x||^4 \mu(dx) < \infty\right\}$$

and define

$$T_t\mu(\Gamma) := \int_H \bar{p}(x, t, \Gamma)\mu(dx)$$

for each $\mu \in \mathcal{S}$, $t \in [0, T]$. This mapping has the following properties:

(a) $T_t : \mathcal{S} \to \mathcal{S}$,

(b) $T_0\mu = \mu$ for each $\mu \in \mathcal{S}$,

(c) $T_{t+s} = T_t \circ T_s = T_s \circ T_t$ for $s, t, s+t \in [0, T]$.

We deduce the result in (a) by observing that $T_t\mu$ is the distribution of $U(t)$ if the initial condition x_0 has the distribution μ ($\mu \in \mathcal{S}$ because of hypothesis **(vii)**). Then by using Lemma 2.3 we have $T_t\mu \in \mathcal{S}$. For (b) one can make some easy calculations and (c) follows from (10). Hence $(T_t)_t$ satisfies the *semigroup property*.

4. Some results from stochastic analysis

Proposition 4.1. *Let $(\mathcal{T}_M)$ and $\mathcal{T}$ be stopping times, such that*

$$\lim_{M\to\infty} P(\mathcal{T}_M < \mathcal{T}) = 0.$$

Let (Q_n) be a sequence of processes from the space $\mathcal{L}^2_{\mathbb{R}}([0,T]\times\Omega)$ such that for each fixed M we have

$$\lim_{n\to\infty} E|Q_n(\mathcal{T}_M)| = 0$$

and there exists a positive constant c independent of n such that

$$E|Q_n(\mathcal{T})|^2 < c \quad \textit{for all} \quad n \in \mathbb{N}.$$

Then

$$\lim_{n\to\infty} E|Q_n(\mathcal{T})| = 0.$$

Proof. Let $\varepsilon, \delta > 0$. There exists $M_0 \in \mathbb{N}$ such that

$$P(\mathcal{T}_{M_0} < \mathcal{T}) \le \frac{\varepsilon}{2}.$$

By the hypothesis it follows that for this M_0 we have $\lim\limits_{n\to\infty} E|Q_n(\mathcal{T}_{M_0})| = 0$. Consequently, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\delta}E|Q_n(\mathcal{T}_{M_0})| \le \frac{\varepsilon}{2}$$

for all $n \ge n_0$. We write

$$\begin{aligned} P\Big(|Q_n(\mathcal{T})| \ge \delta\Big) &\le P\Big(\mathcal{T}_{M_0} < \mathcal{T}\Big) + P\Big(\{\mathcal{T}_{M_0} = \mathcal{T}\} \wedge \Big\{|Q_n(\mathcal{T})| \ge \delta\Big\}\Big) \\ &\le \frac{\varepsilon}{2} + P\Big(|Q_n(\mathcal{T}_{M_0})| \ge \delta\Big) \le \frac{\varepsilon}{2} + \frac{1}{\delta}E|Q_n(\mathcal{T}_{M_0})| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $n \ge n_0$. Hence for all $\delta > 0$ we get $\lim\limits_{n\to\infty} P\Big(|Q_n(\mathcal{T})| \ge \delta\Big) = 0$. Therefore, the sequence $\Big(|Q_n(\mathcal{T})|\Big)$ converges in probability to zero. From the hypothesis it follows that this sequence is uniformly integrable (with respect to $\omega \in \Omega$). Hence it converges also in mean to zero

$$\lim_{n\to\infty} E|Q_n(\mathcal{T})| = 0.$$

□

Proposition 4.2. *Let $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ be a σ-algebra, (Q_n) be a sequence of H-valued random variables which converges for a.e. $\omega \in \Omega$ to Q. If each random variable Q_n is independent of $\widehat{\mathcal{F}}$, then Q is independent of $\widehat{\mathcal{F}}$.*

Proof. The random variable Q is independent of $\widehat{\mathcal{F}}$ if

$$P(\{||Q|| < x\} \cap A) = P(||Q|| < x)P(A) \tag{11}$$

for all $x \in \mathbb{R}, A \in \widehat{\mathcal{F}}$. The hypothesis implies that the sequence $\Big(||Q_n||\Big)$ converges in probability to $||Q||$. Therefore, the sequence of their distribution functions is convergent

$$\lim_{n\to\infty} F_{||Q_n||}(x) = F_{||Q||}(x) \tag{12}$$

for each $x \in \mathbb{R}$ which is a continuity point of $F_{||Q||}$.

Let $x \in \mathbb{R}, A \in \widehat{\mathcal{F}}, \delta > 0$. First we consider that $F_{||Q||}$ is continuous in x. Then using the hypothesis and (12) we get

$$\lim_{n\to\infty} P(\{||Q_n|| < x\} \cap A) = \lim_{n\to\infty} P(||Q_n|| < x)P(A) = P(||Q|| < x)P(A). \tag{13}$$

We write

$$\begin{aligned} P(\{||Q|| < x-\delta\} \cap A) &\leq P(\{||Q|| < x-\delta\} \cap \{||Q_n|| < x\} \cap A) \\ &+ P(\{||Q|| < x-\delta\} \cap \{||Q_n|| \geq x\} \cap A) \\ &\leq P(\{||Q_n|| < x\} \cap A) + P\Big(\Big|||Q|| - ||Q_n||\Big| > \delta\Big). \end{aligned}$$

Analogously we have

$$P(\{||Q_n|| < x\} \cap A) \leq P(\{||Q|| < x+\delta\} \cap A) + P\Big(\Big|||Q|| - ||Q_n||\Big| > \delta\Big).$$

Consequently,

$$P(\{\|Q\| < x - \delta\} \cap A) - P\Big(\big|\|Q\| - \|Q_n\|\big| > \delta\Big) \leq P(\|Q_n\| < x)P(A)$$
$$\leq P(\{\|Q\| < x + \delta\} \cap A) + P\Big(\big|\|Q\| - \|Q_n\|\big| > \delta\Big).$$

In the inequalities above we take the limit $n \to \infty$ and use (13) to obtain

$$P(\{\|Q\| < x - \delta\} \cap A) \leq P(\|Q\| < x)P(A) \leq P(\{\|Q\| < x + \delta\} \cap A).$$

Let $\delta \searrow 0$ in the inequalities above. Then

$$P(\{\|Q\| \leq x\} \cap A) \leq P(\|Q\| < x)P(A) \leq P(\{\|Q\| \leq x\} \cap A).$$

Since x is a point of continuity for $F_{\|Q\|}$ we have

$$P(\{\|Q\| \leq x\} \cap A) = P(\{\|Q\| < x\} \cap A).$$

Consequently, (11) holds and Q is independent of $\widehat{\mathcal{F}}$.

Now we consider that x is not a point of continuity of $F_{\|Q\|}$. Let (x_n) be a monotone increasing sequence of continuity points of $F_{\|Q\|}$ which converges to x. Then

$$\lim_{n\to\infty} F_{\|Q\|}(x_n) = F_{\|Q\|}(x),$$

and because x_n is a point of continuity for $F_{\|Q\|}$, we have

$$P(\{\|Q\| < x_n\} \cap A) = P(\|Q\| < x_n)P(A).$$

Now we take the limit $n \to \infty$ and conclude that (11) holds. Hence Q is independent of $\widehat{\mathcal{F}}$. □

Proposition 4.3. *Let $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ be a σ-algebra, $f : H \times \Omega \to H$ be a mapping such that for each $x \in H$ the random variable $f(x, \cdot)$ is bounded, measurable and independent of $\widehat{\mathcal{F}}$. Let ξ be a H-valued $\widehat{\mathcal{F}}$-measurable random variable. Then*

$$E\big(f(\xi,\omega)\big|\widehat{\mathcal{F}}\big) = E\big(f(\xi,\omega)\big|\sigma_{[\xi]}\big),$$

where $\sigma_{[\xi]}$ is the σ-algebra generated by the random variable ξ.

This proposition is proved in [3] (see Lemma 1, p. 63).

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