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COUNTERPARTS OF ARITHMETIC MEAN-GEOMETRIC MEAN-HARMONIC MEAN INEQUALITY

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Abstract. Some converse inequalities for the celebrated arithmetic mean-geometric mean-harmonic mean inequality are given.

1. Introduction

Recall the means

1) *weighted arithmetic mean* $A_n(\mathbf{w},\mathbf{a})$,

$$A_n(\mathbf{w},\mathbf{a}) := \frac{1}{W_n}\sum_{i=1}^{n} w_i a_i;$$

2) *weighted geometric mean* $G_n(\mathbf{w},\mathbf{a})$,

$$G_n(\mathbf{w},\mathbf{a}) := \left(\prod_{i=1}^{n} a_i^{w_i}\right)^{\frac{1}{W_n}}$$

and

3) *weighted harmonic mean* $H_n(\mathbf{w},\mathbf{a})$,

$$H_n(\mathbf{w},\mathbf{a}) := \frac{W_n}{\sum_{i=1}^{n}\frac{w_i}{a_i}}$$

where

$$\mathbf{a} = (a_1, ..., a_n), \mathbf{w} = (w_1, ..., w_n), a_i, w_i > 0\ (i = 1, ..., n)$$

and $W_n := \sum_{i=1}^{n} w_i$.

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The following inequality is well known in the literature as *arithmetic mean - geometric mean - harmonic mean* inequality

$$A_n(\mathbf{w},\mathbf{a}) \geq G_n(\mathbf{w},\mathbf{a}) \geq H_n(\mathbf{w},\mathbf{a}). \quad (1.1)$$

The equality holds in (1.1) if and only if $a_1 = ... = a_n$. Note that (1.1) is equivalent to

$$1 \leq \frac{A_n(\mathbf{w},\mathbf{a})}{G_n(\mathbf{w},\mathbf{a})}, \quad 1 \leq \frac{G_n(\mathbf{w},\mathbf{a})}{H_n(\mathbf{w},\mathbf{a})}. \quad (1.2)$$

The main aim of this note in to point out upper bounds for the quotients

$$\frac{A_n(\mathbf{w},\mathbf{a})}{G_n(\mathbf{w},\mathbf{a})}, \quad \frac{G_n(\mathbf{w},\mathbf{a})}{H_n(\mathbf{w},\mathbf{a})}.$$

2. The Results

In the recent paper [1], Dragomir and Goh, by the use of an inequality for convex functions, have proved the following analytic inequality for the logarithmic mapping.

Lemma 1. *Let $\xi, p_i > 0$ $(i = 1, ..., n)$ where $\sum_{i=1}^{n} p_i = 1$. Then*

$$0 \leq \sum_{i=1}^{n} p_i \ln \xi_i - \ln\left(\sum_{i=1}^{n} p_i \xi_i\right) \quad (2.1)$$

$$\leq \sum_{i=1}^{n} \frac{p_i}{\xi_i} \sum_{i=1}^{n} p_i \xi_i - 1 = \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \frac{(\xi_i - \xi_j)^2}{\xi_i \xi_j}$$

$$= \sum_{1 \leq i < j \leq n} p_i p_j \frac{(\xi_i - \xi_j)^2}{\xi_i \xi_j}.$$

The equalities hold iff $\xi_1 = ... = \xi_n$

Using this result, we can state the following theorem containing a converse of A.-G.-H. inequalities.

Theorem 2. *Let* $\mathbf{w}, \mathbf{a}$ *be as in Introduction. Then*

$$1 \leq \frac{A_n(\mathbf{w},\mathbf{a})}{G_n(\mathbf{w},\mathbf{a})} \leq \exp\left[\frac{A_n(\mathbf{w},\mathbf{a})}{H_n(\mathbf{w},\mathbf{a})} - 1\right] \tag{2.2}$$

$$= \exp\left[\frac{1}{2W_n^2}\sum_{i,j=1}^{n} w_i w_j \frac{(a_i - a_j)^2}{a_i a_j}\right]$$

$$= \exp\left[\frac{1}{W_n^2}\sum_{1\leq i<j\leq n} w_i w_j \frac{(a_i - a_j)^2}{a_i a_j}\right] =: B_n(\mathbf{w},\mathbf{a})$$

and

$$1 \leq \frac{G_n(\mathbf{w},\mathbf{a})}{H_n(\mathbf{w},\mathbf{a})} \leq B_n(\mathbf{w},\mathbf{a}). \tag{2.3}$$

The equalities hold in both inequalities iff $a_1 = ... = a_n$.

Proof. The proof of (2.2) follow by (2.1) choosing $p_i = \frac{w_i}{W_n}$ and $\xi_i = a_i \; (i = 1, ..., n)$.

The proof of (3) follows by (2.2) choosing $\frac{1}{\mathbf{a}}$ instead of $\mathbf{a}$ and taking into account that $B_n\left(\mathbf{w}, \frac{1}{\mathbf{a}}\right) = B_n(\mathbf{w},\mathbf{a})$. □

We point out another results which does not use the concavity property of log -mapping, but an inequality between the geometric and logarithmic mean of two positive numbers.

Theorem 3. *Let* $\mathbf{w}, \mathbf{a}$ *be as in Introduction. Then*

$$1 \leq \frac{A_n(\mathbf{w},\mathbf{a})}{G_n(\mathbf{w},\mathbf{a})} \leq \exp\left[\frac{1}{[A_n(\mathbf{w},\mathbf{a})]^{1/2}} \cdot \frac{1}{W_n}\sum_{i=1}^{n} \frac{w_i}{\sqrt{a_i}} |A_n(\mathbf{w},\mathbf{a}) - a_i|\right]$$

$$\leq \exp\left[\frac{\left[\frac{1}{W_n}\sum_{i=1}^{n} w_i [A_n(\mathbf{w},\mathbf{a}) - a_i]^2\right]^{1/2}}{[A_n(\mathbf{w},\mathbf{a})\,H_n(\mathbf{w},\mathbf{a})]^{1/2}}\right]. \tag{2.4}$$

The equality holds iff $a_1 = ... = a_n$.

Proof. We recall the following well known inequality between the *geometric mean* $G(a,b) := \sqrt{ab}\,(a,b>0)$ and the *logarithmic mean* (see for example [2, p. 346])

$$L(a,b) := \begin{cases} a \text{ if } b=a \\ \frac{b-a}{\ln b-\ln a} \text{ if } b \neq a \end{cases} \quad (a,b>0)$$

i.e.,

$$G(a,b) \leq L(a,b) \text{ for all } a,b>0. \tag{2.5}$$

Note that (2.5) is equivalent to

$$|\ln b-\ln a| \leq \frac{|b-a|}{\sqrt{ab}},\ a,b>0. \tag{2.6}$$

The equality holds in (2.6) iff $a=b$.

Now, choose in (2.6)

$$b := A_n(\mathbf{w},\mathbf{a}),\ a=a_i\ (i=1,...,n)$$

to get

$$|\ln A_n(\mathbf{w},\mathbf{a})-\ln a_i| \leq \frac{|A_n(\mathbf{w},\mathbf{a})-a_i|}{\sqrt{A_n(\mathbf{w},\mathbf{a})\,a_i}} \tag{2.7}$$

for all $i \in \{1,...,n\}$.

Multiplying by $w_i>0$ and summing over $i \in \{1,...,n\}$, we deduce

$$\left|W_n \ln A_n(\mathbf{w},\mathbf{a}) - \sum_{i=1}^n w_i \ln a_i\right|$$

$$\leq \sum_{i=1}^n w_i\,|\ln A_n(\mathbf{w},\mathbf{a})-\ln a_i|$$

$$\leq \sum_{i=1}^n w_i \frac{|A_n(\mathbf{w},\mathbf{a})-a_i|}{\sqrt{A_n(\mathbf{w},\mathbf{a})\,a_i}}$$

$$= \frac{1}{[A_n(\mathbf{w},\mathbf{a})]^{1/2}} \sum_{i=1}^n \frac{w_i}{\sqrt{a_i}}\,|A_n(\mathbf{w},\mathbf{a})-a_i|$$

from where results the first inequality in (4).

Using the Cauchy-Buniakowski-Schwarz's discrete inequality, we get

$$\frac{1}{W_n}\sum_{i=1}^{n}\frac{w_i}{\sqrt{a_i}}\left|A_n(\mathbf{w},\mathbf{a})-a_i\right|$$

$$\leq\left[\frac{1}{W_n}\sum_{i=1}^{n}w_i\left[A_n(\mathbf{w},\mathbf{a})-a_i\right]^2\right]^{1/2}\left(\frac{1}{W_n}\sum_{i=1}^{n}\frac{w_i}{a_i}\right)^{1/2}$$

$$=\frac{\left[\frac{1}{W_n}\sum_{i=1}^{n}w_i\left[A_n(\mathbf{w},\mathbf{a})-a_i\right]^2\right]^{1/2}}{\left[H_n(\mathbf{w},\mathbf{a})\right]^{1/2}}$$

and the second inequality in (4) also holds.

The case of equality is obvious. □

The following corollary which provides an upper bound for the quotient

$$\frac{G_n(\mathbf{w},\mathbf{a})}{H_n(\mathbf{w},\mathbf{a})}$$

holds.

Corollary 4. *Let* $\mathbf{w},\mathbf{a}$ *be as in Introduction. Then*

$$1\leq\frac{G_n(\mathbf{w},\mathbf{a})}{H_n(\mathbf{w},\mathbf{a})}\leq\exp\left[\frac{1}{\left[H_n(\mathbf{w},\mathbf{a})\right]^{1/2}}\cdot\frac{1}{W_n}\sum_{i=1}^{n}\frac{w_i}{\sqrt{a_i}}\left|H_n(\mathbf{w},\mathbf{a})-a_i\right|\right]$$

$$\leq\exp\left[\left[\frac{A_n(\mathbf{w},\mathbf{a})}{H_n(\mathbf{w},\mathbf{a})}\right]^{1/2}\cdot\frac{1}{W_n}\sum_{i=1}^{n}\frac{w_i}{a_i^2}\left[H_n(\mathbf{w},\mathbf{a})-a_i\right]^2\right].$$

The equality holds iff $a_1=\ldots=a_n$.

The proof follows by the inequality (4) putting $\frac{1}{\mathbf{a}}$ instead of $\mathbf{a}$ and taking into account that

$$A_n\left(\mathbf{w},\frac{1}{\mathbf{a}}\right)=H^{-1}(\mathbf{w},\mathbf{a}).$$

We omit the details.

For an extensive literature on weighted means and their inequalities, the author recommends the monograph [2].

References

[1] S.S. Dragomir and C.J. Goh, A counterpart of Jensen's discrete inequality for differentiable convex mappings and applications in information theory, *Math. Compute. Modelling*, **24**(2)(1996), 1-11.
[2] P.S. Bullen, D.S. Mitrinović and P.M. Vasić, *Means and Their Inequalities*, D. Reidel Publishing Company, 1988.

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