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A SUFFICIENT CONDITION FOR UNIVALENCE

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Abstract. In this paper we obtain an univalence criterion for holomorphic mappings in the unit ball of $\mathbb{C}^n$.

1. Introduction

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$ with the usual inner product

$$\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w_j}$$

and norm $\|z\| = \langle z, z \rangle^{\frac{1}{2}}$. The unit ball $\{z \in \mathbb{C}^n : \|z\| < 1\}$ is denoted B^n.

We let $\mathcal{L}(\mathbb{C}^n)$ denote the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$, i.e. the $n \times n$ complex matrices $A = (A_{jk})$, with the standard operator norm

$$\|A\| = \sup\{\|Az\| : \ \|z\| < 1\}, \quad A \in \mathcal{L}(\mathbb{C}^n).$$

$I = (I_{jk})$ denotes the identity in $\mathcal{L}(\mathbb{C}^n)$.

We denote by $H(B^n)$ the class of holomorphic mappings

$$f(z) = (f_1(z), \ldots, f_n(z)), z \in B^n$$

from B^n into $\mathbb{C}^n$. We say that $f \in H(B^n)$ is *locally biholomorphic* in B^n if f has a local inverse at each point in B^n or equivalently if the derivative

$$Df(z) = \left(\frac{\partial f_k(z)}{\partial z_j}\right)_{1 \le j,k \le n}$$

is nonsingular at each point $z \in B^n$.

The second derivative of a function $f \in H(B^n)$ is a symmetric bilinear operator $D^2 f(z)(\cdot,\cdot)$ on $\mathbb{C}^n \times \mathbb{C}^n$. $D^2 f(z)(z,\cdot)$ is the linear operator obtained by restricting $D^2f(z)$ to $\{z\} \times \mathbb{C}^n$ and has the matrix representation

$$D^2 f(z)(z,\cdot) = \left(\sum_{m=1}^{n} \frac{\partial^2 f_k(z)}{\partial z_j \partial z_m} z_m \right)_{1 \le j,k \le n}$$

A mapping $v \in H(B^n)$ is called a *Schwarz function* if $\|v(z)\| \le \|z\|, z \in B^n$. If $f, g \in H(B^n)$ we say that f is *subordinate* to g $(f \prec g)$ in B^n, if there exists a Schwarz function v such that $f(z) = g(v(z)), z \in B^n$.

A function $L : B^n \times [0,\infty) \to \mathbb{C}^n$ is an *univalent subordination chain* if $L(\cdot,t) \in H(B^n), L(\cdot,t)$ is univalent in B^n for all $t \in [0,\infty)$ and $L(\cdot,s) \prec L(\cdot,t)$, whenever $0 \le s < t < \infty$.

We shall use only normalized functions in an univalent subordination chain, i.e $DL(0,t) = e^t I$, for all $t \ge 0$.

The following theorem is due to J.A. Pfaltzgraff and we shall use it to prove our results.

Theorem 1. [3] *Let* $L(z,t) = e^t z + \ldots$, *be a function from* $B^n \times [0,\infty)$ *into* $\mathbb{C}^n$ *such that:*

(i) *For each* $t \ge 0$, $L(\cdot,t) \in H(B^n)$.

(ii) $L(z,t)$ *is a locally absolutely continuous function of* t, *locally uniformly with respect to* $z \in B^n$.

Let $h(z,t)$ *be a function from* $B^n \times [0,\infty)$ *into* $\mathbb{C}^n$ *such that:*

(iii) *For each* $t \ge 0, h(\cdot,t) \in H(B^n)$, $h(0,t), h(0,t) = 0$, $Dh(0,t) = I$ *and* $\mathrm{Re} < h(z,t), z \ge 0$, $z \in B^n$.

(iv) *For each* $T > 0$ *and* $r \in (0,1)$ *there is a number* $K = K(r,T)$ *such that* $\|h(z,t)\| \le K(r,T)$, *where* $\|z\| \le r$ *and* $t \in [0,T]$.

(v) *For each* $z \in B^n$, $h(z,t)$ *is a measurable function of* t *on* $[0,\infty)$.

Suppose $h(z,t)$ *satisfies*

$$\frac{\partial L(z,t)}{\partial t} = DL(z,t)\, h(z,t) \quad a.e \quad t \ge 0, \quad for\ all \quad z \in B^n. \tag{1}$$

Further, suppose there is a sequence $(t_m)_{m\geq 0}$, $t_m > 0$ *increasing to* ∞ *such that*

$$\lim_{m\to\infty} e^{-t_m} L(z, t_m) = F(z) \tag{2}$$

locally uniformly in B^n.

Then for each $t \geq 0$, $L(\cdot, t)$ *is univalent on* B^n.

2. Main results

Theorem 2. *Let* $f, g \in H(B^n)$ *such that* $f(0) = g(0) = 0$, $Df(0) = Dg(0) = I$ *and* g *is locally univalent in* B^n. *If*

$$\left\|(Dg(z))^{-1} Df(z) - I\right\| < 1 \tag{3}$$

and

$$\left\|\|z\|^2 \left[(Dg(z))^{-1} Df(z) - I\right] + \left(1 - \|z\|^2\right)(Dg(z))^{-1} D^2 g(z)(z, \cdot)\right\| < 1 \tag{4}$$

for all $z \in B^n$, *then* f *is an univalent function in* B^n.

Proof. We define

$$L(z, t) = f(e^{-t} z) + (e^t - e^{-t}) Dg(e^{-t} z)(z), \quad (z, t) \in B^n \times [0, \infty)$$

We shall prove that $L(z, t)$ satisfies the conditions of Theorem 1 and hence $L(\cdot, t)$ is univalent in B^n, for all $t \in [0, \infty)$. Since $f(z) = L(z, 0)$ we obtain that f is an univalent function in B^n.

We have $L(z, t) = e^t z +$ (holomorphic term). Thus $\lim_{t\to\infty} e^{-t} L(z, t) = z$, locally uniformly with respect to B^n and hence (2) holds for $F(z) = z$.

Clearly $L(z, t)$ satisfies the absolute continuity requirements of Theorem 1.

From (5) we obtain

$$DL(z, t) = e^t Dg(e^{-t} z)[I - E(z, t)] \tag{5}$$

where, for all $(z, t) \in B^n \times [0, \infty)$, $E(z, t)$ is the linear operator defined by

$$\begin{aligned} E(z, t) &= e^{-2t}\left[(Dg(e^{-t} z))^{-1} Df(e^{-t} z) - I\right] - \\ &\quad - (1 - e^{-2t})(Dg(e^{-t} z))^{-1} D^2 g(e^{-t} z)(e^{-t} z, \cdot). \end{aligned} \tag{6}$$

We consider

$$A\left(e^{-t}z\right) = \left(Dg\left(e^{-t}z\right)\right)^{-1} Df\left(e^{-t}z\right) - I$$

$$B\left(e^{-t}z\right) = \left(Dg\left(e^{-t}z\right)\right)^{-1} D^2g\left(e^{-t}z\right)\left(e^{-t}z,\cdot\right) \quad \text{and}$$

$$F\left(z,t,\lambda\right) = \lambda A\left(e^{-t}z\right) + \left(1-\lambda\right)B\left(e^{-t}z\right), \quad \lambda \in [0,1]$$

From (3) and (4) it results $\|A\left(e^{-t}z\right)\| < 1$ and $\|F\left(z,t,\lambda_z\right)\| < 1$, where $\lambda_z = e^{-2t}\|z\|^2, z \in B^n, t \geq 0$. Since $1 \geq e^{-2t} > \lambda_z$, for all $z \in B^n$ and $t \geq 0$ we can write $e^{-2t} = u + (1-u)\lambda_z$, where $u \in [0,1)$. Then

$$-E\left(z,t\right) = uA\left(e^{-t}z\right) + \left(1-u\right)F\left(z,t,\lambda_z\right), \quad u \in [0,1).$$

We obtain

$$\|E\left(z,t\right)\| \leq u\left\|A\left(e^{-t}z\right)\right\| + \left(1-u\right)\|F\left(z,t,\lambda_z\right)\| < 1, \quad (z,t) \in B^n \times [0,\infty)$$

and hence $I - E\left(z,t\right)$ is an invertible operator.

Further calculation shows that

$$\begin{aligned}\frac{\partial L\left(z,t\right)}{\partial t} &= e^t Dg\left(e^{-t}z\right)\left[I + E\left(z,t\right)\right](z) = \\ &= DL\left(z,t\right)\left[I - E\left(z,t\right)\right]^{-1}\left[I + E\left(z,t\right)\right](z).\end{aligned}$$

It results that $L\left(z,t\right)$ satisfies the differential equation (1) for all $t \geq 0$ and $z \in B^n$, where

$$h\left(z,t\right) = \left[I - E\left(z,t\right)\right]^{-1}\left[I + E\left(z,t\right)\right](z). \tag{7}$$

We shall show that $h\left(z,t\right)$ satisfies the condition (iii), (iv) and (v) of Theorem 1. Clearly, $h\left(z,t\right)$ satisfies the holomorphy and measurability requirements, $h\left(0,t\right) = 0$ and $Dh\left(0,t\right) = I$. The inequality

$$\|h\left(z,t\right) - z\| = \|E\left(z,t\right)\left(h\left(z,t\right) + z\right)\| \leq \|E\left(z,t\right)\|\cdot\|h\left(z,t\right) + z\| \leq \|h\left(z,t\right) + z\|$$

implies $\operatorname{Re}\langle h\left(z,t\right), z\rangle \geq 0$, for $z \in B^n$ and $t \geq 0$.

For a fixed $t \geq 0$, $E(\cdot, t)$ defined by (7) is an holomorphic function from B^n into $\mathcal{L}(\mathbb{C}^n)$, $E(0,t) = 0$ and $\|E(z,t)\| < 1, z \in B^n$.

By using Schwarz lemma for $\mathbb{C}^n$ we obtain $\|E(z,t)\| \leq \|z\|, z \in B^n$.

It follows

$$\|h(z,t)\| \leq \|z\| \frac{1+\|z\|}{1-\|z\|}, \quad \text{for all} \quad \in B^n.$$

The conditions of Theorem 1 being satisfied we obtain that the functions $L(z,t)$, $t \geq 0$ are univalent in B^n. In particular $f(z) = L(z,0)$ is univalent in B^n.

□

Remark. If $g = f$, then Theorem 2 becomes the n-dimensional version of Becker's univalence criterion [3].

References

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