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FIBER PICARD OPERATORS THEOREM AND APPLICATIONS

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Abstract. In this paper we study the following problem: Let (X_k, d_k), $k = \overline{0,p}$, $p \geq 1$, be metric spaces and $A_k : X_0 \times \cdots \times X_k \to X_k$, $k = \overline{0,p}$ be operators. We suppose that

(a) the operators A_k are continuous, $k = \overline{0,p}$;

(b) the operators $A_0, A_k(x_0, \ldots, x_{k-1}, \cdot)$, $k = \overline{1,p}$ are (weakly) Picard operators.

Establish conditions which imply that the operator

$$B_p : X_0 \times \cdots \times X_p \to X_0 \times \cdots \times X_p$$

$$B_p(x_0, \ldots, x_p) := (A_0(x_0), A_1(x_0, x_1), \ldots, A_p(x_0, \ldots, x_p)),$$

is a (weakly) Picard operator.

1. Introduction

Let (X, d) be a metric space and $A : X \to X$ an operator. In this paper we shall use the following notations:

$P(X) := \{Y \subset X | \ Y \neq \emptyset\}$,

$F_A := \{x \in X | \ A(x) = x\}$ - the fixed point set of A,

$I(A) := \{Y \in P(X) | \ A(Y) \subset Y\}$.

Definition 1.1 (Rus [9], [11]). An operator $A : X \to X$ is weakly Picard operator (WPO) if the sequence

$$(A^n(x))_{n \in N}$$

converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.

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Definition 1.2 (Rus [9], [11]). If A is WPO, then we consider the operator A^∞ defined by

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

We remark that, $A^\infty(X) = F_A$.

Definition 1.3 (Rus [9], [11]). If A is WPO and $F_A = \{x^*\}$, then by definition the operator A is a Picard operator.

Example 1.1. Let (X, d) be a complete metric space and $A : X \to X$ such that

$$d(A^2(x), A(x)) \leq ad(x, A(x))$$

for all $x \in X$ and for some $a \in]0, 1[$. Then A is weakly Picard operator (see [8], [9], [11]).

Example 1.2. Let (X, d) be a complete metric space and $A, B : X \to X$ such that

$$d(A(x), B(y)) \leq a[d(x, A(x)) + d(y, B(y))]$$

for all $x, y \in X$ for some $a \in \left]0, \frac{1}{2}\right[$. Then A and B are Picard operators.

Example 1.3. $X = C[0,1]$, $d(x, y) = \|x - y\|_C$,

$$A(x)(t) = x(0) + \int_0^t K(t,s)x(s)ds, \quad t \in [0,1]$$

where $K \in C([0,1] \times [0,1])$. Then A is WPO.

For other examples see [13], [10], [1], [2], [20],...

We have the following characterization theorem for WPO.

Theorem 1.1. *Let (X, d) be a metric space and $A : X \to X$ an WPO. Then there exist $X_i \in I(A)$, $i \in I$, such that*

(i) $X = \bigcup_{i\in I} X_i$, $X_i \cap X_j = \emptyset$, $i \neq j$,

(ii) $A\big|_{X_i}$ *is a Picard operator,* $i \in I$.

Proof. Let $x \in F_A$. Let X_x be the domain of attraction of x. It is clear that

$$X = \bigcup_{x\in F_A} X_x$$

is a partition of X and that $X_x \in I(A)$. By the definition of X_x, we have that

$$F_A \cap X_x = \{x\}.$$

In this paper we study the following problem:

Problem 1.1. Let (X, d) and (Y, ρ) be the metric spaces and $A = (B, C) : X \times Y \to X \times Y$ a triangular operator, i.e.

$$A(x, y) = (B(x), C(x, y)), \quad x \in X, \quad y \in Y.$$

We suppose that the operators $B : X \to X$, $C(x, \cdot) : Y \to Y$, $x \in X$, are Picard operators. Establish conditions which imply that the operator A is Picard operator.

If the operators, $B : X \to X$, $C(x, \cdot) : Y \to Y$, $x \in X$, are WPO, establish conditions which imply that the operator A is WPO.

2. Fiber Picard operators theorem

The following result is given by M.W. Hirsch and C.C. Pugh ([5], 1970):

Theorem 2.1 (Fiber contraction theorem). *Let (X, d) be a metric space and $B : X \to X$ be an operator having an atractive fixed point $p \in X$. Let (Y, ρ) be a metric space and $C : X \times Y \to Y$ an operator such that*

(i) there exists $\lambda \in [0, 1[$, such that the operator $C(x, \cdot)$ is a λ-contraction for all $x \in X$;

(ii) the operator $A : X \times Y \to X \times Y$, $A(x, y) := (B(x), C(x, y))$ is continuous.

Let $q \in Y$ be a fixed point for $C(p, \cdot)$.

Then (p, q) is an attractive fixed point for A.

For some generalization of this theorem see [10]-[15], [18] and [19].

We have

Theorem 2.2. *Let (X, d) and (Y, ρ) be two metric space and $A = (B, C)$ a triangular operator. We suppose that*

(i) (Y, ρ) is a complete metric space;

(ii) the operator $B : X \to X$ is WPO;

(iii) there exists $\alpha \in [0, 1[$, such that $C(x, \cdot)$ is an α-contraction, for all $x \in X$;

(iv) if $(x^*, y^*) \in F_A$, then $C(\cdot, y^*)$ is continuous in x^*.

Then the operator A is WPO.

If B is Picard operator, then A is Picard operator.

Proof. Let $(x, y) \in X \times Y$. Let y^* the unique fixed point of $C(B^\infty(x), \cdot)$. We shall prove that $A^n(x, y) \to (B^\infty(x), y^*)$ as $n \to \infty$. Let $A^n(x, y) = (x_n, y_n)$. Then

$$x_n = B^n(x), \quad y_n = C(x_{n-1}, y_{n-1}).$$

The proof that $y_n \to y^*$ as $n \to \infty$ is similarly with the proof given in [5] for the Theorem 1.

Remark 2.1. The proof that $y_n \to y^*$ as $n \to \infty$ follows, also, from the following

Lemma 2.1 (see [13]). *Let (X, d) be a complete metric space and A_n, $A : X \to X$, $n \in N$, some operators. We suppose that*

(a) the sequence $(A_n)_{n \in N}$ pointiwse converges to A;

(b) there exist $\alpha \in [0, 1[$ such that the operators A_n and A, $n \in N$, are α-contractions.

Then the sequence $(A_n \circ A_{n-1} \circ \cdots \circ A_0)_{n \in N}$ poinwise converges to A^∞.

Remark 2.2. In the proof of Lemma 2.2 on uses the following

Lemma 2.2 (see [13], [14] and [15]). *Let $a_n, b_n \in R_+$, $n \in N$. We suppose that*

(a) $a_n \to 0$ as $n \to \infty$;

(b) $\sum_{k=0}^{\infty} b_k < +\infty$.

Then

$$\sum_{k=0}^{n} a_k b_{n-k} \to 0 \text{ as } n \to \infty.$$

Remark 2.3. For to have a generalization of the Theorem 2.2, we need suitable generalization for Lemma 2.1 and Lemma 2.2. For some generalization of these Lemmas, see [15] and [19].

Remark 2.4. By induction, from the Theorem 2.2 we have

Theorem 2.3 (see [13]). *Let (X_k, d_k), $k = \overline{0, p}$, $p \geq 1$, be some metric spaces. Let*

$$A_k : X_0 \times \cdots \times X_k \to X_k, \quad k = \overline{0, p}$$

be some operators. We suppose that:

(a) the spaces (X_k, d_k), $k = \overline{1, p}$ are complete metric spaces;

(b) the operator A_0 *is WPO;*

(c) there exist $\alpha_k \in [0,1[$ *such that the operators* $A_k(x_0, \ldots, x_{k-1}, \cdot)$ *are* α_k*-contractions;*

(d) if $(x_0^*, \ldots, x_p^*) \in F_{B_p}$, $B_p = (A_0, \ldots, A_p)$, *then the operators* $A_k(\cdot, \ldots, \cdot, x_k^*)$,
$k = \overline{1,p}$, *are continuous in* $(x_0^*, x_1^*, \ldots, x_{k-1}^*)$.

Then the operator B_p *is WPO.*

Remark 2.5. The next conjecture is in connection with our results.

Discrete Markus-Yamabe Conjecture (see [3], [6], [1]). Let A be a C^1 function from R^n into itself such that $A(0) = 0$ and for any $x \in R^n$, $JA(x)$ (the Jacobian matrix of A at x) has all its eigenvalues with modulus less than one. Then A is a Picard function.

From the fiber Picard operators theorem we have

Theorem 2.4. *Let* $A : R^n \to R^n$ *be a* C^1 *triangular function,* $A = (A_1, \ldots, A_n)$.

If there exists $\alpha \in]0,1[$ *such that*

$$\left|\frac{\partial A_i}{\partial x_i}\right| \leq \alpha, \quad i = \overline{1,n}.$$

Then the function A *is Picard function.*

A. Cima, A. Gasull, F. Mañosas prove that the Discrete Markus-Yamabe Conjecture ([3], 1999) is a theorem for A provided

$$\left|\frac{\partial A_i}{\partial x_j}\right| < 1, \quad j = \overline{1,i}, \quad i = \overline{1,n}.$$

3. Applications

The fiber Picard operators theorem is very useful for proving solutions of operatorial equations to be differentiable with respect to parameters (see [17], [12], [13], [14], [15], [20], [18]). For example:

- (J. Sotomayor) differentiability with respect to initial data for the solution of differential equations

$$x' = f(t,x), \quad x(t_0) = x_0, \quad f : \Omega \to R^n, \quad \Omega \subset R^{n+1};$$

- (I.A. Rus [12]) differentiability with respect to λ for the solution of the integral equation

$$x(t) = 1 + \lambda \int_t^1 x(s)x(s-t)ds, \quad t \in [0,1],$$

where $\lambda \in R$;

- (A. Tămăşan) differentiability with respect to lag function for pantograph equation

$$x\prime(t) = f(t), x(t), x(\lambda t)), \quad t > 0; \quad 0 < \lambda < 1, \quad x(0) = 0.$$

In what follow we apply the fiber Picard operators theorem to study the following integral equations modelling population growth in a periodic environment (see [10], [7])

$$x(t) = \int_{t-\tau}^{t} f(s, x(s); \lambda) ds \tag{1}$$

where $f \in C(R \times [\alpha, \beta] \times J, [m, M]$, with $\tau, \alpha, \beta, m, M \in R_+^*$ and $J \subset R$ a compact interval.

Let

$$X_\omega := \{x \in C(R \times J, [\alpha, \beta]) | \; x(t+\omega, \lambda) = x(t, \lambda),$$

$$\text{for all } t \in R, \; \lambda \in J\}, \; \omega > 0.$$

We consider on X_ω the metric $d(x,y) := \|x - y\|_C$. We have

Theorem 3.1. *We suppose that*

(a) $0 < m < M$, $0 < \alpha < \beta$; $\alpha \leq m\tau$, $\beta \geq M\tau$;

(b) $m \leq f(t, u; \lambda) \leq M$, *for* $t \in R$, $u \in [\alpha, \beta]$, $\lambda \in J$;

(c) $f(t+\omega, u; \lambda) = f(t, u; \lambda)$, $t \in R$, $u \in [\alpha, \beta]$, $\lambda \in J$;

(d) there exists $l(t)$, *such that*

$$|f(t, u; \lambda) - f(t, v; \lambda)| \leq l(t)|u - v|$$

for all $t \in R$, $u, v \in [\alpha, \beta]$;

(e) there exists $q \in]0,1[$ *such that*

$$\int_{t-\tau}^{t} l(s) ds \leq q, \text{ for all } t \in R.$$

Then

(i) the equation (1) has in X_ω a unique solution x^*;

(ii) for all $x_0 \in X_\omega$, the sequence defined by

$$x_{n+1}(t,\lambda) = \int_{t-\tau}^{t} f(s, x_n(s,\lambda))ds$$

converges uniformly to x^*;

(iii) if $f(t,\cdot,\cdot) \in C^1$, then $x^*(t,\cdot) \in C^1(J)$.

Proof. (i)+(ii). We consider the operator

$$B : X_\omega \to C(R \times J), \quad B(x)(t,\lambda) := \int_{t-\tau}^{t} f(s, x(s,\lambda))ds.$$

From (a) and (c) we have that $X_\omega \in I(B)$. From (d) it follows that B is a contraction.

By the contraction principle we have that B is a Picard operator.

(iii). Let we prove that there exists $\frac{\partial x^*}{\partial \lambda}$ and $\frac{\partial x^*}{\partial \lambda} \in C(R \times J)$.

If we suppose that there exists $\frac{\partial x^*}{\partial \lambda}$, then from

$$x(t,\lambda) = \int_{t-\tau}^{t} f(s, x(s,\lambda);\lambda)ds$$

we have

$$\frac{\partial x(t,\lambda)}{\partial \lambda} = \int_{t-\tau}^{t} \frac{\partial f(s,x(s,\lambda);\lambda)}{\partial x} \cdot \frac{\partial x(s,\lambda)}{\partial \lambda} ds + \int_{t-\tau}^{t} \frac{\partial f(s,x(s,\lambda);\lambda)}{\partial \lambda} ds.$$

This relation suggest us to consider the following operator

$$A : X_\omega \times Y_\omega \to X_\omega \times Y_\omega$$

defined by

$$A = (B, C), \quad A(x,y) = (B(x), C(x,y)),$$

where

$$C(x,y)(t,\lambda) := \int_{t-\tau}^{t} \frac{\partial f(s,x(s,\lambda);\lambda)}{\partial x} y(s,\lambda)ds + \int_{t-\tau}^{t} \frac{\partial f(s,x(s,\lambda);\lambda)}{\partial \lambda} ds$$

and $Y_\omega := \{y \in C(R \times J) | \ y(t+\omega,\lambda) = y(t,\lambda), \ t \in R, \ \lambda \in J\}$.

Now we are in the condition of the fiber Picard operators theorem. From this theorem, the operator A is a Picard operator and the sequences

$$x_{n+1} = B(x_n)$$

and

$$y_{n+1} = C(x_n, y_n)$$

converge uniformly to $(x^*, y^*) \in F_A$, for all $x_0 \in X_\omega$, $y_0 \in Y_\omega$.

If we take $x_0 \in X_\omega$, $y_0 \in Y_\omega$ such that $y_0 = \frac{\partial x_0}{\partial \lambda}$, then we have that $y_n = \frac{\partial x_n}{\partial \lambda}$, for all $n \in N$.

So

$$x_n \stackrel{unif.}{\longrightarrow} x^* \text{ as } n \to \infty,$$

$$\frac{\partial x_n}{\partial \lambda} \stackrel{unif.}{\longrightarrow} y^* \text{ as } n \to \infty.$$

Using a Weierstrass argument we conclude that x^* is differentiable and $y^* = \frac{\partial x^*}{\partial \lambda}$.

References

[1] G.R. Belitskii, Yu.I. Lyubich, *Matrix Norms and their Applications*, Birkhäuser, Basel, 1988.

[2] A. Buică, *Old and new Gronwall-type inequalities as consequences of an abstract lemma of Rus* (to appear).

[3] A. Cima, A. Gasul, F. Mañosas, *The discrete Markus-Yamabe problem*, Nonlinear Analysis, 35(1999), 343-354.

[4] A. Domokos, *On the continuity and differentiability of the implicit functions*, Studia Univ. Babeş-Bolyai (to appear).

[5] M.W. Hirsch, C.C. Pugh, *Stable manifolds and hyperbolic sets*, Proceed. of Symposia in pure mathematics, 14(1970), A.M.S., 1970, 133-143.

[6] G. Meisters, *A biography of the Markus-Yamabe conjecture*, presented at the conference Aspects of Mathematics-Algebra, Geometry and Several Complex Variables, held June 10-13, 1996, at the University Hong Kong.

[7] R. Precup, E. Kirr, *Analysis of a nonlinear integral equation modelling infections diseases*, Proc. Int. Conf., Timişoara, 1997, 178-195.

[8] I.A. Rus, *Metrical fixed point theorems*, Univ. of Cluj-Napoca, 1979.

[9] I.A. Rus, *Generalized contractions*, Seminar on Fixed Point Theory, Cluj-Napoca, 1983.

[10] I.A. Rus, *A delay integral equation from biomathematics*, Seminar on Diff. Eq., Babeş-Bolyai Univ., Cluj-Napoca, 1989, 87-90.

[11] I.A. Rus, *Weakly Picard mappings*, Comment. Math. Univ. Caroline, 34, 4(1993), 769-773.

[12] I.A. Rus, *An abstract point of view for some integral equations from applied mathematics*, Proc. Int. Conf. on Analysis and Numerical Computation, Univ. of Timişoara, 1997, 256-270.

[13] I.A. Rus, *Fiber Picard operators and applications*, Novi Sad J. Math. (to appear).

[14] I.A. Rus, *A Fiber generalized contraction theorem and applications*, Mathematica (to appear).
[15] I.A. Rus, *Fiber Picard operator on generalized metric spaces and application*, Scripta Scientiarum (to appear).
[16] S. Smale, *Mathematical problems for the next century*, Math. Intelligencer, 20(1998), Nr.2, 7-15.
[17] J. Sotomayor, *Smooth dependence of solutions of differential equations on initial data: a simple proof*, Bol. Soc. Bras. Mat., 4(1973), 55-59.
[18] A. Tămăşan, *Differentiability with respect to lag function for nonlinear pantograph equations* (to appear).
[19] M. Şerban, *Fiber φ-contraction theorem* (to appear).
[20] P. Ver Eecke, *Application du calcul différentiel*, Presses Univ. de France, Paris, 1985.

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