ABSOLUTELY F/U-PURE MODULES

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Abstract. Let R be an associative ring with non-zero identity. A submodule A of a right R-module B is said to be F/U-pure if $f \otimes_R 1_{F/U}$ is a monomorphism for every free left R-module F and for every cyclic submodule U of F, where $f : A \to B$ is the inclusion monomorphism. A right R-module D is said to be absolutely F/U-pure if D is F/U-pure in every right R-module which contains it as a submodule. We characterize absolutely F/U-purity by injectivity with respect to a certain monomorphism. We also prove that the class of absolutely F/U-pure right R-modules is closed under taking direct products, direct sums and extensions. Moreover, we consider absolutely F/U-pure right modules over right noetherian rings and regular (von Neumann) rings.

1. Introduction

In this paper we denote by R an associative ring with non-zero identity and all R-modules are unital. By a homomorphism we understand an R-homomorphism. The category of right R-modules is denoted by $Mod-R$. The injective envelope of a right R-module A is denoted by $E(A)$.

Let

$$0 \longrightarrow A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \longrightarrow 0 \tag{1}$$

be a short exact sequence of right R-modules and homomorphisms. The monomorphism f is said to be F/U-pure if the tensor product $f \otimes 1_{F/U} : A \otimes_R F/U \to B \otimes_R F/U$ is a monomorphism for every free left R-module F and for every cyclic submodule U of F [1, Definition 2.1]. If f is F/U-pure, then the short exact sequence (1) is called

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F/U-pure. If A is a submodule of B and f is the inclusion monomorphism, then A is said to be an F/U-pure submodule of B.

Let $M \in Mod-R$. Then M is said to be projective with respect to the short exact sequence (1) if the natural homomorphism $Hom_R(M,B) \to Hom_R(M,C)$ is surjective. The right R-module M is said to be injective with respect to the short exact sequence (1) (or with respect to the monomorphism f) if the natural homomorphism $Hom_R(B,M) \to Hom_R(A,M)$ is surjective.

Following Maddox [3], a right R-module M is said to be absolutely pure if M is pure in every right R-module which contains M as a submodule.

In the present paper we introduce the notion of absolutely F/U-pure right R-module and we establish some properties for such modules.

2. Basic results

We shall begin with two results which will be used later in the paper.

Theorem 2.1. [1, Theorem 2.8] *Let A be a submodule of a right R-module B. Then the following statements are equivalent:*

(i) A is F/U-pure in B;

(ii) If $a_1, \ldots, a_n \in R$, $r_1, \ldots, r_n \in R$ and the system of equations $a_i = xr_i$, $i = 1, \ldots, n$ has a solution $b \in B$, then it has a solution $a \in A$.

Theorem 2.2. [2, Theorem 2.3] *A short exact sequence (1) is F/U-pure if and only if for every finitely generated right ideal of R the right R-module R/I is projective with respect to the short exact sequence (1).*

We shall give now the following definition.

Definition 2.3. A right R-module A is said to be *absolutely F/U-pure* if A is F/U-pure in each right R-module which contains A as a submodule.

In the sequel we shall denote by $\mathcal{A}$ the class of absolutely F/U-pure right modules.

Theorem 2.4. *Let $A \in Mod - R$. Then the following statements are equivalent:*

(i) $A \in \mathcal{A}$;

(ii) A is F/U-pure in $E(A)$;

(iii) If A is a finitely generated right ideal of R and $i : I \to R$ is the inclusion monomorphism, then A is injective with respect to i.

Proof. Let I be a finitely generated right ideal of R and consider the short exact sequence of right R-modules

$$0 \longrightarrow I \xrightarrow{i} R \xrightarrow{p} R/I \longrightarrow 0 \tag{2}$$

where i is the inclusion monomorphism and p the natural epimorphism. Since R is projective, we have $Ext^1_R(R, A) = 0$. Hence the short exact sequence (2) induces the following short exact sequence of abelian groups:

$$Hom_R(R, A) \xrightarrow{Hom_R(i, 1_A)} Hom_R(I, A) \longrightarrow Ext^1_R(R/I, A) \longrightarrow 0 \tag{3}$$

Let $D \in Mod - R$ such that A is a submodule of D and consider the short exact sequence

$$0 \longrightarrow A \xrightarrow{j} E(D) \xrightarrow{q} E(D)/A \longrightarrow 0 \tag{4}$$

where j is the inclusion monomorphism and q the natural epimorphism. By injectivity of $E(D)$, we have $Ext^1_R(R/I, E(D)) = 0$. Hence the short exact sequence (4) induces the following short exact sequence of abelian groups:

$$Hom_R(R/I, E(D)) \xrightarrow{Hom_R(1_{R/I}, q)} Hom_R(R/I, E(D)/A) \longrightarrow$$

$$\longrightarrow Ext^1_R(R/I, A) \longrightarrow 0 \tag{5}$$

$(i) \Longrightarrow (ii)$ This is clear.

$(ii) \Longrightarrow (iii)$ Suppose that A is F/U-pure in $E(A)$ and consider $D = A$ in the short exact sequence (4). By Theorem 2.2, $Hom_R(1_{R/I}, q)$ is surjective. Hence $Ext^1_R(R/I, A) = 0$, because the sequence (5) is exact. By the exactness of the sequence

(3), it follows that $Hom_R(i, 1_A)$ is surjective. Therefore A is injective with respect to i.

$(iii) \implies (i)$ Suppose that A is injective with respect to i. Then $Hom_R(i, 1_A)$ is surjective. Since the short exact sequence (3) is exact, it follows that $Ext_R^1(R/I, A) = 0$. By the exactness of the sequence (5), $Hom_R(1_{R/I}, q)$ is surjective. By Theorem 2.2, A is F/U-pure in $E(D)$. By Theorem 2.1, A is F/U-pure in D. Therefore $A \in \mathcal{A}$. □

Remark. Every injective right R-module is absolutely F/U-pure.

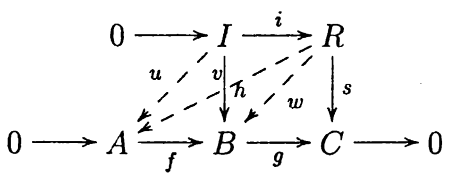
Corollary 2.5. *The class $\mathcal{A}$ is closed under taking direct products and direct summands.*

Lemma 2.6. *The class $\mathcal{A}$ is closed under taking direct sums.*

Proof. Let $(A_j)_{j \in J}$ be a family of absolutely F/U-pure right R-modules and let $A = \oplus_{j \in J} A_j$. Let I be a finitely generated right ideal of R, $i : I \to R$ the inclusion monomorphism and $f : I \to A$ an homomorphism. Since $f(I)$ is finitely generated, there exists a finite subset $K \subseteq J$ such that $f(I) \subseteq \oplus_{k \in K} A_k = B$. By Corollary 2.5, $B \in \mathcal{A}$. Therefore by Theorem 2.4, there exists a homomorphism $g : R \to B$ such that $gi = v$, where $v : I \to B$ is the homomorphism defined by $v(r) = f(r)$ for every $r \in I$. Let $u : B \to A$ be the inclusion monomorphism. Then $ugi = uv = f$. By Theorem 2.4, $A \in \mathcal{A}$. □

Theorem 2.7. *Let (1) be a short exact sequence of right R-modules and let $A, C \in \mathcal{A}$. Then $B \in \mathcal{A}$.*

Proof. Let I be a right ideal of R, $i : I \to R$ the inclusion monomorphism and $h : I \to B$ a homomorphism. Consider the following diagram of right R-modules with exact rows:



where u, v, w, s are homomorphisms which will be defined. Since $C \in \mathcal{A}$, by Theorem 2.4 there exists a homomorphism $s : R \to C$ such that $si = gh$. By projectivity of R, there exists a homomorphism $w : R \to B$ such that $gw = s$. We have $gwi = si = gh$, hence $g(wi - h) = 0$. Let $r \in I$. Then $g((wi - h)(r)) = 0$, therefore $(wi - h)(r) \in Ker\, g = Im\, f$. Since f is a monomorphism, there exists a unique element $a \in A$ such that $(wi - h)(r) = f(a)$. Hence we can define a homomorphism $u : I \to A$ by $u(r) = a$. We have also $h(r) = (wi)(r) - f(a)$. Since $A \in \mathcal{A}$, there exists a homomorphism $v : R \to A$ such that $vi = u$. Then

$$((w - fv)i)(r) = (wi)(r) - (fu)(r) = (wi)(r) - f(a) = h(r) \,.$$

Hence there exists the homomorphism $w - fv : R \to B$ such that $(w - fv)i = h$. By Theorem 2.4, $B \in \mathcal{A}$. □

3. Absolutely F/U-pure modules over particular rings

In this section we shall consider absolutely F/U-pure R-modules over right noetherian rings and regular(von Neumann) rings.

Theorem 3.1. *The following statements are equivalent:*

(i) R is right noetherian;

(ii) If $A \in \mathcal{A}$, then A is injective.

Proof. $(i) \Longrightarrow (ii)$ Suppose that R is noetherian. Let $A \in \mathcal{A}$, let I be a right ideal of R and let $i : I \to R$ be the inclusion monomorphism. Since R is noetherian, I is finitely generated. By Theorem 2.4, A is injective with respect to i.Therefore by Baer's criterion, A is injective.

$(ii) \Longrightarrow (i)$ Suppose that every absolutely F/U-pure right R-module is injective. Let $(A_j)_{j \in J}$ be a family of injective right R-modules and let $A = \oplus_{j \in J} A_j$. Then $A_j \in \mathcal{A}$ for every $j \in J$. By Lemma 2.6, $A \in \mathcal{A}$, hence A is injective. Since every direct sum of injective right R-modules is injective, it follows that R is right noetherian [5, Chapter 4, Theorem 4.1]. □

Remark. If R is not right noetherian, there exist absolutely F/U-pure right R-modules which are not injective.

Lemma 3.2. *Let I be a finitely generated right ideal of R. If $I \in \mathcal{A}$, then I is a direct summand of R.*

Proof. Suppose that $I \in \mathcal{A}$ and let $i : I \to R$ be the inclusion monomorphism. By Theorem 2.4, there exists a homomorphism $p : R \to I$ such that $pi = 1_I$ Therefore I is a direct summand of R. □

Theorem 3.3. *The following statements are equivalent:*

(i) $A \in \mathcal{A}$ for every $A \in Mod - R$;

(ii) $I \in \mathcal{A}$ for every finitely generated right ideal I of R;

(iii) R is regular (von Neumann).

Proof. $(i) \Longrightarrow (ii)$ This is clear.

$(ii) \Longrightarrow (iii)$ It follows by Lemma 3.2, because R is regular if and only if every finitely generated right ideal I of R is a direct summand of R [4, Chapter I, Theorem 14.7.8 and Proposition 4.6.1].

$(iii) \Longrightarrow (i)$ Suppose that R is regular. Let $A \in Mod - R$, let I be a finitely generated right ideal of R and let $f : I \to A$ be a homomorphism. Then I is a direct summand of R. Hence there exists a finitely generated right ideal J of R such that $R = I \oplus J$. Then there exist a unique $r \in I$ and a unique $s \in J$ such that $1 = r + s$. Therefore we can define a unique homomorphism $h : R \to A$ such that $h(1) = f(r)$. It follows that $hi = f$. By Theorem 2.4, $A \in \mathcal{A}$. □

Corollary 3.4. *Let R be regular (von Neumann) and let I be a right ideal of R which is not finitely generated. Then $I \in \mathcal{A}$, but I is not injective.*

Example 3.5. Let $\mathbb{Z}$ be the ring of integers and let $\mathcal{P}$ be the set of all primes. Then $R = \prod_{p \in \mathcal{P}} \mathbb{Z}/p\mathbb{Z}$ is a commutative regular (von Neumann) ring and $R = \oplus_{p \in \mathcal{P}} \mathbb{Z}/p\mathbb{Z}$ is an ideal of R. Since I is not finitely generated, it follows that $I \in \mathcal{A}$, but I is not injective.

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