ON THE UNIVALENCE OF FUNCTIONS RELATED TO HYPERGEOMETRIC FUNCTIONS

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Abstract. In lucrare se studiază univalenţa unei clase de funcţii exprimată prin intermediul funcţiei hipergeometrice.

1. Introduction

Let A be the class of function f which are analytic in the unit disk $U = \{ z \in C : |z| < 1 \}$ with $f(0) = 0$ and $f'(0) = 1$. In this note we improve the result from [2] using another univalence criterion.

2. Preliminaries

Theorem 2.1. ([2]). *Let $f \in A$ and let α be a complex number, $Re\alpha > 0$. If*

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \qquad (\forall)\ z \in U \tag{1}$$

then for all complex numbers β with $Re\beta \geq Re\alpha$, the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)du\right]^{1/\beta} \tag{2}$$

is analytic and univalent in U.

3. Main results

It is easy to prove the following:

Lemma 3.1. *Let α, γ be complex numbers and let the function*

$$E(\alpha,\ \gamma, z, \bar{z}) = \frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\gamma\frac{z}{1-z}\right|, \qquad |z| < 1. \tag{3}$$

If $0 < Re\alpha < 1$, *then*

$$E(\alpha,\ \gamma, z, \bar{z}) \leq \frac{|\gamma|}{Re\alpha}, \qquad (\forall) z \in U. \tag{4}$$

If $Re\alpha \geq 1$, *then*

$$E(\alpha,\ \gamma, z, \bar{z}) \leq 2|\gamma|, \qquad ,(\forall) z \in U. \tag{5}$$

Theorem 3.1. *Let* α, β, γ *be complex numbers. If*

$$|\gamma| \leq Re\alpha < 1\ , \tag{6}$$

$$|\gamma| \leq \frac{1}{2} \ \ and \ \ Re\alpha \geq 1\ , \tag{7}$$

$$Re\beta \geq Re\alpha\ , \tag{8}$$

then the function

$$F_\beta(z) = z \cdot [\ F(\ \beta,\ \gamma,\ \beta+1,\ z\ )]^{1/\beta} \tag{9}$$

is analytic and univalent in U, *where by* $F(a, b, c, z)$ *we denoted the hypergeometric function.*

Proof. If in (2) we make the change $u = tz$, we obtain

$$F_\beta(z) = z \cdot \left[\beta \int_0^1 t^{\beta-1} f'(tz) dt\right]^{1/\beta}. \tag{10}$$

In the following we consider the function

$$f(z) = \int_0^z (1-u)^{-\gamma} du \tag{11}$$

For this function we obtain

$$\frac{zf''(z)}{f'(z)} = \gamma \frac{z}{1-z} \ \ and$$

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha} \left|\frac{zf''(z)}{f'(z)}\right| = \frac{1-|z|^{2Re\alpha}}{Re\alpha} \left|\gamma \frac{z}{1-z}\right|.$$

According to Lemma 3.1 we deduce that the condition (1) from Theorem 2.1 , for the function (11) is verified in the cases

$$(i) \qquad |\gamma| \leq Re\alpha < 1\ ;$$

$$(ii) \qquad |\gamma| \le \frac{1}{2} \ and \ Re\alpha \ge 1 .$$

Replacing in (10) the function f defined by (11) we obtain

$$F_\beta(z) = z \cdot \left[\beta \int_0^1 t^{\beta-1}(1-tz)^{-\gamma} dt\right]^{1/\beta} =$$

$$= z \cdot [\, F(\,\beta,\ \gamma,\ \beta+1,\ z\,)\,]^{1/\beta} .$$

where by $F(a,b,c,z)$ we noted the hypergeometric function.

References

[1] E.Cazacu, N.N.Pascu, *On the univalence of functions related to hypergeometric functions*, Preprint nr.5(1986),Cluj-Napoca,25-26.

[2] N.N.Pascu, *An improvement of Becker's univalence criterion*, Preprint nr.1(1987), Braşov,43-48.

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