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NON-ANALYTIC n-STARLIKE AND n-SPIRALLIKE FUNCTIONS

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Abstract. In this paper properties of geometric kind for non-analytic n-starlike and n-spirallike functions, $n \in \mathbb{N} \cup \{0\}$, are obtained. They are extensions of some results in the case of non-analytic (usual) starlike and convex functions proved in [3] and in the case of analytic n-starlike functions in [2].

1. Introduction

A well-known method in complex analysis by which can be introduced new classes of functions is by using differential inequalities of the form

$$F(f, D(f)(z), \ldots, D^n(f)(z)) > 0, z \in U = \{z \in \mathbb{C};\ |z| < 1\},$$

where $D(f)(z) = z\frac{\partial f}{\partial z} - \overline{z}\frac{\partial f}{\partial \overline{z}}$, $D^n(f)(z) = D[D^{n-1}(f)](z)$ (if f is analytic then $D(f)(z) = zf'(z)$).

In this sense let us mention the following two examples:

(i) the class of analytic n-starlike ($n \in \mathbb{N} \cup \{0\}$) functions on U, introduced in [6];

(ii) the class of analytic logarithmically n-starlike functions on U, introduced in [2].

In Section 2 we extend to non-analytic n-starlike functions some properties of non-analytic usual starlike and convex functions in [3].

Section 3 is concerned with the case of non-analytic n-spirallike functions.

2. Non-analytic n-starlike functions

We say that $f : U \to \mathbb{C}$ is in $\mathbb{C}^n(U)$, $n \in \mathbb{N}$ fixed, if it is continuous and has continuous all partial derivatives of order n with respect $x = \text{Re}z$ and $y = \text{Im}z$. For

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$f \in C^{n+1}(U)$, $n \in \mathbb{N} \cup \{0\}$, we consider the operator (see [3])

$$D(f)(z) = z\frac{\partial f}{\partial z} - \overline{z}\frac{\partial f}{\partial \overline{z}},$$

where

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right], \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right], \quad i = \sqrt{-1},$$

and the iterates $D^{n+1}(f) = D[D^n(f)]$, $D^0(f) \equiv f$.

Definition 2.1. Let $f \in C^{n+1}(U)$. We say that f is n-starlike in U if $f(0) = 0$, f is univalent on U, $D^n(f)(z) \neq 0$, $\forall\, z \in U \setminus \{0\}$ and

$$\mathrm{Re}\frac{D^{n+1}(f)(z)}{D^n(f)(z)} > 0, \ \forall\, z \in U \setminus \{0\}. \tag{1}$$

Remarks.

1) For $n = 0$ and $n = 1$, the condition (1) means the geometric condition of starlikeness and of convexity in the case of non-analytic functions, respectively, considered for the first time in [3].
2) If f is analytic then $D(f)(z) = zf'(z)$ and the classes of functions satisfying (1) were considered for the first time in [6].

The following result can be considered, in a certain sense, a property of geometric kind of the left hand-side in (1).

Lemma 2.2. *For fixed $r \in (0,1)$, let us denote $C_r = \partial U_r$, $U_r = \{z \in U;\ |z| < r\}$, $\gamma_r(\theta) = f(\partial U_r)$, $z = re^{i\theta}$, $\theta \in [0, 2\pi)$. If $f \in C^{n+1}(U)$ and $D^n(f)(z) \neq 0$ for all $z \in U \setminus \{0\}$, then*

$$\mathrm{Re}\frac{D^{n+1}(f)(z)}{D^n(f)(z)} = \frac{\partial}{\partial \theta}[\arg \gamma_r^{(n)}], \quad z = re^{i\theta}, \quad \theta \in [0, 2\pi).$$

Proof. Let $f = A + iB$. We have

$$D(f)(z) = x\frac{\partial B}{\partial y} - y\frac{\partial B}{\partial x} + i\left[y\frac{\partial A}{\partial x} - x\frac{\partial A}{\partial y}\right],$$

$$z = x + iy = r[\cos\theta + i\sin\theta],$$

$$\gamma_r'(\theta) = \frac{\partial A}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial A}{\partial y}\cdot\frac{\partial y}{\partial \theta} + i\left[\frac{\partial B}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial B}{\partial y}\cdot\frac{\partial y}{\partial \theta}\right] =$$

$$= x\frac{\partial A}{\partial y} - y\frac{\partial A}{\partial x} + i\left[x\frac{\partial B}{\partial y} - y\frac{\partial B}{\partial x}\right] = iD(f)(z).$$

By the general formula $\frac{\partial g}{\partial \theta} = iD(g)$ (see [3]), we get

$$\gamma_r''(\theta) = \frac{\partial}{\partial \theta}[iD(f)(z)] = i^2 D^2(f)(z) = -D^2(f)(z),$$

$$\gamma_r'''(\theta) = \frac{\partial}{\partial \theta}[-D^2(f)(z)] = -iD^3(f)(z),$$

$$\gamma_r^{(4)}(\theta) = \frac{\partial}{\partial \theta}[-iD^3(f)(z)] = D^4(f)(z),$$

and finally,

$$\frac{\partial}{\partial \theta}[\arg \gamma_r^{(n)}] = \frac{\partial}{\partial \theta}[\arg(cD^n(f)(z)] = (\text{see } [3]) = \mathrm{Re}\frac{D[cD^n(f)(z)}{cD^n(f)(z)} = \mathrm{Re}\frac{D^{n+1}(f)(z)}{D^n(f)(z)},$$

where $c \in \{-1, +1, i, -i\}$, which proves the theorem.

In the analytic case, in [6] was proved that the condition (1) and $f(0) = f'(0) - 1 = 0$, imply the univalence of f and that as function of n, the classes of n-starlike functions form a decreasing sequence (in respect with the inclusion).

In the non-analytic case, as was pointed out in [3] for $n = 0$ and $n = 1$, these conditions do not imply the univalence of f and additional conditions are required for that. □

In this order of ideas, concerning the classes introduced by Definition 2.1, we have the followiong extension to non-analytic case of Corollary 2.1 in [6].

Theorem 2.3. *Let $m, n \in \mathbb{N} \cup \{0\}$, $m < n$. If $f \in C^{n+1}(U)$ satisfies:*

$$f(0) = 0, \quad f(z)\prod_{i=1}^{n} D^i(f)(z) \neq 0, \quad z \in U \setminus \{0\}, \quad J(f)(z) > 0, \quad z \in U$$

(here $J(f)$ means the Jacobian of f), then the condition $\mathrm{Re}\frac{D^{n+1}(f)(z)}{D^n(f)(z)} > 0$, $\forall\ z \in U \setminus \{0\}$, implies $\mathrm{Re}\frac{D^{m+1}(f)(z)}{D^m(f)(z)} > 0$, $\forall\ z \in U \setminus \{0\}$ and the fact that f is univalent on U.

Proof. A direct consequence of the proofs of Theorem 1 and of Lemmas 1, 2, 3 in [4], is the following:

if $F \in C^2(U)$, $F(0) = 0$, $J(F)(0) > 0$, $F(z)D(F)(z) \neq 0$, $\forall\ z \in U \setminus \{0\}$ and $\mathrm{Re}\frac{D^2(F)(z)}{D(F)(z)} > 0$, $\forall\ z \in U \setminus \{0\}$, then $\mathrm{Re}\frac{D(F)(z)}{F(z)} > 0$, $\forall z \in U \setminus \{0\}$. Also, we have $f(0) = D(f)(0) = \cdots = D^n(f)(0) = 0$.

On the other hand, we have $J[D(f)](0) = J(f)(0)$, $J[D^k(f)](0) = [J(f)(0)] > 0$, $k = \overline{1,n}$.

Applying the above result for $F = D^{n-1}(f)$, $F = D^{n-2}(f), \ldots, F = f$, we easily obtain $\mathrm{Re}\dfrac{D^{m+1}(f)(z)}{D^m(f)(z)} > 0$, $\forall\, z \in U \setminus \{0\}$, for any $m \in \{0, 1, \ldots, n-1\}$.

Taking $m = 0$ and applying Theorem 1 in [3] it follows that f is univalent on U, which proves the theorem. □

3. Non-analytic n-spirallike functions

Keeping the notations in Section 2, we introduce the following.

Definition 3.1. Let $f \in C^{n+1}(U)$, $n \in \mathbb{N} \cup \{0\}$. We say that f is logarithmically n-spirallike of type $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $f(0) = 0$, $D^n(f)(z) \neq 0$, $\forall\, z \in U \setminus \{0\}$, f is univalent on U and

$$\mathrm{Re}\left[e^{i\gamma}\frac{D^{n+1}(f)(z)}{D^n(f)(z)}\right] > 0, \ \forall\, z \in U \setminus \{0\}. \tag{2}$$

We say that f is Archimedean n-spirallike on U if $f(0) = 0$, $D^n(f)(z) \neq 0$, $\forall\, z \in U \setminus \{0\}$, f is univalent on U and

$$\mathrm{Re}\left[(1 - i|D^n(f)(z)|)\frac{D^{n+1}(f)(z)}{D^n(f)(z)}\right] > 0, \ \forall\, z \in U \setminus \{0\}. \tag{3}$$

We say that f is hyperbolic n-spirallike on U if $f(0) = 0$, $D^n(f)(z) \neq 0$, $\forall\, z \in U \setminus \{0\}$, f is univalent on U and

$$\mathrm{Re}\left[(|D^n(f)(z)| + i)\frac{D^{n+1}(f)(z)}{D^n(f)(z)}\right] > 0, \ \forall\, z \in U \setminus \{0\}. \tag{4}$$

Remarks.

1) Taking $n = 0$ in Definition 3.1, we obtain the classes of usual non-analytic spirallike functions studied in [1].

2) For $n = 1$ and f analytic on U, the class of functions defined by (2) was for the first time considered in [5]. In this case, the situation is different from the starlike case, because as was pointed out in [5], even for analytic functions f, the conditions (2), $f(0) = 0$, $f'(z) \neq 0$, $z \in U$, do not imply in general the univalence of f. However, in e.g. the paper [5], was proved that for $0 < \cos\gamma \leq 0.2315$, these above conditions imply the univalence of f.

3) The analytic logarithmically n-spirallike functions were considered in [2].

4) For $\gamma = 0$ in (2) we obtain the classes of n-starlike functions introduced by Definition 2.1.

The following result can be considered in a certain sense, of geometric kind for the relations (2), (3), (4), respectively.

Theorem 3.2. *For fixed $r \in (0,1)$, let us denote $C_r = \partial U_r$, $U_r = \{z \in \mathbb{C};\ |z| < r\}$, $\gamma_r(\theta) = f(\partial U_r)$, $z = re^{i\theta}$, $\theta \in [0, 2\pi)$. Let $f \in C^{n+1}(U)$ be with $D^n(f)(z) \neq 0$, $\forall\ z \in U \setminus \{0\}$, $D(f)(z) = z\dfrac{\partial f}{\partial z} - \overline{z}\dfrac{\partial f}{\partial \overline{z}}$.*

(i) Let us consider the family of logarithmically spirals

$$w_\phi(t) = e^{t\cos\gamma} e^{i(\phi - t\sin\gamma)}, \quad t \in (-\infty, +\infty),$$

where $\phi \in [0, 2\pi)$ is a parameter. Then (2) is equivalent with

$$\frac{\partial \phi}{\partial \theta} > 0,\ \forall\ \theta \in [0, 2\pi), \tag{5}$$

where $\phi = \phi(\theta, r)$ is the solution of the equation

$$w_\phi(t) = \gamma_r^{(n)}(\theta) \tag{6}$$

(ii) Let us consider the family of Archimedean spirals

$$w_\phi(t) = te^{i(t+\phi)}, \quad t \in (0, +\infty),$$

where $\phi \in [0, 2\pi)$ is a parameter. Then (3) is equivalent with (5), where ϕ is given by (6).

(iii) Let us consider the family of hyperbolic spirals

$$w_\phi(t) = e^{i(t+\phi)}/t, \quad t \in (0, +\infty), \quad \phi \in [0, 2\pi).$$

Then (4) is equivalent with (5), where ϕ is given by (6).

Proof. From the proof of Lemma 2.2 we have

$$\gamma_r^{(n)}(\theta) = cD^n(f)(z), \quad c \in \{-1, +1, i, -i\}.$$

(i) By (6) we get (as in the proof of Theorem 1 in [1])

$$t\cos\gamma = \log|cD^n(f)(z)|$$

$$\phi - t\sin\gamma = \arg(cD^n(f)(z)),$$

and consequently

$$\phi = \arg[cD^n(f)(z)] + \operatorname{tg}\gamma\log|cD^n(f)(z)|,$$

$$\frac{\partial\phi}{\partial\theta} = \frac{1}{\cos\gamma}\operatorname{Re}\left[e^{i\gamma}\frac{cD^{n+1}(f)(z)}{cD^n(f)(z)}\right] = \frac{1}{\cos\gamma}\operatorname{Re}\left[e^{i\gamma}\frac{D^{n+1}(f)(z)}{D^n(f)(z)}\right].$$

(ii) Replacing in the statement (iii) of Theorem 2 in [1] $f(z)$ by $cD^n(f)(z)$, we obtain (5).

(iii) Replace $f(z)$ by $cD^n(f)(z)$ in the statement (iii) of Theorem 3 in [1].

The theorem is proved. □

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