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AN EXTENSION OF THE BANACH FIXED-POINT THEOREM AND SOME APPLICATIONS IN THE THEORY OF DYNAMICAL SYSTEMS

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Abstract. In this paper we present an extension of the Banach Fixed-Point Theorem and we apply this new result to find the attractors of some classes of discrete dynamical processes. By associating a convergent sequence of Iterated Function Systems (IFS) to a dynamical process, we derive some applications in the approximation of (IFS) attractors.

1. Introduction

Let's remember the celebrated Banach Fixed-Point Theorem:

Theorem 1.1. *Each contraction f of a complete metric space (X, d) has an unique fixed-point.*

It is well-known that this fixed-point, ξ, is the limit of the sequence $(x_n)_{n\in\mathbf{N}}$, $x_n = f(x_{n-1})$ with an arbitrary $x_0 \in X$ (Picard's method) .

In Section 2 we propose an extension of this result: the contraction f is replaced by a sequence of contractions, $(f_n)_{n\in\mathbf{N}}$. We analyse three cases:

- the sequence $(f_n)_{n\in\mathbf{N}}$ is convergent.
- all the applications f_n , $n \in \mathbf{N}$ have the same fixed-point.
- the sequence $(f_n)_{n\in\mathbf{N}}$ is k-periodic.

In each case we obtain a similar result to Theorem 1.1. (Theorem 2.1., 2.2. and 2.3.).

Banach's classical theorem has some important applications in the theory of Dynamical Systems, namely in the theory of Iterated Function Systems (IFS). The

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existence of (IFS) attractors and of the Hutchinson measure attached to an (IFS), for example, are consequences of Theorem 1.1. Let's see more details.

If (X, d) is a metric space, one can consider $dist_X : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbf{R}_+$ by

$$dist_X(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y).$$

This application is not quite a metric because $dist_X(A, B) \neq dist_X(B, A)$ for many $A, B \in \mathcal{P}(X)$, but the celebrated Pompeiu-Hausddorff metric can be obtained by

$$h : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbf{R}_+ , \; h(A, B) = \max(dist_X(A, B), dist_X(B, A))$$

It is clear that $h(\{x\}, \{y\}) = d(x, y)$.

(see [HS] for details)

Using this metric, one can see, by Picard's method, that $\lim_{n \to \infty} h(\{x_n\}, \{\xi\}) = 0$ for the recurrent sequence $x_{n+1} = f(x_n)$ with arbitrary $x_0 \in X$.

If we should consider the discrete dynamical system (X, f), the previous relation means that $\{\xi\}$ is the global attractor of the system.

The results presented in the second paragraph of our paper may be applied to the theory of dynamical processes (a kind of dynamical systems' generalization).

One can consider that the pair $\left(X, (f_n)_{n \in \mathbf{N}}\right)$ may be thought of as a discrete dynamical process and the corresponding recurrent equation, $x_n = f_n(x_{n-1})$, is used to define the process attractor (a good survey on this problem is [Vis]). If $f_n = f$, for all $n \in \mathbf{N}$ we obtain the classical case. Using the above mentioned results we obtain some characterisations of the dynamical processes' attractors (Theorem 3.1., 3.2.).

This way, we extend to dynamical processes some well-known results.

One can obtain, as a particular case, some well-known results in (IFS) theory and some important applications in the approximation of an (IFS) attractor.

In order to approximate the attractor of an (IFS) using computer facilities, we associate the sequence of truncated (IFS) to a dynamical process and we prove that the initial (IFS) attractor, which is in fact the attractor of the associated process, is the limit of the truncated attractors (Theorem 3.4), so it may be approximated as deep as we want by choosing an appropriate number of decimals for the truncation operator.

2. Some extensions of Banach's Fixed-Point Theorem

The next results are not generalizations of Banach's Fixed-Point Theorem, because we sometimes use the classical result in the proofs.

For punctually convergent generating sequences we can prove:

Proposition 2.1. *Let (X,d) be a metric space and $(f_n)_{n\in\mathbf{N}}$ a sequence of s-contractions, punctually convergent on X to f. Then f is an s-contraction.*

Proof. Because the inequalities

$$\begin{aligned} d(f(x),f(y)) &\leq d(f(x),f_n(x))+d(f_n(x),f_n(y))+d(f_n(y),f(y))\leq \\ &\leq sd(x,y)+d(f_n(x),f(x))+d(f_n(y),f(y)) \end{aligned}$$

hold for every $n\in\mathbf{N}$ and every $x,y\in X$ it is clear that

$$d(f(x),f(y))\leq \lim_{n\to\infty}[sd(x,y)+d(f_n(x),f(x))+d(f_n(y),f(y))]=sd(x,y).$$

□

Proposition 2.2. *Let (X,d) be a complete metric space, $(f_n)_{n\in\mathbf{N}}$ a sequence of s-contractions punctually convergent on X to f , ξ_n the fixed points of $f_n, n\in\mathbf{N}$, and $\xi\in X$. Then $\xi_n\to\xi$ if and only if $f(\xi)=\xi$.*

Proof. "$\Longleftarrow$" From

$$\begin{aligned} d(\xi_n,\xi) &= d(f_n(\xi_n),f(\xi))\leq d(f_n(\xi_n)f_n(\xi))+d(f_n(\xi),f(\xi))\leq \\ &\leq sd(\xi_n,\xi)+d(f_n(\xi),f(\xi)) \end{aligned}$$

results that

$$d(\xi_n,\xi)\leq\frac{1}{1-s}d(f_n(\xi),f(\xi))\to 0$$

so $d(\xi_n, \xi) \to 0$. It is clear now that $\xi_n \to \xi$.

" $\Longrightarrow$ " $d(f(\xi), \xi) \leq d(f(\xi), f_n(\xi)) + d(f_n(\xi), f_n(\xi_n)) + d(f_n(\xi_n), \xi) \leq$

$\leq d(f(\xi), f_n(\xi)) + (s+1)\, d(\xi_n, \xi) \to 0.$

Hence $f(\xi) = \xi$. □

The next Lemma (a classical result in mathematical analysis) will be used in the proof of Theorem 2.1.

Lemma 2.1. *If $(a_n)_{n\in\mathbf{N}}$ and $(b_n)_{n\in\mathbf{N}}$ are sequences of positive numbers and there is $s \in (0,1)$ such that $a_{n+1} - s a_n \leq b_n$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Theorem 2.1. *Let (X, d) be a metric space, $(f_n)_{n\in\mathbf{N}}$ a sequence of s-contraction of X, punctually convergent on X to f and $\xi \in X$. Let also consider the recurrent sequence $x_n = f_n(x_{n-1})$, $n \in \mathbf{N}^*$ with arbitrary $x_0 \in X$. Then $x_n \to \xi$ if and only if ξ is the fixed point of f.*

Proof. "$\Longrightarrow$" From

$$d(f(\xi), \xi) \leq d(f(\xi), f_n(\xi)) + d(f_n(\xi), \xi) \leq$$

$$\leq d(f(\xi), f_n(\xi)) + d(f_n(\xi), f_n(x_{n-1})) + d(f_n(x_{n-1}), \xi) \leq$$

$$d(f(\xi), f_n(\xi)) + s d(\xi, x_{n-1}) + d(x_{n-1}, \xi)$$

for all $n \in \mathbf{N}$, it results that $d(f(\xi), \xi) \leq \lim_{n\to\infty} d(f(\xi), f_n(\xi)) + s d(\xi, x_{n-1}) + d(x_{n-1}, \xi) = 0$.

So $f(\xi) = \xi$.

"$\Longleftarrow$" Let us notice that

$$d(x_n, \xi) = d(f_n(x_{n-1}), f(\xi)) \leq d(f_n(x_{n-1}), f_n(\xi)) + d(f_n(\xi), f(\xi)) \leq$$

$$\leq s \cdot d(x_{n-1}, \xi) + d(f_n(\xi), f(\xi)),$$

so $d(x_n, \xi) - s \cdot d(x_{n-1}, \xi) \leq d(f_n(\xi), f(\xi))$. One can now apply Lemma 2.1. for $a_n = d(x_n, \xi)$ and $b_n = d(f_n(\xi), f(\xi))$. It results that $\lim_{n\to\infty} d(x_n, \xi) = 0$, so $\lim_{n\to\infty} x_n = \xi$. □

The previous result etablishes that $\lim_{n\to\infty} h(x_n, \xi) = 0$. It can be formulated in terms of dynamical systems theory:

Corollary 2.1. *Let (X, d) be a complete metric space, $(f_n)_{n\in\mathbf{N}}$ a sequence of s-contraction of X, punctually convergent on X to f, and let be $\xi \in X$ the unique fixed-point of f. Then $\{\xi\}$ is the global attractor of the dynamical process $\mathcal{P} = (X, (f_n)_{n\in\mathbf{N}})$.*

It is a natural result and it has some interesting applications.

For periodic generating sequences we can prove:

Proposition 2.3. *If (X, d) is a complete metric space, $(f_n)_{n\in\mathbf{N}}$ is a k-periodic sequence ($f_{n+k}, = f_n$ for all $n \in \mathbf{N}$) and $\xi_1, \xi_2, ...\xi_k$ are the fixed points of $f_1, f_2, ..., f_k$ then $\{\xi_1, ...\xi_k\}$ is Lyapunov stable.*

Proof. We must find $U \in \mathcal{V}(\{\xi_1, ...\xi_k\})$ such that, for every $x \in X$ there is $n_x \in \mathbf{N}$ with the property $\{f_n(x), n \geq n_x\} \subset U$.

Let's consider $a = \max\{d(\{\xi_1, \xi_2), d(\xi_2, \xi_3) ...d(\xi_{k-1}, \xi_k), d(\xi_k, \xi_1)\}$.

It is quite simple to see that

$$d(x_j, \xi_j) \leq s^j d(x_1, \xi_1) + \frac{1}{1-s}a$$

But $\xi_{k+j} = \xi_j$ for all $j \in \mathbf{N}$, so

$$d(x_{nk+j}, \xi_j) \leq s^{nk+j} d(x_1, \xi_1) + \frac{1}{1-s}a \text{ for all } n \in \mathbf{N} \text{ and } j \in \mathbf{N}$$

We now choose $U = \bigcup_{j=1}^{k} B\left(\xi_j, \frac{2}{1-s}a\right)$,

Because $\lim_{n\to\infty} s^{nk+j} = 0$ there is $n_0 \in \mathbf{N}$ such that, for all $n \geq n_0$ the inequality $s^{nk} d(x_1, \xi_1) \leq \frac{a}{1-s}$ should hold. Then

$$d(x_{nk+j}, \xi_j) \leq \frac{2}{1-s}a \text{ for all } n \geq n_0 \text{ and } j \in \{1, 2, ..., k\}$$

If $n_x \overset{not}{=} n_0 \cdot k$, then $x_n \in U$ for all $n \geq n_x$.

If $(f_n)_{n\in\mathbf{N}}$is a k-periodic sequence of s-contraction on X then the application $f_k \circ f_{k-1} \circ ... \circ f_1$ is a s^k- contraction on X and has an unique fixed point, namely ξ.

The sequence $(x_n)_{n\in\mathbf{N}}$, $x_n(\xi) = f_n(x_{n-1})$, with $x_0(\xi) = \xi$ is also k-periodic, so $x_n = x_{n \mod k}$ for all $n \in \mathbf{N}$. □

Theorem 2.2. *Let (X,d) be a complete metric space, $(f_n)_{n\in\mathbf{N}}$ a k-periodic sequence of s-contractions, ξ the fixed point of $f_k \circ ... \circ f_1$ and $x_n = f_n(x_{n-1})$ with arbitrary $x_0 \in X$. Then*

$$\lim_{n\to\infty} h\left(\{x_n\}, \{f_1(\xi), (f_2 \circ f_1)(\xi), ..., (f_k \circ f_{k-1} \circ ... \circ f_1)(\xi)\}\right) = 0.$$

Proof. Because $f_1, f_2, ..., f_k$ are s-contractions it is clear that

$$\lim_{n\to\infty} d\left(x_n, (f_n \circ ... \circ f_1)(\xi)\right) = 0,$$

so

$$\lim_{n\to\infty} h\left(\{x_n\}, \{(f_n \circ f_{n-1} \circ ..., f_1)(\xi)\ ,\ n \in \mathbf{N}^*\}\right) = 0.$$

We may use now the periodicity of $(f_n)_{n\in\mathbf{N}}$ to obtain that

$$\lim_{n\to\infty} h\left(\{x_n\}, \{f_1(\xi), (f_2 \circ f_1)(\xi), ..., (f_k \circ f_{k-1} \circ ... \circ f_1)(\xi)\}\right) = 0$$

□

Corollary 2.2. *Let $S = (\mathbf{N}, X, f)$ be a contractive dynamical system on the complete metric space X, ξ the fixed point of f and $x_n = f(x_{n-1})$ with arbitrary $x_0 \in X$. Then $\lim_{n\to\infty} h(\{x_n\}, \{\xi\}) = 0$.*

Proof. In Theorem 2.2. we choose $k = 1$. □

Now let's see what happens when all $f_n, n \in \mathbf{N}$ have the same fixed-point.

Theorem 2.3. *Let (X,d) be a complete metric space, $(f_n)_{n\in\mathbf{N}}$ sequences of s-contractions of X and $x_n = f_n(x_{n-1})$ with arbitrary $x_0 \in X$. If the applications $f_n, n \in \mathbf{N}$ have the same fixed point ξ, then $\lim_{n\to\infty} h(\{x_n\}, \{\xi\}) = 0$ and each sphere centered in $\{\xi\}$ is Lyapunov stable.*

Proof. Because $d(x_n, \xi) \le s^n d(x_0, \xi)$ and $h(\{x_n\}, \{\xi\}) = d(x_n, \xi)$ we obtain immediately the results of the Theorem.

Let's notice that, even if $f_n, n \in \mathbf{N}$ have the same fixed point, we know nothing about the convergence or periodicity of $(f_n)_{n\in\mathbf{N}}$.

For example, the applications $f_n(x) = \frac{1}{n}x$ have the same fixed-point, 0, for all $n \in \mathbf{N}$ and $f_n \overset{punctually}{\underset{\mathbf{R}}{\rightarrow}} 0$, still $(f_n)_{n\in\mathbf{N}}$ is not periodic.

The applications $g_n(x) = \frac{2+(-1)^n}{4}x$ have also the same fixed-point, 0, for all $n \in \mathbf{N}$, but the sequence $(g_n)_{n\in\mathbf{N}}$ is punctually convergent only on $\{0\}$ and it is periodic ($k = 2$). □

It is clear now that the situations analyzed in the previous theorems are different.

3. Applications to the Theory of Dynamical Systems

We shall apply the previous results to the theory of Iterated Function Systems (IFS), which are classical examples of chaotic dynamical systems (in the sense of the Devaney definition) and whose attractors are fractals (see [Hut]).

An IFS on the complete metric space (X, d) is

$S = (X, w_1, w_2, ..., w_n)$ where $w_1, w_2, ..., w_n : X \to X$ are s-contractions of X.

On the family of compact subsets of X with $\mathcal{H}(X)$, we consider

$h : \mathcal{H}(X) \times \mathcal{H}(X) \to \mathbf{R}_+$, Pompeiu-Hausdorff's metric.

It is well-known that $(\mathcal{H}(X), h)$ is a complete metric space if (X, d) is so.

Using the s-contractions $w_1, w_2, ..., w_n$ one can obtain another s-contraction, namely $\overline{w} : \mathcal{H}(X) \to \mathcal{H}(X)$, $\overline{w}(B) = w_1(B) \cup w_2(B) \cup ... \cup w_n(B)$ for each $B \in \mathcal{H}(X)$ which has (see Banach's Fixed Point Theorem) a single fixed point $A \in \mathcal{H}(X)$, so $A = w_1(A) \cup w_2(A) \cup ... \cup w_n(A)$.

The Iterated Function System S is associated to the contractive dynamical system $\widetilde{S} = (\mathcal{H}(X), \overline{w})$.

The single fixed-point of $\overline{w}$, $A \in \mathcal{H}(X)$, is in fact the global attractor of $\widetilde{S}$ (it is a compact set and $\lim_{n\to\infty} h(\overline{w}^n(x), A) = 0$ for every $x \in X$). It is called the attractor of S and it is interesting to prove that S exhibits chaotic dynamics on A (see [Ba] for details).

We associate now an (IFS) sequence to a discrete dynamical process.

Definition 3.1. Let's consider $k \in \mathbf{N}$ and $(S_n)_{n\in\mathbf{N}}$, $S_n = (X, w_{1,n}, w_{2,n}, ..., w_{k,n})$a sequence of s-IFS . $\overline{w}_n : \mathcal{H}(X) \to \mathcal{H}(X)$ is defined by

$$\overline{w}_n (B) = w_{1,n} (B) \cup w_{2,n} (B) \cup ... \cup w_{k,n} (B)$$

for all $B \in \mathcal{H}(X)$ then $\mathcal{P} = (\mathcal{H}(X), (\overline{w}_n)_{n\in\mathbf{N}})$ is the contractive dynamical process associated to the sequence $(S_n)_{n\in\mathbf{N}}$.

Let's notice that, if $S_n = S$ for all $n \in \mathbf{N}$, then $\mathcal{P} = (\mathcal{H}(X), \overline{w})$ is the contractive dynamical system associated to S.

We shall study the properties of the dynamical process' attractor if $(S_n)_{n\in\mathbf{N}}$ is a convergent or a periodic sequence.

Proposition 3.1. *Let $(w_n)_{n\in\mathbf{N}}$ be a sequence of s-contractions of the compact metric space (X, d). Then $w_n \underset{(X,d)}{\overset{p}{\to}} w$ if and only if $\overline{w}_n \underset{(\mathcal{H}(X),h)}{\overset{p}{\to}} \overline{w}$.*

Proof. "$\Rightarrow$" From the previous definitions, it results that

$$\begin{aligned} d(\bar{w}_n(B), \overline{w}(B)) &= \max_{y\in\bar{w}_n(B)} (\min_{z\in\overline{w}(B)} d(y,z)) = \\ &= \max_{x\in B}(\min_{x'\in B} d(w_n(x), w(x'))) \end{aligned}$$

Suppose that $d(\overline{w}_n(B), \overline{w}(B)) \not\to 0$. Then there is $\epsilon > 0$ and $n_k \to \infty$ so that $d(\overline{w}_{n_k}(B), \overline{w}(B)) > \epsilon$. For this $\epsilon > 0$ and for every $k \in \mathbf{N}$ there is $x_{n_k} \in B$ such that

$$d(w_{n_k}(x_{n_k}), w(x_{n_k})) > \epsilon.$$

But $(x_{n_k})_{k\in\mathbf{N}} \subset B$ and B is a compact set, so it has a convergent subsequence, equally denoted by $(x_{n_k})_{k\in\mathbf{N}}$ for the simplicity of writing.

So there are $\epsilon > 0$, a sequence of natural numbers $(n_k)_{k\in\mathbf{N}}$ tending to ∞ and a sequence $(x_{n_k})_{k\in\mathbf{N}} \subset X$ convergent to $x \in X$ such that

$$d(w_{n_k}(x_{n_k}), w(x_{n_k})) > \epsilon,$$

for all $k \in \mathbf{N}$. Then

$$\begin{aligned} \epsilon &< d\left(w_{n_k}\left(x_{n_k}\right), w\left(x_{n_k}\right)\right) < \\ &< d\left(w_{n_k}\left(x_{n_k}\right), w_{n_k}(x)\right) + d\left(w_{n_k}(x), w(x)\right) + d\left(w(x), w\left(x_{n_k}\right)\right) < \\ &< sd\left(x_{n_k}, x\right) + d\left(w_{n_k}(x), w(x)\right) + sd\left(x_{n_k}, x\right) \end{aligned}$$

This is a contradiction, because

$$\lim_{n_k \to \infty} d\left(x_{n_k}, x\right) = 0$$

and

$$\lim_{n_k \to \infty} d\left(w_{n_k}(x), w(x)\right) = 0.$$

It results that $d\left(\overline{w}_n(B), \overline{w}(B)\right) \to 0$. In the same way we can prove that

$$d\left(\overline{w}(B), \overline{w}_n(B)\right) \to 0.$$

We may now see that

$$\lim_{n \to \infty} h\left(\overline{w}_n(B), \overline{w}(B)\right) = 0$$

for every $B \in \mathcal{H}(X)$. It means that $\overline{w}_n \underset{(\mathcal{H}(X),h)}{\overset{p}{\to}} \overline{w}$.

"$\Leftarrow$" Because $\{x\} \in \mathcal{H}(X)$ for every $x \in X$ and $\overline{w}_n(\{x\}) \to w(\{x\})$ it results that

$$\lim_{n \to \infty} d\left(w_n(x), w(x)\right) = \lim_{n \to \infty} h\left(\overline{w}_n(x), \overline{w}(x)\right) = 0$$

so $w_n \underset{(X,d)}{\overset{p}{\to}} w$. □

Corollary 3.1. *Let (X, d) be a compact metric space and*

$$\left(S_n\right)_{n \in \mathbf{N}} = \left(\left(X, w_{i,n}, w_{2,n}, \ldots, w_{k,n}\right)\right)_{n \in \mathbf{N}}$$

a sequence of s-(I.F.S.) such that $w_{i,n} \underset{X}{\overset{p}{\to}} u_i$ for every $i \in \{1, 2, \ldots k\}$ and let's denote $S = (X, u_1, u_2, \ldots, u_k)$. Then the sequence of the associated contractive dynamical systems $\widetilde{S}_n = \left(\mathcal{H}(X), \overline{w}_n\right)$, $n \in \mathbf{N}$ is convergent to $S = \left(\mathcal{H}(X), \overline{u}\right)$ in the Pompeiu-Hausdorff metric.

Using this result and a previous theorem we can prove

Theorem 3.1. *Let (X,d) be a compact metric space and*

$$(S_n)_{n\in\mathbf{N}} = ((X, w_{i,n}, w_{2,n}, ..., w_{k,n}))_{n\in\mathbf{N}}$$

a sequence of s-(I.F.S.) such that $w_{i,n} \underset{X}{\overset{p}{\rightarrow}} u_i$ *for every* $i \in \{1,2,...k\}$ *and let's note* $S = (X, u_1, u_2, ..., u_k)$. *Then the attractor of the contractive dynamical process associated to* $(S_n)_{n\in\mathbf{N}}$ *is the very attractor of* S.

Proof. Let $\widetilde{S}_n = (\mathcal{H}(X), \overline{w}_n)$ be the contractive dynamical system associated to S_n.

In the theorem's hypothesis it is clear that A_n, the attractor of $\widetilde{S}_n$, is the fixed point of the s-contraction $\overline{w}_n : \mathcal{H}(X) \to \mathcal{H}(X)$.

From Proposition 3.1. it results that $\overline{w}_n \underset{(\mathcal{H}(X),h)}{\overset{p}{\rightarrow}} \overline{u}$ (here $\widetilde{S} = (\mathcal{H}(X), \overline{u})$ is the contractive dynamical system associated to S). Let's notice that A is the fixed point of $\overline{u}$ and Corollary 2.1. shows that A is the attractor of $\mathcal{P} = (\mathcal{H}(X), (\overline{w}_n)_{n\in\mathbf{N}})$, the contractive dynamical process associated to $(S_n)_{n\in\mathbf{N}}$. □

A direct method to obtain the attractor A is the following:

- we choose $A_0 \in \mathcal{H}(X)$ (usually with a single element).
- we construct the sequence $A_n = \overline{w}_n(A_{n-1})$ and we see that $\lim_{n\to\infty} A_n = A$, so A may be approximated by A_n for $n \in \mathbf{N}$ large enough.

One may say that the attractor of the approximating system is the approximation of the attractor. The random procedure presented in [Ba] can be easily adapted to this situation.

If $(\overline{w}_n)_{n\in\mathbf{N}}$ is a periodic sequence we may apply Theorem 2.3. in order to prove:

Theorem 3.2. *If* $(S_n)_{n\in\mathbf{N}} = ((X, w_{1,n}, w_{2,n}, ..., w_{k,n}))_{n\in\mathbf{N}}$ *is a k-periodic sequence of s-iterated function systems (so* $w_{i,n} = w_{i,n+k}$ *for all* $i \in \{1,2,...,k\}$ *and all* $n \in \mathbf{N}$*) and* $\overline{w}_n : \mathcal{H}(X) \to \mathcal{H}(X)$ *is* $\overline{w}_n = \overline{w}_{1,n} \cup \overline{w}_{2,n} \cup ... \cup \overline{w}_{k,n}$ *then the contractive dynamical process associated to* $(S_n)_{n\in\mathbf{N}}$ *, namely* $\mathcal{P} = \left(\mathcal{H}(X), (\overline{w}_n)_{n\in\mathbf{N}}\right)$ *is k-periodic and its attractor is the orbit of the unique fixed point of the application* $\overline{w}_k \circ ... \circ \overline{w}_1$.

More precisely there is an unique set $A \in \mathcal{H}(X)$ *such that* $\{\overline{w}_1(A), (\overline{w}_2 \circ \overline{w}_1)(A), ..., (\overline{w}_k \circ ... \circ \overline{w}_1)(A)\}$ *is the attractor of* $\mathcal{P}$.

For $k = 1$ this is a well known result in the theory of iterated function systems.

Theorem 3.1. also contains the basic ideas of the approximation of an IFS attractor using computer facilities. In this case, the repeated truncations can dramatically modify the attractor's properties.

If T_k is a 10^{-k}-truncation operator on the metric space (X, d) then $d(T_k(x), T_k(y)) \leq 2 \cdot 10^{-k}$ if $d(x, y) \leq 10^{-k}$ and $\lim\limits_{k \longrightarrow \infty} T_k(x) = x$ for all x, y in X.

Let's consider $S = (X, w_1, w_2, ... w_n)$ an IFS and T_k a 10^{-k}-truncation operator on X.

Let's denote $S_k = (\mathbf{N}, X, T_k \circ w_1, ..., T_k \circ w_n)$.

Simple computations show that $T_k \circ w_i \underset{X}{\overset{p}{\rightarrow}} w_i$. Unfortunately, the previous result may not be applied, because $T_k \circ w_i$ is not a contraction but, using the mentioned properties of T_k, we can easily obtain a result similar to Theorem 3.1.

Theorem 3.3. *Let w be an s-contraction in the compact metric space (X, d) and $(T_k)_{k \in \mathbf{N}}$ a sequence of $10^{-k}, k \in \mathbf{N}$ truncation operators on X.*

Then $T_k \circ w \underset{(X,d)}{\overset{p}{\rightarrow}} w$ if and only if $\overline{T_k \circ w} \underset{(\mathcal{H}(X),h)}{\overset{p}{\rightarrow}} \overline{w}$.

Using this result we can prove

Theorem 3.4. *Let's consider $S = (X, w_1, w_2, ... w_n)$ an IFS on the complete metric space (X, d) and $(T_k)_{k \in \mathbf{N}}$ a sequence of (10^{-k})-truncation operators on X. If A_k, A are the attractors of $S_k = (\mathbf{N}, X, T_k \circ w_1, ..., T_k \circ w_n)$ and S, respectively, then, in $(\mathcal{H}(X), h)$, we have*

$$\lim_{k \to \infty} A_k = A$$

From Theorem 3.4. it results that A can be well approximated, by choosing an appropriate number of decimals for the truncation operator.

It is very important, because there are no general relations between the real attractor and the truncated one.

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