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ON A SUBCLASS OF CERTAIN STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. This work presents the class of functions, note by $P(n,\lambda,\alpha)$, which contain univalent functions with negative coefficients, satisfying:

$$Re\{\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda zf'(z)+(1-\lambda)f(z)}\} > \alpha.$$

If $f_j(z) \in P(n,\lambda,\alpha)$, $j=\overline{1,m}$, then the convolution of theese functions, $h(z)$, lies to the class $P(n,\lambda,\beta)$, where we have β.

The author obtain the order of starlikeness of a convex function of order α, with negative coefficients. The theorems **2,3,4** and corrolaris **1,2,4,5** are original results of the author.

Let A(n) denote the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k,$$

$a_k \geq 0, n \in N = \{1,2,...,n\}$, which are analytic in the unite disk:

$$U = \{z \in C : |z| < 1\}.$$

The function $f(z) \in A(n)$ is said to be in the class $P(n,\lambda,\alpha)$ if it satisfies:

$$Re\{\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda zf'(z)+(1-\lambda)f(z)}\} > \alpha,$$

for some α $(0 \leq \alpha < 1)$, λ $(0 \leq \lambda \leq 1)$, and for all $z \in U$.

The classes $P(n,0,\alpha) \equiv T_\alpha(n)$ and $P(n,1,\alpha) \equiv C_\alpha(n)$ were studied by Srivastava,Owa and Chatterjea in [3], and the classes $P(1,0,\alpha) \equiv T^\star(\alpha)$ and $P(1,1,\alpha) \equiv C(\alpha)$ by Silvermann in [2].

Theorem 1 ([1]). *The function $f(z) \in A(n)$ is in the class $P(n, \lambda, \alpha)$ if and only if:*

$$\sum_{k=n+1}^{\infty} (k-\alpha)(\lambda k - \lambda + 1)a_k \leq 1 - \alpha.$$

For $\lambda = 0$ and $\lambda = 1$ we obtain two Lemmas in [3], and if $n = 1$ too, we obtain two Lemmas in [2]. We have the following theorem :

Theorem 2. *If the function $f \in C_\alpha(n)$, then $f \in P(n, \lambda, \beta)$, where:*

$$\beta = 1 - \frac{n(1-\alpha)(\alpha n + 1)}{(n+1)(n+1-\alpha) - (1-\alpha)(\lambda n + 1)}.$$

The result is sharp, the extremal function is:

$$f(z) = z - \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1}.$$

Proof. We know that:

$$f \in C_\alpha(n) \Leftrightarrow \sum_{k=n+1}^{\infty} k(k-\alpha)a_k \leq 1 - \alpha.$$

and

$$f \in P(n, \lambda, \beta) \Leftrightarrow \sum_{k=n+1}^{\infty} (k-\beta)(\lambda k - \lambda + 1)a_k \leq 1 - \beta.$$

We have to find the largest β such that

$$\frac{(k-\beta)(\lambda k - \lambda + 1)}{1-\beta} \leq \frac{k(k-\alpha)}{1-\alpha}. \tag{1}$$

The inequality (1) is equivalent to

$$\beta \leq \frac{k(k-\alpha) - k(1-\alpha)(\lambda k - \lambda + 1)}{k(k-\alpha) - (1-\alpha)(\lambda k - \lambda + 1)} = 1 - \frac{(k-1)(1-\alpha)(\lambda k - \lambda + 1)}{k(k-\alpha) - (1-\alpha)(\lambda k - \lambda + 1)}.$$

We define the function $g(k)$by:

$$g(k) = 1 - \frac{(k-1)(1-\alpha)(\lambda k - \lambda + 1)}{k(k-\alpha) - (1-\alpha)(\lambda k - \lambda + 1)}.$$

Therefore $g(k) \leq g(k+1)$ we have that the function $g(k)$ is an increasing function on k, $k \geq n+1$.

Finaly we have :

$$\beta = g(n+1) = 1 - \frac{n(1-\alpha)(\lambda n + 1)}{(n+1)(n+1-\alpha) - (1-\alpha)(\lambda n + 1)},$$

which completes the proof of our theorem. □

Convolution of functions

Let the functions $f_j(z)$ be defined by :

$$f_j(z) = z - \sum_{k=n+1}^{\infty} a_{j,k}\, z^k,$$

$a_{j,k} \geq 0$, $j = 1, 2, \ldots, m$. Then we define the function $h(z)$ by:

$$h(z) = z - \sum_{k=n+1}^{\infty} (a_{1,k}^2 + a_{2,k}^2 + \cdots + a_{m,k}^2) z^k. \tag{2}$$

Theorem 3. *If $f_j(z) \in P(n, \lambda, \alpha)$, $j = 1, 2, \ldots, m$, then the function $h(z)$ given by (2) is in the class $P(n, \lambda, \beta)$, where:*

$$\beta = 1 - \frac{mn(1-\alpha)^2}{(n+1-\alpha)^2(\lambda n + 1) - m(1-\alpha^2)}.$$

The result is sharp, the extremal functions are:

$$f_j(z) = z - \frac{1-\alpha}{(n+1-\alpha)(\lambda n + 1)} z^{n+1}, \qquad j = 1, 2, \ldots, m.$$

Proof. By using Theorem 1 we have

$$\sum_{k=n+1}^{\infty} [\frac{(k-\alpha)(\lambda k - \lambda + 1)}{1-\alpha}]^2 a_{j,k}^2 \leq [\sum_{k=n+1}^{\infty} \frac{(k-\alpha)(\lambda k - \lambda + 1}{1-\alpha)} a_{j,k}]^2 \leq 1, \tag{3}$$

$j = 1, 2, \ldots, m$. (3) implies:

$$\frac{1}{m} \sum_{k=n+1}^{\infty} [\frac{(k-\alpha)(\lambda k - \lambda + 1)}{1-\alpha}]^2 (a_{1,k}^2 + \cdots + a_{m,k}^2) \leq 1.$$

We have to find the largest β such that:

$$\frac{(k-\beta)(\lambda k - \lambda + 1)}{(1-\beta)} \leq \frac{1}{m} \frac{(k-\alpha)^2(\lambda k - \lambda + 1)^2}{(1-\alpha)^2}. \tag{4}$$

The inequality (4) is equivalent to

$$\beta \leq \frac{(k-\alpha)^2(\lambda k - \lambda + 1) - mk(1-\alpha)^2}{(k-\alpha)^2(\lambda k - \lambda + 1) - m(1-\alpha)^2} =$$

$$= 1 - \frac{m(k-1)(1-\alpha)^2}{(k-\alpha)^2(\lambda k - \lambda + 1) - m(1-\alpha)^2}.$$

Let the function $s(k)$ be :

$$s(k) = 1 - \frac{m(k-1)(1-\alpha)^2}{(k-\alpha)^2(\lambda k - \lambda + 1) - m(1-\alpha)^2},$$

We prove that $s(k) \leq s(k+1)$ for k, $k \geq n+1$, inequality wich is equivalent to

$$g(k) \geq 0,$$

where

$$g(k) = 2\lambda k^3 + (1 - \lambda - 2\alpha\lambda)k^2 + (-1 - \lambda + 2\alpha\lambda)k + (m-1)(1-\alpha)^2.$$

We have

$$g(2) = 6\lambda + 4\lambda(1-\alpha) + 2 + (m-1)(1-\alpha)^2 \geq 0.$$

By calculating the derivate of the $g(k)$, we obtain :

$$g'(k) = 6\lambda k^2 + 2(1 - \lambda - 2\alpha\lambda)k - 1 - 1 + 2\alpha\lambda.$$

We also have :

$$g'(2) = 13\lambda + 6\lambda(1-\alpha) + 3 > 0 \quad (5)$$

$$g''(k) = 12\lambda k + 2(1 - \lambda - 2\alpha\lambda) \quad (6)$$

$$g''(2) = 18\lambda + 4\lambda(1-\alpha) + 2 > 0 \quad (7)$$

$$g'''(k) = 12\lambda > 0, \; for \; 0 < \lambda \leq 1 \quad (8)$$

For $\lambda = 0$ we have $g(k) = k(k-1) + (m-1)(1-\alpha)^2 \geq 0$

So that (8) implies that the function $g''(k)$ is an increasing function on k, and by using (7) we have $g''(k) > 0$. This implies that the function $g'(k)$ is increasing on k. Using (5) we have $g'(k) > 0$ so that the function $g(k)$ is increasing on k. But $g(2) \geq 0$ so $g(k) \geq 0$ for $k \geq n+1$.

Therefore $s(k) \leq s(k+1)$, the function $s(k)$ is an increasing function in k , $k \geq n+1$, and this implies that :

$$\beta \leq s(n+1) = 1 - \frac{mn(1-\alpha)^2}{(n+1-\alpha)^2(\lambda n + 1) - m(1-\alpha)^2}.$$

For the functions :

$$f_j(z) = z - \frac{1-\alpha}{(n+1-\alpha)(\lambda n+1)} z^{n+1}, \qquad j = 1, 2, \ldots, m,$$

the result is sharp. □

Corollary 1. *If $f_j(z) \in P(n, \lambda, \alpha)$, $j = 1, 2$, then the function :*

$$h(z) = z - \sum_{k=n+1}^{\infty} (a_{1,k}^2 + a_{2,k}^2) z^k$$

is in the class $P(n, \lambda, \beta)$, where:

$$\beta = 1 - \frac{2n(1-\alpha)^2}{(n+1-\alpha)^2)(\lambda n+1) - 2(1-\alpha)^2}.$$

The result is sharp for the functions :

$$f_1(z) = f_2(z) = z - \frac{1-\alpha}{(n+1-\alpha)(\lambda n+1)} z^{n+1}.$$

Corollary 2. *Let $f_j(z) \in T_\alpha(n)$, $j = 1, 2, \ldots, m$. Then the function $h(z)$ given by (2) is in the class $T_\beta(n)$, where*

$$\beta = 1 - \frac{mn(1-\alpha)^2}{(n+1-\alpha)^2 - m(1-\alpha)^2}.$$

The result is sharp, the extremal functions are :

$$f_j(z) = z - \frac{1-\alpha}{n+1-\alpha} z^{n+1} \qquad j = 1, 2, \ldots, m.$$

Corollary 3. *Let $f_j(z) \in C_\alpha(n), j = 1, 2, \ldots, m$. Then the function $h(z)$ given by (2) lies to the class $C_\beta(n)$, where:*

$$\beta = 1 - \frac{mn(1-\alpha)^2}{(n+1)(n+1-\alpha)^2 - m(1-\alpha)^2}.$$

The result is sharp for the functions :

$$f_j(z) = z - \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1} \qquad j = 1, 2, \ldots, m.$$

The order of starlikeness of a convex function of order α from the class $A(n)$

We know that the class $P(n,1,\alpha) \equiv C_\alpha(n)$ contain convex functions of order α, with :

$$Re(1+\frac{zf''(z)}{f'(z)}) > \alpha, \qquad z \in U,$$

and the class $P(n,0,\beta) \equiv T_\beta(n)$ contain starlike functions of order β, with :

$$Re\frac{zf'(z)}{f(z)} > \beta, \qquad z \in U.$$

Theorem 4. *If $f \in C_\alpha(n)$,then $f \in T_\beta(n)$,where :*

$$\beta = \frac{n(n+1)}{(n+1)(n+1-\alpha)-(1-\alpha)}.$$

The result is sharp for the function :

$$f(z) = z - \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1}.$$

Proof. Using the Theorem 1. for $\lambda = 1$ we have:

$$\sum_{k=n+1}^{\infty} k(k-\alpha)a_k \leq 1-\alpha. \tag{9}$$

From the Theorem 1. for $\lambda = 0$ we have:

$$f \in T_\beta(n) \Leftrightarrow \sum_{k=n+1}^{\infty} (k-\beta)a_k \leq 1-\beta. \tag{10}$$

We have to find the largest β such that:

$$\frac{k-\beta}{1-\beta} \leq \frac{k(k-\alpha)}{1-\alpha}. \tag{11}$$

The inequality (11) is equivalent to:

$$\beta \leq \frac{k(k-1)}{k(k-\alpha)-(1-\alpha)}.$$

Let the function $g(k)$ be:

$$g(k) = \frac{k(k-1)}{k(k-\alpha)-(1-\alpha)}.$$

Therefore $g'(k) \geq 0$ for k, $k \geq n+1$, the function $g(k)$ is an increasing function on k, $k \geq n+1$, we have :

$$\beta \leq g(n+1) = \frac{n(n+1)}{(n+1)(n+1-\alpha)-(1-\alpha)},$$

which completes the proof of our theorem.

The inequality in (9) and (10) are attained for the function:

$$f(z) = z - \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1}.$$

□

Corollary 4. *For $\alpha = 0$ we obtain $\beta = \frac{n+1}{n+2}$. Thus a convex function from class $A(n)$ is starlike of order $\beta = \frac{n+1}{n+2}$.*

Corollary 5. *For $n = 1$ we have $\beta = \frac{2}{3-\alpha}$. If $\alpha = 0$, then we have $\beta = \frac{2}{3}$, so a convex function of the form:*

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k$$

is starlike of order $\frac{2}{3}$, and $\frac{2}{3} > \frac{1}{2}$.

We know, that in case of the functions of the form :

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

not necessary with negative coefficients, the theorem of Marx and Strohhacker tell us that a convex function is starlike of order $\frac{1}{2}$.

The same theorem, for $n = 2$, tell us that a convex function of the form

$$f(z) = z + \sum_{k=3}^{\infty} a_k z^k,$$

is starlike of order $\frac{2}{\pi}$.

From Theorem 4., for $n = 2$ we have $\beta = \frac{3}{4-\alpha}$, and if $\alpha = 0$, we obtain $\beta = \frac{3}{4}$.

Finally, a convex function of the form :

$$f(z) = z - \sum_{k=3}^{\infty} a_k z^k$$

is starlike of order $\frac{3}{4}$, and $\frac{3}{4} > \frac{2}{\pi}$.

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