SOME APPROXIMATION IDEALS

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Abstract. We consider some approximation ideals of operators on operator spaces. The method used is similar to that from [8], [10], [11], in the case of the classical Banach spaces, or [2], [7] for the case of Hilbert spaces.

1. Introduction

The theory of the approximation ideals is well known for the case of linear and bounded operators on Hilbert, or Banach, spaces [2], [6], [7], [8], [11].

Here we consider the special case of the completely bounded operators on operator spaces. For these notions it can be seen [1], [3], [5].

We begin by recalling some definitions.

An operator space E , in short O.S, is a Banach space, or a normed space before completion, given with an isometric embedding $J : E \to L(H)$, where $L(H)$ is the space of all linear and bounded operators $T : H \to H$, H being a Hilbert space. We shall identify often E with $J(E)$ and so we shall say that an O.S is a (closed) subspace of $L(H)$.

If $E \subset L(H)$ is an operator space then $M_n \otimes E$ can be identified with the space of all $n \times n$ matrices having entries in E, that it will be denoted by $M_n(E)$.

Clearly $M_n(E)$ can be seen as an o.s. embedded in $L(H^n)$, where

$H^n = H \otimes ... \otimes H$ (number of H is n) .

Let us denote by $\|\cdot\|_n$ the norm induced by $L(H^n)$ on $M_n(E)$, in the particular case $n = 1$ we get the norm of E. Taking the natural embedding $M_n(E) \to M_{n+1}(E)$ we can consider $M_n(E)$ included in $M_{n+1}(E)$, and $\|\cdot\|_n$ induced by $\|\cdot\|_{n+1}$.

Thus we may consider $\bigcup\limits_n M_n(E)$ a normed space equipped with it's natural norm $\|\cdot\|_\infty$.

We denote by $K[E]$ the completion of $\bigcup_n M_n(E)$. If we denote by

$K_0 = \bigcup_n M_n$, the case $E = C$, then the completion of K_0 coincides isometrically with the C^*-algebra, $C(l_2)$, of all compact operators on the space l_2.

It is easy to check that $\bigcup_n M_n(E)$ can be identified isometrically with $K_0 \otimes E$.

The basic idea of o.s. is that the norm of the Banach space E is replaced by a sequence of norms $\{\|\cdot\|_n\}_n$ on $\{M_n(E)\}_n$ or by a single norm $\|\cdot\|_\infty$ on the space $K[E]$.

Definition 1. Let $E_1 \subset L(H_1)$ and $E_2 \subset L(H_2)$ be operator spaces,

$u : E_1 \to E_2$ be a linear map and

$u_n : (x_{ij}) \in M_n(E_1) \to (u(x_{ij})) \in M_n(E_2)$. We say that u is **completely bounded, c.b.,** if $\sup_n \|u_n\| < \infty$ and we define $\|u\|_{c.b.} := \sup_n \|u_n\|$.

Definition 2. (equivalent) u is **completely bounded** if the maps u_n can be extended to a single bounded map $u_\infty : K[E_1] \to K[E_2]$ and we have $\|u\|_{c.b.} = \|u_\infty\|$.

Definition 3. $c.b.(E_1, E_2) := \{u : E_1 \to E_2 : \ u \text{ is c.b.}\}$. We shall consider the $c.b.(E_1, E_2)$ equipped with $\|\cdot\|_{c.b.}$.

Remark 1. The similar definition of the uniform norm for the bounded operators can be written as follows:

$$\|u\|_{c.b.} = \sup\{\|u_\infty\| : x \in K[E], \|x\| \leq 1\}.$$

Remark 2. Likewise the case of an isomorphism between two Banach spaces, we say that two o.s. E_1, E_2 are **completely isomorphic, completely isometric,** if there is an c.b. isomorphism $u : E_1 \to E_2$ with c.b. inverse and in addition $\|u\|_{c.b.} = \|u^{-1}|_{u(E_1)}\|_{c.b.} = 1$

Let $E_1 \subset L(H_1)$ and $E_2 \subset L(H_2)$ be operator spaces. There is an embedding $J : E_1 \odot E_2 \to L(H_1 \otimes H_2)$ defined by $J(x_1 \otimes x_2)(h_1 \otimes h_2) = x_1(h_1) \otimes x_2(h_2)$.

We denote by $E_1 \otimes_{\min} E_2$ the completion of $E_1 \odot E_2$ equipped with the norm $x \to \|Jx\|$.

Obviously J can be extend to an isometric embedding. So we can see $E_1 \otimes_{\min} E_2$ as an o.s. embedded into $L(H_1 \otimes_\sigma H_2)$. This space is called the **minimal, spatial, tensor product** of E_1 and E_2. ($H_1 \otimes_\sigma H_2$ is the hilbertian tensor product, [7], [11].)

If $E \subset L(H)$ is an o.s. then $M_n \otimes_{\min} E$ can be identified with the space $M_n(E)$ and $K[E]$ can be identified isometrically with $K \otimes_{\min} E$. Thus, for any linear map $u : E_1 \to E_2$ we have $\|u\|_{c.b.} = \|I \otimes u : K \otimes_{\min} E_1 \to K \otimes_{\min} E_2\| = \|I \otimes u : K \otimes_{\min} E_1 \to K \otimes_{\min} E_2\|_{c.b.}$. More generaly it can be shown that, for any o.s. $F \subset L(\widetilde{H})$, we have $\|I_F \otimes u : F \otimes_{\min} E_1 \to F \otimes_{\min} E_2\| \leq \|u\|_{c.b.}$. Further on, if $v : F_1 \to F_2$ is another c.b. map, we obtain

$\|u \otimes v : F_1 \otimes_{\min} E_1 \to F_2 \otimes_{\min} E_2\|_{c.b.} \leq \|v\|_{c.b.} \cdot \|u\|_{c.b.}$.

This relation will be very useful in the sequel.

For others properties of the minimal tensor product it can be seen the papers [1], [5], etc.

2. Approximation numbers of completely bounded operators

Definition 4. Let $u : E \to F$ be a completely bounded map, $u \in c.b.(E, F)$. The **approximation numbers**, $a_n^{c.b.}(u)$ will be defined as follows

$$a_n^{c.b.}(u) := \inf \{\|u - a\|_{c.b.} : a \in c.b.(E, F), rank(a) < n\}, \; n = 1, 2, \ldots.$$

Remark 3. From this definition it results that $\|u\|_{c.b.} = a_1^{c.b.}(u) \geq a_2^{c.b.}(u) \geq \ldots \geq 0$.

Proposition 1. *The approximation numbers $a_n^{c.b.}(u)$ verify the following inequalities:*

1. $\sum_{n=1}^{k} a_n^{c.b.}(u_1 + u_2) \leq 2 \cdot \sum_{n=1}^{k} \left(a_n^{c.b.}(u_1) + a_n^{c.b.}(u_2)\right)$, for $k = 1, 2, \ldots$
2. $\sum_{n=1}^{k} a_n^{c.b.}(u_1 \circ u_2) \leq 2 \cdot \sum_{n=1}^{k} \left(a_n^{c.b.}(u_1) \cdot a_n^{c.b.}(u_2)\right)$, for $k = 1, 2, \ldots$

Proof. 1) Let $\varepsilon > 0$. There are a_i, $i = 1, 2$, such that $rank\,(a_i) < n$ and

$\|u_i - a_i\|_{c.b.} \le a_n^{c.b.}(u_i) + \frac{\varepsilon}{2}$.

We obtain:

$$a_{2\cdot n-1}^{c.b.}(u_1 + u_2) \le \|(u_1 + u_2) - (a_1 + a_2)\|_{c.b.} \le$$
$$\le \|u_1 - a_1\|_{c.b.} + \|u_2 - a_2\|_{c.b.} \le$$
$$\le a_n^{c.b.}(u_1) + a_n^{c.b.}(u_2) + \varepsilon.$$

Since ε is arbitrary it follows that:

$$a_{2\cdot n-1}^{c.b.}(u_1 + u_2) \le a_n^{c.b.}(u_1) + a_n^{c.b.}(u_2).$$

Further on it results:

$$\sum_{n=1}^{k} a_n^{c.b.}(u_1 + u_2) \le \sum_{n=1}^{k} a_{2\cdot n-1}^{c.b.}(u_1 + u_2) + \sum_{n=1}^{k} a_{2\cdot n}^{c.b.}(u_1 + u_2) \le$$
$$\le 2 \cdot \sum_{n=1}^{k} a_{2\cdot n-1}^{c.b.}(u_1 + u_2) \le 2 \cdot \sum_{n=1}^{k} \left(a_n^{c.b.}(u_1) + a_n^{c.b.}(u_2)\right).$$

2) We consider also a_i, $i = 1, 2$, such that $rank\,(a_i) < n$ and

$\|u_i - a_i\|_{c.b.} \le a_n^{c.b.}(u_i) + \frac{\varepsilon}{2}$.

We obtain:

$$a_n^{c.b.}(u_1 \circ u_2) \le \|(u_1 \circ u_2) - [u_1 \circ a_2 + a_1 \circ (u_2 - a_2)]\|_{c.b.} =$$
$$= \|(u_1 - a_1) \circ (u_2 - a_2)\|_{c.b.} \le \left(a_n^{c.b.}(u_1) + \frac{\varepsilon}{2}\right) \cdot \left(a_n^{c.b.}(u_2) + \frac{\varepsilon}{2}\right).$$

Since ε is arbitrary it follows that:

$$a_{2\cdot n-1}^{c.b.}(u_1 \circ u_2) \le a_n^{c.b.}(u_1) \cdot a_n^{c.b.}(u_2).$$

Likewise the 1) results 2). □

Remark 4. For the case of the linear and bounded operators between Banach spaces the above inequalities are known, [8], [11].

In the sequel we deduce an inequality for the case of the c.b. operator $u_1 \otimes_{\min} u_2$ using a similar method with that from [9], [10], used for the classical case of the bounded operators on Banach spaces.

Proposition 2. *The approximation numbers $a_n^{c.b.}(u_1 \otimes_{\min} u_2)$ verify the inequalities:*

$$\sum_{n=1}^{k} \frac{a_n^{c.b.}(u_1 \otimes_{\min} u_2)}{n} \le 6 \cdot \sum_{n=1}^{k} \frac{a_n^{c.b.}(u_1) \cdot \|u_2\|_{c.b.} + a_n^{c.b.}(u_2) \cdot \|u_1\|_{c.b.}}{n}, \text{ for } k = 1, 2, \dots.$$

Proof. Let $\varepsilon > 0$. There are a_i, $i = 1, 2$, such that $rank\,(a_i) < n$ and

$\|u_i - a_i\|_{c.b.} \leq a_n^{c.b.}(u_i) + \frac{\varepsilon}{2}$.

We obtain:

$$a_{n^2}^{c.b.}(u_1 \otimes_{\min} u_2) \leq \|u_1 \otimes_{\min} u_2 - a_1 \otimes_{\min} a_2\|_{c.b.} =$$
$$= \|(u_1 - a_1) \otimes_{\min} u_2 - a_1 \otimes_{\min} (u_2 - a_2)\|_{c.b.} \leq$$
$$\leq \|u_1 - a_1\|_{c.b.} \cdot \|u_2\|_{c.b.} + \|a_1\|_{c.b.} \cdot \|u_2 - a_2\|_{c.b.} \leq$$
$$\leq \left(a_n^{c.b.}(u_1) + \tfrac{\varepsilon}{2}\right) \cdot \|u_2\|_{c.b.} + \|a_1 - u_1 + u_1\|_{c.b.} \cdot \left(a_n^{c.b.}(u_2) + \tfrac{\varepsilon}{2}\right) \leq$$
$$\leq \left(a_n^{c.b.}(u_1) + \tfrac{\varepsilon}{2}\right) \cdot \|u_2\|_{c.b.} + (\|a_1 - u_1\|_{c.b.} + \|u_1\|_{c.b.}) \cdot \left(a_n^{c.b.}(u_2) + \tfrac{\varepsilon}{2}\right) \leq$$
$$\leq \left(a_n^{c.b.}(u_1) + \tfrac{\varepsilon}{2}\right) \cdot \|u_2\|_{c.b.} + 2 \cdot \|u_1\|_{c.b.} \cdot \left(a_n^{c.b.}(u_2) + \tfrac{\varepsilon}{2}\right).$$

Since ε is arbitrary we obtain:

$$a_{n^2}^{c.b.}(u_1 \otimes_{\min} u_2) \leq 2 \cdot \left(a_n^{c.b.}(u_1) \cdot \|u_2\|_{c.b.} + a_n^{c.b.}(u_2) \cdot \|u_1\|_{c.b.}\right).$$

Taking account that the sequence of the aproximation numbers is decreasing we can writte:

$$\sum_{n=1}^{k} \frac{a_n^{c.b.}(u_1 \otimes_{\min} u_2)}{n} \leq \sum_{n=1}^{j} (2 \cdot n + 1) \frac{a_{n^2}^{c.b.}(u_1 \otimes_{\min} u_2)}{n^2}, \text{ where } j^2 \leq k < (j+1)^2.$$

Now we obtain:

$$\sum_{n=1}^{k} \frac{a_n^{c.b.}(u_1 \otimes_{\min} u_2)}{n} \leq \sum_{n=1}^{j} (2 \cdot n + 1) \frac{a_{n^2}^{c.b.}(u_1 \otimes_{\min} u_2)}{n^2} \leq 3 \cdot \sum_{n=1}^{j} n \cdot \frac{a_{n^2}^{c.b.}(u_1 \otimes_{\min} u_2)}{n^2} \leq$$
$$\leq 6 \cdot \sum_{n=1}^{j} \frac{a_n^{c.b.}(u_1) \cdot \|u_2\|_{c.b.} + a_n^{c.b.}(u_2) \cdot \|u_1\|_{c.b.}}{n} \leq 6 \cdot \sum_{n=1}^{k} \frac{a_n^{c.b.}(u_1) \cdot \|u_2\|_{c.b.} + a_n^{c.b.}(u_2) \cdot \|u_1\|_{c.b.}}{n}.$$

This finishes the proof. □

Remark 5. By means of these approximation numbers we can define special approximation ideals in $c.b.(E, F)$.

3. Special approximation ideals

Definition 5. Let $x = \{x_1, x_2, ...\}$ be a real sequence and let $card\,(x)$ be $card\,\{i \in N : x_i \neq 0\}$.

Let K be the set of all real sequences $x \in l_\infty$ having the following two properties:

1. $card\,(x) < n\,(x)$, $n\,(x)$ is a natural number

2. $x_1 \geq x_2 \geq ... \geq x_{n(x)} \geq 0$.

A function $\Phi : K \to R$ is called a **symmetric norming function** if:

1. $\Phi(x) > 0$ if $x \in K$ and $x \neq 0$;
2. $\Phi(\alpha \cdot x) = \alpha \cdot \Phi(x)$, for every $\alpha \geq 0$ and $x \in K$;
3. $\Phi(x + y) \leq \Phi(x) + \Phi(y)$, for every $x, y \in K$;
4. $\Phi(\{1, 0, 0...\}) = 1$;
5. If $x, y \in K$ and $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$, for every $k = 1, 2, ...$, then $\Phi(x) \leq \Phi(y)$.

Remark 6. The above definition can be extend on the whole space l_∞ taking

$\Phi(x) := \lim_{n\to\infty} \Phi(\{x_1^*, ..., x_n^*, 0, 0...\})$, where $x^* = \{x_i^*\}_{i\in N}$ is the sequence $\{|x_i|\}_{i\in N}$ rearranged in decreasing order.

Definition 6. In the sequel we shall consider a subclass of $c.b.(E, F)$ which is defined as follows:

$$\Phi - c.b.(E, F) := \left\{ u \in c.b.(E, F) : \|u\|_\Phi^{c.b.} := \Phi\left(\left\{a_n^{c.b.}(u)\right\}_n\right) < \infty \right\}.$$

Remark 7. We prove that this class has similar properties with the similar classes defined for linear and bounded operators. (For the case of the Hilbert spaces it can be seen [2], [7] and for the case of the Banach spaces it can be seen [8], [9], [10], [11].)

Proposition 3. $\left(\Phi - c.b., \|.\|_\Phi^{c.b.}\right)$ *is a quasi-normed operator ideal.*

Proof. 1. Any unidimensional operator , $u \in c.b.(E, F)$, belongs to

$\Phi - c.b.(E, F)$ because, in this case, the sequence $\left\{a_n^{c.b.}(u)\right\} = \{\|u\|_{c.b.}, 0, 0, ...\}$ and hence $\Phi\left(\left\{a_n^{c.b.}(u)\right\}_n\right) = \|u\|_{c.b.} < \infty$.

2. If $u_1, u_2 \in c.b.(E, F)$ then $u_1 + u_2 \in c.b.(E, F)$. This results from the proposition 5 (1). and from the properties of Φ, as follows:

$$\Phi\left(\left\{a_n^{c.b.}(u_1 + u_2)\right\}_n\right) \leq 2 \cdot \Phi\left(\left\{a_n^{c.b.}(u_1) + a_n^{c.b.}(u_2)\right\}_n\right) \leq$$
$$\leq 2 \cdot \left(\Phi\left(\left\{a_n^{c.b.}(u_1)\right\}_n\right) + \Phi\left(\left\{a_n^{c.b.}(u_2)\right\}_n\right)\right).$$

3. If $v \in c.b.(E, E)$, $u \in \Phi - c.b.(E, F)$ and $w \in c.b.(F, F)$ then

$w \circ u \circ v \in \Phi - c.b.(E,F)$.

From the proposition 5 (2) and from the definition of $a_n^{c.b.}(u)$ it follows that $a_n^{c.b.}(w \circ u \circ v) \leq \|w\|_{c.b.} \cdot a_n^{c.b.}(u) \cdot \|v\|_{c.b.}$ and hence $\Phi\left(\left\{a_n^{c.b.}(w \circ u \circ v)\right\}_n\right) \leq \|w\|_{c.b.} \cdot \Phi\left(\left\{a_n^{c.b.}(u)\right\}_n\right) \cdot \|v\|_{c.b.}$. □

Remark 8. We present now some properties similar to the properties of the classical approximation ideals L_Φ, [8], [11].

Lemma 1. *The approximation numbers $a_n^{c.b.}(u)$ verify the inequalities:*

$$\sum_{n=1}^{k} a_{2\cdot n-1}^{c.b.}(u) \leq \sum_{n=1}^{k} a_n^{c.b.}(u) \leq 2 \cdot \sum_{n=1}^{k} a_{2\cdot n-1}^{c.b.}(u), k = 1, 2, \ldots.$$

Proof. The first inequality is a consequence of the fact that the sequence $\left\{a_n^{c.b.}(u)\right\}_n$ is decreasing.

The second inequality results as follows:
$\sum_{n=1}^{k} a_n^{c.b.}(u) \leq \sum_{n=1}^{2\cdot k} a_n^{c.b.}(u) \leq \sum_{n=1}^{k} a_{2\cdot n-1}^{c.b.}(u) + \sum_{n=1}^{k} a_{2\cdot n}^{c.b.}(u) \leq$
$\leq 2 \cdot \sum_{n=1}^{k} a_{2\cdot n-1}^{c.b.}(u)$. □

Corollary 1. $\|u\|_{\Phi}^{c.b.} := \Phi\left(\left\{a_{2\cdot n-1}^{c.b.}(u)\right\}_n\right)$ *is a quasi-norm equivalent*

with $\|u\|_{\Phi}^{c.b.} = \Phi\left(\left\{a_n^{c.b.}(u)\right\}_n\right)$.

Remark 9. Since $\sum_{n=1}^{k} \left(a_n^{c.b.}(u)\right)^p \leq \sum_{n=1}^{k} \left(\frac{1}{n} \cdot \sum_{i=1}^{n} a_i^{c.b.}(u)\right)^p \leq$

$\leq c(p) \cdot \sum_{n=1}^{k} \left(a_n^{c.b.}(u)\right)^p$, $1 < p < \infty, k = 1, 2, \ldots,$
see the Hardy inequality [11], it follows that

$$\Phi\left(\left\{\left(a_n^{c.b.}(u)\right)^p\right\}_n\right) \leq \Phi\left(\left\{\left(\frac{1}{n} \cdot \sum_{i=1}^{n} a_i^{c.b.}(u)\right)^p\right\}_n\right) \leq c(p) \cdot \Phi\left(\left\{\left(a_n^{c.b.}(u)\right)^p\right\}_n\right)$$

and hence $\|u\|_{\Phi_{(p)}}^{c.b.}$ is equivalent with $\overline{\|u\|_{\Phi_{(p)}}^{c.b.}} := \Phi_{(p)}\left(\left\{\frac{1}{n} \cdot \sum_{i=1}^{n} a_i^{c.b.}(u)\right\}_n\right)$, where $\Phi_{(p)}(\{x_i\}) = \Phi(\{x_i^p\})^{\frac{1}{p}}$ is a symmetric norming function, [2], [6], [7], [11], for $1 < p < \infty$.

Because $a_n^{c.b.}(u) \leq \left(a_1^{c.b.}(u) \cdot \ldots \cdot a_n^{c.b.}(u)\right)^{\frac{1}{n}} \leq \frac{1}{n} \cdot \sum_{i=1}^{n} a_i^{c.b.}(u)$ it follows, also, that

$$\left\{a_n^{c.b.}(u)\right\}_n \in l_{\Phi_{(p)}} \text{ if } g_n(u) := \left\{\left(a_1^{c.b.}(u) \cdot \ldots \cdot a_n^{c.b.}(u)\right)^{\frac{1}{n}}\right\}_n \in l_{\Phi_{(p)}},$$
$$\{x_n\} \in l_{\Phi_{(p)}} \Longleftrightarrow \Phi_{(p)}(\{x_n\}) < \infty.$$

Remark 10. If we consider the special case of the function

$$\overline{\Phi}_{(p)} : \left(\left\{a_n^{c.b.}(u)\right\}\right) \to \Phi\left(\left\{\frac{\left(a_n^{c.b.}(u)\right)^p}{n}\right\}\right)^{\frac{1}{p}},$$

where $1 \leq p < \infty$, the classes $\overline{\Phi}_{(p)} - c.b.(E, F)$ are tensor product stable.

Proposition 4. *If* $u_k \in \overline{\Phi}_{(p)} - c.b.(E_k, F_k)$, $k = 1, 2.$, *then*

$$u_1 \otimes_{\min} u_2 \in \overline{\Phi}_{(p)} - c.b.(E_1 \otimes_{\min} E_2, F_1 \otimes_{\min} F_2).$$

Proof. It is similar to that for the classical approximation ideals [10], [11].

First we remark that, using the relation

$$a_{n^2}^{c.b.}(u_1 \otimes_{\min} u_2) \leq 2 \cdot \left(a_n^{c.b.}(u_1) \cdot \|u_2\|_{c.b.} + a_n^{c.b.}(u_2) \cdot \|u_1\|_{c.b.}\right)$$

we can obtain

$$\sum_{n=1}^{k} \frac{\left(a_n^{c.b.}(u_1 \otimes_{\min} u_2)\right)^p}{n} \leq c(p) \cdot \sum_{n=1}^{k} \frac{\left(a_n^{c.b.}(u_1) \cdot \|u_2\|_{c.b.}\right)^p + \left(a_n^{c.b.}(u_2) \cdot \|u_1\|_{c.b.}\right)^p}{n},$$

for $k = 1, 2, \ldots$, see Proposition 6 for $p = 1$.

Now taking into account the properties of the functions Φ it follows that

$$\Phi\left(\left\{\frac{\left(a_n^{c.b.}(u_1 \otimes_{\min} u_2)\right)^p}{n}\right\}\right) \leq$$
$$c(p) \cdot \Phi\left(\left\{\frac{\left(a_n^{c.b.}(u_1) \cdot \|u_2\|_{c.b.}\right)^p}{n} + \frac{\left(a_n^{c.b.}(u_2) \cdot \|u_1\|_{c.b.}\right)^p}{n}\right\}\right).$$

Hence

$$
\begin{aligned}
&\overline{\Phi}_{(p)}\left(\left\{a_n^{c.b.}\left(u_1 \otimes_{\min} u_2\right)\right\}\right) \leq \\
&\leq c_1(p) \cdot \left(\overline{\Phi}_{(p)}\left(\left\{a_n^{c.b.}\left(u_1\right)\right\}\right) \cdot \left\|u_2\right\|_{c.b.} + \overline{\Phi}_{(p)}\left(\left\{a_n^{c.b.}\left(u_2\right)\right\}\right) \cdot \left\|u_1\right\|_{c.b.}\right) \leq \\
&\leq 2 \cdot c_1(p) \cdot \overline{\Phi}_{(p)}\left(\left\{a_n^{c.b.}\left(u_1\right)\right\}\right) \cdot \overline{\Phi}_{(p)}\left(\left\{a_n^{c.b.}\left(u_2\right)\right\}\right), \\
&c_1(p) = c(p)^{\frac{1}{p}} \text{ and } \left\|u_k\right\|_{c.b} \leq \overline{\Phi}_{(p)}\left(\left\{a_n^{c.b.}\left(u_k\right)\right\}\right),\ k = 1, 2.
\end{aligned}
$$

The proof is fulfiled. □

Remark 11. The above result remains true if we consider **the maximal tensor product.**

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