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FIBER φ-CONTRACTIONS

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Abstract. This paper containes some conditions for proving that if an operator is fiber Picard operator then this operator is Picard operator. The result obtained is used for proving the differentiability with respect some parameters.

1. Introduction

Let X be a nonempty set and $A : X \to X$ an operator. We note by:

$$P(X) := \{Y \subset X \,|Y \neq \emptyset\}$$

$$F_A := \{x \in X \,|A(x) = x\} \quad - \quad \text{the fixed point set of A.}$$

Definition 1.1. (I.A. Rus [6]). Let (X,d) be a metric space. An operator $A : X \to X$ is (uniformly) Picard operator if there exists $x^* \in X$ such that:

(a) $F_A = \{x^*\}$,

(b) the sequence $(A^n(x))_{n \in N}$ converges (uniformly) to x^*, for all $x \in X$.

Definition 1.2. (I.A. Rus [6]). Let (X,d) be a metric space. An operator $A : X \to X$ is (uniformly) weakly Picard operator if:

(a) the sequence $(A^n(x))_{n \in N}$ converges (uniformly), for all $x \in X$,

(b) the limit (which may depend on x) is a fixed point of A.

If A is weakly Picard operator then we consider the following operator:

$$A^\infty : X \to X,$$

$$A^\infty(x) = \lim_{n \to \infty} A^n(x).$$

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In this paper we consider the following class of operators:

$$A : X_1 \times \ldots \times X_p \to X_1 \times \ldots \times X_p$$

$$(x_1, \ldots, x_p) \mapsto (A_1(x_1), A_2(x_1, x_2), \ldots, A_p(x_1, \ldots, x_p)),$$

where (X_1, d_i), $i = \overline{1,p}$, are metrical spaces and $A_k : X_1 \times \ldots \times X_k \to X_k$, $k = \overline{1,p}$, are such that the operators

$$A_k(x_1, \ldots, x_{k-1}, \cdot) : X_k \to X_k$$

are weakly Picard operators, for all $x_i \in X_i$, $i = \overline{1,k}$, $k = \overline{1,p}$.

The aim of this paper is to give an answer of Problem 4.2 from I. A. Rus [5]. We replace the condition that $A_k(x_1, \ldots, x_{k-1}, \cdot)$ is $\alpha-$contraction with $A_k(x_1, \ldots, x_{k-1}, \cdot)$ is φ_k-contraction and we give the conditions for φ_k to obtain that operator Λ is a Picard operator.

2. Comparison functions and (c)-comparison function

Definition 2.1.(I.A. Rus [5]). A function $\varphi : \Re_+ \to \Re_+$ is called comparison function if:

(a) φ is monotone increasing: $t_1 \leq t_1 \Longrightarrow \varphi(t_1) \leq \varphi(t_2)$, $\qquad t_1, t_1 \in \Re_+$.

(b) $(\varphi^n(t))_{n \in N}$ converges to 0, as $n \to \infty$, for each t.

We are interested in finding that comparison functions which satisfies the condition:

$$\sum_{k=0}^{\infty} \varphi^k(t) < \infty. \tag{1}$$

V. Berinde in [2] gave a necessary and sufficient result for the convergence of the series of decreasing positive terms.

Theorem 2.1.(V. Berinde [2]). *A series* $\sum\limits_{k=0}^{\infty} u_k$ *of decreasing positive terms converges if and only if there exists a convergent series of nonnegative terms* $\sum\limits_{k=0}^{\infty} v_k$ *such that:*

$$\frac{u_{n+1}}{u_n + v_n} \leq \alpha < 1, for \quad n \geq n_0, \tag{2}$$

is satisfied.

Using this result we obtain which comparison functions satisfy the condition (1).

Corollary 2.1.(V. Berinde [3]) *Let $\varphi : \Re_+ \to \Re_+$ be a comparison function. The series $\sum_{k=0}^{\infty} \varphi^k(t)$, $t \in \Re_+$, is convergent if and only if there exists a number α, $0 < \alpha < 1$, and there exists a convergent series of nonnegative terms $\sum_{i=0}^{\infty} v_k$ such that:*

$$\frac{\varphi^{k+1}(t)}{\varphi^k(t) + v_k} \leq \alpha \leq 1, \quad for \quad k \geq k_0. \tag{3}$$

Remark 2.1. If $\sum_{i=0}^{\infty} v_i$ is a convergent series of nonnegative terms and $0 < \alpha$ then also $\sum_{i=0}^{\infty} \alpha v_i$ is a convergent series of nonnegative terms, so we can write the condition (3) in equivalent form:

$$\varphi^{k+1}(t) \leq \alpha\varphi^k(t) + v'_k, \tag{4}$$

where $0 < \alpha < 1$, $\sum_{i=0}^{\infty} v'_k$ is a convergent series of nonnegative terms.

By Corollary 2.1 and Remark 2.1 we obtain a new class of comparison functions.

Definition 2.2.(V. Berinde [1], [2]) A function $\varphi : \Re_+ \to \Re_+$ is called (c)-comparison function if the following condition hold:

(a) φ is monotone increasing: $t_1 \leq t_1 \Longrightarrow \varphi(t_1) \leq \varphi(t_2)$, $\quad t_1, t_1 \in \Re_+$.

(b) there exist two numbers k_0, α, $0 < \alpha < 1$, and a convergent series of nonnegative terms $\sum_{i=0}^{\infty} v_k$ such that:

$$\varphi^{k+1}(t) \leq \alpha\varphi^k(t) + v_k,$$

for each t and $k \geq k_0$.

Theorem 2.2. (V. Berinde [1], [3]) *If $\varphi : \Re_+ \to \Re_+$ is a* (c)-comparison function then:

(i) $\varphi(t) < t$, for each $t > 0$;

(ii) φ is continuous in 0;

(iii) the series $\sum_{k=0}^{\infty} \varphi^k(t)$ converges for each $t \in \Re_+$;

(iv) the sum of the series (1), s(t), is monotone increasing and continuous in 0;

(v) $(\varphi^n(t))_{n\in N}$ converges to 0, as $n \to \infty$, for each t.

Example 2.1. The function $\varphi : \Re_+ \to \Re_+$, $\varphi(t) = \frac{t}{t+1}$ is a comparison function, but is not a (c)-comparison function.

Example 2.2. The function $\varphi : \Re_+ \to \Re_+$, $\varphi(t) = \alpha t$, $0 < \alpha < 1$, is a (c)-comparison function.

Example 2.3. The function $\varphi : \Re_+ \to \Re_+$,

$\varphi(t) = \begin{cases} at & t \in [0; 2a] \\ bt + c & t > 2a \end{cases}$, where $0 \leq a < 1$, $a - \frac{c}{2a} \leq b \leq 1$ and $c < 0$ is a comparison function.

Example 2.4. For the function from Example 2.3, if $a = \frac{1}{2}$, $b = 1$, $c = -\frac{1}{3}$, we obtain a (c)-comparison function.

Definition 2.3. (I.A. Rus [8]) Let (X,d) be a metric space and $\varphi : \Re_+ \to \Re_+$ is a comparison function. A mapping $f : X \to X$ is a $\varphi-$contraction if:

$$d(f(x), f(y)) \leq \varphi(d(x,y)),$$

for every $x, y \in X$.

3. Fiber Picard operators problem

We'll start with a result which generalize Lemma 3.2 from I.A. Rus [5].

Lemma 3.1. *Let $\alpha_n \in \Re_+$, $n \in N$, and $\varphi : \Re_+ \to \Re_+$ such that:*

(i) $\alpha_n \to 0$ as $n \to \infty$;

(ii) φ is a (c)-comparison function.

Then $\sum\limits_{k=0}^{\infty} \varphi^{n-k}(\alpha_k) \to 0$ as $n \to \infty$.

Proof. We split the partial sum of the series in two parts:

$$s_n = \sum_{k=0}^{n} \varphi^{n-k}(\alpha_k) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \varphi^{n-k}(\alpha_k) + \sum_{k=\left[\frac{n}{2}\right]+1}^{n} \varphi^{n-k}(\alpha_k).$$

For the first part of partial sum we have:

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} \varphi^{n-k}(\alpha_k) \leq \sum_{k=0}^{\left[\frac{n}{2}\right]} \varphi^{n-k}(\max_{n\in N} \alpha_k) \to 0$$

as $n \to \infty$, because of the fact that φ is a (c)-comparison function and the point (iii) from Theorem 2.2.

For the second part of the partial sum we have:

$$\sum_{k=\left[\frac{n}{2}\right]+1}^{n} \varphi^{n-k}(\alpha_k) \le \sum_{k=\left[\frac{n}{2}\right]+1}^{n} \varphi^{n-k}(\max_{j\le n} \alpha_j) \le s(\max_{j\le n} \alpha_j).$$

Using the continuity of s in 0, (Theorem 2.2, (iv)), and the fact that $\max\limits_{j\le n} \alpha_j \to 0$ as $n \to \infty$ we deduce that the second part also tends to 0 as $n \to \infty$. □

Considering the open problem 3.1 from I.A. Rus [5], we'll give the following result: **Lemma 3.2.** *Let (X,d) be a complete metric space, $\varphi : \Re_+ \to \Re_+$ a (c)-comparison function and A_n, $A : X \to X$, $n \in N$, operators such that:*

(i) φ is subadditive: $\varphi(t_1 + t_2) \le \varphi(t_1) + \varphi(t_2)$, $\quad \forall t_1, t_2 \in \Re_+$;

(ii) the sequence $(A_n)_{n\in N}$ pointwise converges to A;

(iii) A_n and A, $n \in N$, are φ-contractions.

Then the sequence $(A_n \circ A_{n-1} \circ ... \circ A_0)_{n\in N}$ pointwise converges to A^∞.

Proof. From (iii) we deduce that there exists a unique $x^* \in F_A$, so $A^\infty(x) = x^*$, for all $x \in X$. Let $x \in X$. We have:

$$d((A_n \circ A_{n-1} \circ \cdots \circ A_0)(x), x^*) \le$$

$$\le d((A_n \circ A_{n-1} \circ \cdots \circ A_0)(x), (A_n \circ A_{n-1} \circ \cdots \circ A_0)(x^*)) +$$

$$+ d((A_n \circ A_{n-1} \circ \cdots \circ A_0)(x^*), A_n(x^*)) + d(A_n(x^*), x^*) \le \cdots \le$$

$$\le \varphi^{n+1}(d(x, x^*)) + \varphi^n(d(A_0(x^*), x^*)) + \varphi^{n-1}(d(A_1(x^*), x^*)) + \cdots + d(A_n(x^*), x^*).$$

Let $\alpha_k := d(A_k(x^*), x^*)$. It is obvious that $\alpha_k \to 0$ as $k \to \infty$ and the proof of the theorem follows from Lemma 3.1. □

Lemma 3.3. *Let (X,d) and (Y,ρ) be two metric spaces, $x_n, x^* \in X$, $\varphi : \Re_+ \to \Re_+$ a (c)-comparison function and $f : X \times Y \to Y$ an operator such that:*

(i) $x_n \to x^*$ as $n \to \infty$;

(ii) φ is subadditive;

(iii) the operator $f(\cdot, y) : X \to X$ is continuous for all $y \in Y$;

(iv) $f(x,\cdot): Y \to Y$ *is* $\varphi-$*contraction for all* $x \in X$;

(v) (Y,ρ) *is a complete metric space.*

Then the sequence defined by: $y_{n+1} = f(x_n, y_n)$, $y_1 = y$, $n \in N$ *converges to* y^*, *the unique fixed point of* $f(x^*,\cdot)$, *for all* $y \in Y$.

Proof. The proof is a simple application of Lemma 3.3 with $A_n : Y \to Y$, $A_n(y) = f(x_n, y)$, $A : Y \to Y$, $A(y) = f(x^*, y)$. □

The main result of this paper is related to the open problem 4.1, (I.A. Rus [5]). This result is an answer of open problem 4.2, (I.A. Rus [5]), which generalize the Theorem 4.1.

Theorem 3.1. *Let* (X_k, d_k), $k = \overline{0,p}$, $p \geq 1$, *be some metric spaces. Let*

$$A_k : X_0 \times ... \times X_k \to X_k, \qquad k = \overline{0,p},$$

be some operators such that:

(i) *the spaces* (X_k, d_k), $k = \overline{1,p}$, *are complete metric spaces;*

(ii) *the operator* A_0 *is (weakly) Picard operator;*

(iii) *there exist* $\varphi_k : \Re_+ \to \Re_+$ *subadditive (c)-comparison functions such that the operators* $A_k(x_0, ..., x_{k-1}, \cdot)$ *are* φ_k-*contractions,* $k = \overline{1,p}$;

(iv) *the operators* A_k *are continuous with respect to* $(x_0, ..., x_{k-1})$ *for all* $x_k \in X_k$, $k = \overline{1,p}$.

Then the operator $B_p = (A_0, ..., A_p)$ *is (weakly) Picard operator. Moreover if* A_0 *is a Picard operator and*

$$F_{A_0} = \{x_0^*\}, \qquad F_{A_1(x_0^*,\cdot)} = \{x_1^*\}, ..., F_{A_p(x_0^*,...,x_{p-1}^*,\cdot)} = \{x_p^*\}$$

then

$$F_{B_p} = \{(x_1^*, x_2^*, ..., x_p^*).$$

Proof. We prove this theorem by induction respect to $p \in N^*$. First we consider the case of $p = 1$.

Let $x_0 \in X_0$ and $x_1 \in X_1$. We show that

$$B_1^n(x_0, x_1) \to (A_0^\infty(x_0), x_1^*(x_0))$$

as $n \to \infty$, where $x_1^*(x_0)$ is a unique fixed point of $A_1(A_0^\infty(x_0), \cdot)$ It si easy to check that

$$B_1^n(x_0, x_1) = (A_0^n(x_0), y_n),$$

where $y_0 = x_1$, $y_1 = A_1(x_0, y_0)$, ..., $y_{n+1} = A_1(A_0^n(x_0), y_n)$, ...

Using again Lemma 3.3 we obtain the proof in the case $p = 1$.

Now we suppose that the statemant of the theorem is true for the $p \le k$ and we prove the theorem for the $p = k + 1$. We remark that $B_{k+1} = (B_k, A_{k+1})$, where B_k is (weakly) Picard operator, so we are in the case $p = 1$ and thus the proof is complete. □

*Remark 3.1.*The Lemma 3.2, Lemma 3.3, Theorem 4.1 from I.A. Rus [5] can be obtained using φ as in Example 2.2.

4. **Application**

We consider the folowing integral equation:

$$x(t) = g(t) + \lambda \cdot \int_a^b K(t, s, x(s))ds, \qquad t \in [a; b]. \tag{5}$$

*Theorem 4.1. S*uppose that the following conditions hold:

(i) $g \in C[a; b]$, $K \in C([a; b] \times [a; b] \times \Re)$;

(ii) there exists $L_K > 0$ such that: $|K(t, s, u) - K(t, s, v)| \le L_k |u - v|$, for all $t, s \in [a; b]$, $u, v \in \Re$;

(iii) $\lambda_0 L_K(b - a) < 1$, where $\lambda_0 \in \Re_+^*$.

Then

(a) the equation (5) has a unique solution $x^*(\cdot, \lambda)$ in $C([a; b])$, for all $\lambda \in [-\lambda_0; \lambda_0]$;

(b) for all $x_0 \in C([a;b])$ the sequence $(x_n)_{n\in N}$ defined by

$$x_{n+1}(t;\lambda) = g(t) + \lambda \cdot \int_a^b K(t,s,x_n(s))ds,$$

converges uniformly to x^*, for all $t,s \in [a;b]$, $\lambda \in [-\lambda_0;\lambda_0]$;

(c) we have the estimation:

$$\|x_n - x^*\|_C \leq \frac{\alpha^n}{1-\alpha} \cdot \|x_1 - x_0\|_C \,,$$

where $\alpha = \lambda_0 L_K(b-a)$;

(d) the function $x^* : [a;b] \times [-\lambda_0;\lambda_0] \to \Re\ (t,\lambda) \longmapsto x^*(t;\lambda)$ is continuous;

(e) if $K(t,s,\cdot) \in C^1(\Re)$, for all $t,s \in [a;b]$, then

$$x^*(t;\cdot) \in C^1([-\lambda_0;\lambda_0])$$

for all $t \in [a;b]$.

Proof. We consider the Banach space $X := (C([a;b] \times [-\lambda_0;\lambda_0]), \|\cdot\|_C)$, where $\|\cdot\|_C$ is Chebyshev norm, and the operator defined by

$$A_0 : X \to X,$$

$$A_0(x)(t;\lambda) = g(t) + \lambda \cdot \int_a^b K(t,s,x(s;\lambda))ds,$$

for all $t,s \in [a;b]$, $\lambda \in [-\lambda_0;\lambda_0]$.

Using (ii) we obtain:

$$\|A_0(x) - A_0(y)\|_C \leq \lambda_0 L_K\,(b-a) \cdot \|x-y\|_C \tag{6}$$

for all $x,y \in X$, so A_0 is a φ—contraction, where $\varphi(t) = \alpha t$ is a (c)-comparison function because of (iii). From Theorem 3, (V. Berinde, [2]) we conclude (a), (b), (c), (d).

We'll prove that there exists $\frac{\partial x^*}{\partial \lambda}$. If we formally derivate the relation (5) respect to λ we obtain:

$$\frac{\partial x(t;\lambda)}{\partial \lambda} = \int_a^b K(t,s,x(s;\lambda))ds + \lambda \cdot \int_a^b \frac{\partial K(t,s,x(s;\lambda))}{\partial \lambda} \cdot \frac{x(s;\lambda)}{\partial \lambda} ds.$$

This relation sugest to consider the following operator:

$$A_1 : X \times X \to X,$$

$$A_1(x,y)(t;\lambda) = \int_a^b K(t,s,x(s;\lambda))ds + \lambda \cdot \int_a^b \frac{\partial K(t,s,x(s;\lambda))}{\partial \lambda} \cdot y(s;\lambda)ds.$$

We estimate that:

$$\|A_1(x,y_1) - A_1(x,y_2)\|_C \leq \lambda_0 L_K (b-a) \cdot \|y_1 - y_2\|_C,$$

for all $x \in X$. If we take the operator

$$B : X \times X \to X \times X, \qquad B = (A_0, A_1)$$

then we are in the conditions of Theorem 3.1, thus B is a Picard operator and the sequences

$$x_{n+1}(t;\lambda) := g(t) + \lambda \cdot \int_a^b K(t,s,x_n(s))ds$$

$$y_{n+1}(t;\lambda) := \int_a^b K(t,s,x_n(s;\lambda))ds + \lambda \cdot \int_a^b \frac{\partial K(t,s,x_n(s;\lambda))}{\partial \lambda} \cdot y_n(s;\lambda)ds$$

converges uniformly (with respect to $t \in [a;b]$, $\lambda \in [-\lambda_0;\lambda_0]$) to $(x^*,y^*) \in F_B$, for all $x_0, y_0 \in X$. But for fixed $x_0, y_0 \in X$ we have that $y_1 = \frac{\partial x_1}{\partial \lambda}$ and by induction we prove that $y_n = \frac{\partial x_n}{\partial \lambda}$, so we have:

$$x_n \overset{unif.}{\to} x^* \quad \text{as} \quad n \to \infty,$$

$$\frac{\partial x_n}{\partial \lambda} \overset{unif.}{\to} y^*$$

as $n \to \infty$.

These imply that there exists $\frac{\partial x^*}{\partial \lambda}$ and $\frac{\partial x^*}{\partial \lambda} = y^*$. □

*Remark 4.1.*If $K(t,s,\cdot) \in C^m(\Re)$ then $x^*(t;\cdot) \in C^m([-\lambda_0;\lambda_0])$.

Remark 4.2. For other examples of integral equations where Theorem 3.1 is used see I. A. Rus [4], [5], M. A. Şerban [9].

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