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ANALYTIC INVARIANTS AND THE RESOLUTION GRAPHS OF THE SINGULARITIES OF THE TYPE *ADE*

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Abstract. In this note we show that the equality of the maximal ideal cycle and the Artin's fundamental cycle for the complex surface singularities of type ADE can be proved directly, using the resolution graphs.

Let (X, x) be a normal surface singularity, and take a good resolution $\phi : \mathcal{Y} \to X$. The combinatorics/topology of ϕ is codified in the dual resolution graph Γ_ϕ. Using the plumbing construction, it is proved, that the information codified in Γ_ϕ is the same as the information codified in the link L_X. In particular Γ_ϕ is completely equivalent to the topology of (X, x).

We would like to codify numerically some information about the ring of holomorphic functions on (X, x) (actually about the maximal ideal $m_{X,x}$). Therefore, take an arbitrary holomorphic function $f : (X, x) \to (\mathbf{C}, 0)$ (i.e. $f \in m_{X,x}$). Then we take the composed map $f \circ \phi : \mathcal{Y} \to \mathbf{C}$, and denote by $(f \circ \phi)$ its divisor on $\mathcal{Y}$. Recall that $(f \circ \phi)$ is the set of zeros of $f \circ \phi$ with their natural multiplicities.

$$(f \circ \phi) = \sum_i m_i E_i + \sum_j m(\mathrm{St}_j)\mathrm{St}_j.$$

The part of this sum supported by E is $\sum_i m_i E_i$ — where m_i is the vanishing order of $f \circ \phi$ along E_i — and is denoted by $(f \circ \phi)_\Gamma$, while the strict transform $\mathrm{St}(f)$ is $\sum_j m(\mathrm{St}_j)\mathrm{St}_j$.

Therefore, by construction, for any $f \in m_{X,x}$ we get a cycle $(f \circ \phi)_\Gamma = \sum_i m_i(f) E_i$ supported by E.

Definition 1. The set of all these cycles is denoted by

$$\mathcal{Z}_{\mathrm{an}}(\phi) = \{(f \circ \phi)_\Gamma : f \in m_{X,x}\},$$

and is called the set of *analytic cycles.*

Recall, that in the set of cycles we have an ordering. For $Z' = \sum n_i' E_i$ and $Z'' = \sum n_i'' E_i$ we write $Z' \leq Z''$ if and only if $n_i' \leq n_i''$ for all i. With these notations obviously $\min(Z', Z'') = \sum_i \min(n_i', n_i'') \cdot E_i$.

Lemma 2. *If $Z_1, Z_2 \in \mathcal{Z}_{\text{an}}(\phi)$, then*

(1) $Z_1 + Z_2 \in \mathcal{Z}_{\text{an}}(\phi)$,

(2) $\min(Z_1, Z_2) \in \mathcal{Z}_{\text{an}}(\phi)$.

Proof: The proof is easy. For the point (1) just take $f_1 \cdot f_2$. For the point (2), take a generic linear combination $\lambda_1 f_1 + \lambda_2 f_2$. □

The above lemma assures that $\mathcal{Z}(\phi)$ has a unique minimal element with respect to the above ordering. This cycle is denoted by Z_{max}, by S. S.-T. Yau, who introduced it and called it *maximal ideal cycle*; (also it is denoted by Z_f, i.e. *fiber cycle*, by Miles Reid [8]).

Lemma 2 shows that if the linear term of f is sufficiently generic, then $(f \circ \phi) = Z_{\text{max}}$. Actually for any embedding $(X, x) \subset (\mathbf{C}^N, 0)$, a sufficiently general linear function $l : (\mathbf{C}^N, 0) \to (\mathbf{C}, 0)$ induces an $l|_X : (X, x) \to (\mathbf{C}, 0)$ with $(l|_X \circ \phi)_\Gamma = Z_{\text{max}}$.

The main goal is to find — up to the extent which is possible — the lattice $\mathcal{Z}_{\text{an}}(\phi)$ from the topology of (X, x), i.e. only from the graph Γ_ϕ.

We recall the most important property of the cycles $(f \circ \phi)_\Gamma$, $(f \in m_{X,x})$. This is, that $(f \circ \phi)_\Gamma \cdot E_k \leq 0$ for any k. This is a consequence of the fact that $(f \circ \phi) \cdot E_k = 0$ and $\text{St}(f) \cdot E_k \geq 0$ for any k.

Notice also that any cycle $Z = (f \circ \phi)_\Gamma$ is a *positive* cycle, i.e. $Z = \sum n_i E_i$ with $n_i \geq 0$ for any i, and $Z \neq 0$, which we will denote by $Z > 0$.

The topological analog (candidate) for $\mathcal{Z}_{\text{an}}(\phi)$ is

$$\mathcal{Z}_{\text{top}}(\phi) = \{Z \text{ is positive cycle } | Z \cdot E_k \leq 0, \text{ for all } k\}.$$

Lemma 3. *(1) If $Z_1, Z_2 \in \mathcal{Z}_{\text{top}}(\phi)$, then $Z_1 + Z_2 \in \mathcal{Z}_{\text{top}}(\phi)$,*

(2) If $Z_1, Z_2 \in \mathcal{Z}_{\text{top}}(\phi)$, then $\min(Z_1, Z_2) \in \mathcal{Z}_{\text{top}}(\phi)$,

(3) If $Z \in \mathcal{Z}_{\text{top}}(\phi)$, $Z = \sum_i n_i E_i$, then $n_i > 0$ for all i.

For the proof, see [2].

The above lemma shows that in $\mathcal{Z}_{\text{top}}(\phi)$ there is a unique minimal cycle, which has only strict positive coefficients. This cycle is denoted by $Z_{\min}$ and is called *minimal cycle* (or *numerical cycle* or *Artin's fundamental cycle*; it was introduced by Artin [1]).

Notice that $\mathcal{Z}_{\text{top}}(\phi)$ and $Z_{\min}$ is completely described by the graph Γ_ϕ. They are the topological candidates for the set $\mathcal{Z}_{\text{an}}(\phi)$ and the cycle $Z_{\max}$.

Obviously, we have also

$$\begin{cases} \mathcal{Z}_{\text{an}}(\phi) & \subseteq & \mathcal{Z}_{\text{top}}(\phi), \\ Z_{\min}(\phi) & \leq & Z_{\max}(\phi). \end{cases}$$

FACT: In general $Z_{\min}(\phi) \neq Z_{\max}(\phi)$. However for the singularities of type ADE these two cycles agree.

The point is, that for this type of singularities the fact that $Z_{\max} = Z_{\min}$ can be verified using the algorithm described in the chapter 6 in [2]. This algorithm constructs the resolution graph of a surface singularity of type $(X, x) = (\{f(x,y) - z^n = 0\}, 0)$, where $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ is an isolated plane curve singularity.

Let us show this for the singularities of type A_n.

Proposition 4. *(The case A_n, n odd) Take $f(x,y) = x^{n+1} + y^2$, and (X,x) given by $(\{f(x,y) - z^2 = 0\}, 0)$. Suppose n is odd, $n + 1 = 2l$. Then $Z_{\min}(\phi) = Z_{\max}(\phi)$.*

Proof. The embedded resolution graph of f is shown in the figure 1. This is actually the good embedded resolution graph of z. The algorithm described in the chapter

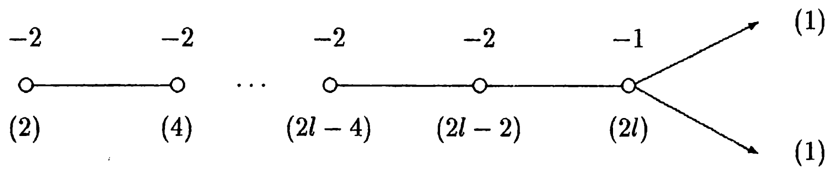

FIGURE 1. The embedded resolution graph of A_n, n odd

6 [2] gives the multiplicities of z, but we need the multiplicities of x, because this corresponds to the generic linear section on $(X, 0)$.

If we follow during the process of blowing up the strict transform of x, we can represent it as the arrow decorated by $(*)$ in figure 2. Notice that we can retain the Euler numbers from the previous graph.

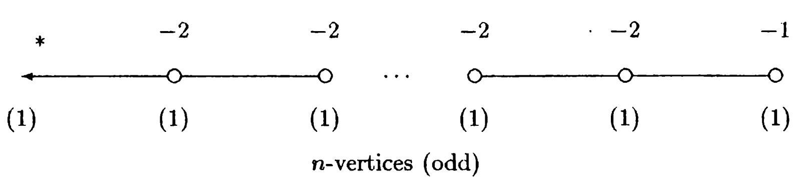

FIGURE 2. The strict transform of x, A_n, n odd

If we apply now the algorithm of chapter 6 [2], we obtain the embedded resolution graph of x, as is shown in the figure 3.

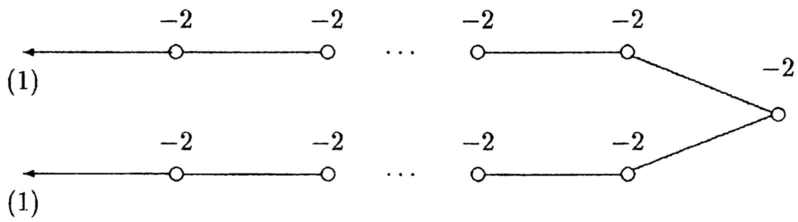

FIGURE 3. The embedded resolution graph of x in the case A_n, n odd

Now, we have the graph of x (see 4), the arrows with multiplicities (1), the Euler numbers, but not all the multiplicities of x in the vertices of the graph.

The trick is, that these multiplicities are uniquely determined by the equations

$$m_w e_w + \sum_{v \in \mathcal{V}_w} m_v = 0, \text{ for every vertex } w,$$

which is a system of linear equations, with nonsingular matrix, hence it has a unique solution. Since the multiplicities $m_v = 1$, for all v, already satisfy the system, they form actually its solution.

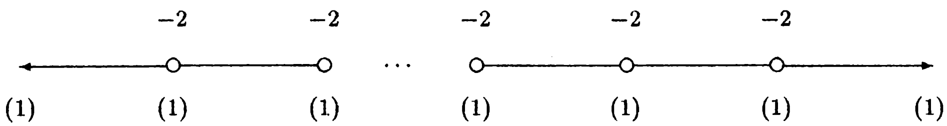

FIGURE 4. The graph of $x : (\{x^{2l} + y^2 + z^2 = 0\}, 0) \to (\mathbf{C}, 0)$

Obviously the cycle given by this set of multiplicities, $Z = E_1 + E_2 + \cdots + E_n$ has the minimal coefficients, and is given by a function (x), hence $Z_{\min} = Z_{\max}$.

The case n even is similar. □

Remark 5. The singularity D_n can be treated analogously.

Remark 6. For the singularities of type E_6, E_7 and E_8, we have to apply in addvance the Laufer's algorithm [3], to get the minimal topological cycle. This is because the multiplicities are not the absolutely minimal one. For all these cases the general linear section represented by the coordinate function which appears on the highest power in the equation of f it turns out to give the minimal cycle given by the Laufer's algorithm.

The computational details are similar for these cases, too. See also [2].

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