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ANALYSIS OF SOME NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. The paper is devoted to the study of the neutral differential equation with delay $x'(t) = f(t, x(t), x(\theta(t)), x'(\theta(t)))$. Our analysis is concerned with the existence, uniqueness and monotone iterative approximation of the nondecreasing global solutions of the initial-value problem. We use fixed point theorems (Schauder, Krasnoselskii, Leray-Schauder) and monotone iterative techniques.

1. Introduction

In this paper, we are concerned with the following nonlinear neutral delay equation

$$x'(t) = f(t, x(t), x(\theta(t)), x'(\theta(t))), \tag{1.1}$$

where $-\tau \leq \theta(t) \leq t$ for some $\tau \geq 0$.

Equations of this type arise when modelling biological, physical, etc., processes whose growth rate at any moment of time t is determined not only by the present state, but also by past states and the past growth rate. For example, such models are described by K. Gopalsamy [4] and Y. Kuang [8], from population dynamics, and by R.D. Driver [3], in connection with the two-body problem.

Basic theory and much literature on differential equations with delay, including the neutral ones, can be found in the monographs by V. Lakshmikantham, L. Wen, B. Zhang [9], V. Kolmanovskii, A. Myshkis [7], D. Bainov, D.P. Mishev [1] and J. Hale [5].

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Recently, T.A. Burton [2] established an analogue of the Peano *local* existence theorem for the Cauchy problem (1.1)-(1.2), where

$$x(t) = \phi(t), \quad -\tau \leq t \leq 0. \tag{1.2}$$

Motivated by the above paper, this article deals with the *global* solvability (on a given interval $[0, T]$) of the Cauchy problem (1.1)-(1.2).

We shall assume that f is nonnegative and continuous, θ is continuous, $\phi \in C^1[-\tau, 0]$ and satisfies the *sewing condition*

$$\phi'(0) = f(0, \phi(0), \phi(\theta(0)), \phi'(\theta(0))). \tag{1.3}$$

We shall look for nondecreasing solutions $x \in C^1[0,T]$ with $x(t) \in [a, R]$ and $x'(0) = b$, where $a = \phi(0)$, $b = \phi'(0)$ and $a < R \leq \infty$. In case that $R = \infty$, all intervals of the form $[c, R]$ should be interpreted as $[c, \infty)$ and all inequalities of the form $c \leq R$, as $c < \infty$.

Let

$$K = \{x \in C^1[0,T]; \quad a \leq x \text{ on } [0,T]\}$$

and

$$K_R = \{x \in K; \quad x \leq R \text{ on } [0,T]\}.$$

Clearly, K is a closed convex set of $C^1[0,T]$ and (1.1)-(1.2) is equivalent to the fixed point problem $A(x) = x$ for the map $A : K_R \to K$,

$$A(x)(t) = a + \int_0^t f(s, x(s), \widetilde{x}(\theta(s)), \widetilde{x}'(\theta(s)))\, ds, \quad 0 \leq t \leq T, \tag{1.4}$$

where $\widetilde{x}(t) = \phi(t)$ on $[-\tau, 0)$ and $\widetilde{x}(t) = x(t)$ on $[0,T]$. Obviously, each fixed point x of A also satisfies $x(0) = a$ and $x'(0) = b$ and so, its prolongation by ϕ is a function in $C^1[-\tau, T]$.

Notice that the dependence of $f(t,x,y,z)$ on the *neutral variable* z is the cause that A is not completely continuous. This is why one tries to represent A as a sum of a completely continuous mapping and a contraction. This happens when f admits the decomposition

$$f(t,x,y,z)=f_0(t,x,y)+f_1(t,x,y,z), \tag{1.5}$$

with f_0 continuous and f_1 satisfying the Lipschitz condition

$$|f_1(t,x,y,z)-f_1(t,\overline{x},\overline{y},\overline{z})|\leq\alpha|x-\overline{x}|+\beta|y-\overline{y}|+\gamma|z-\overline{z}| \tag{1.6}$$

for $\alpha,\beta\geq0$ and $0\leq\gamma<1$. Then A can be represented as $A=A_0+A_1$, where

$$A_0(x)(t)=a+\int_0^t f_0(s,x(s),\widetilde{x}(\theta(s)))\,ds$$

and

$$A_1(x)(t)=\int_0^t f_1(s,x(s),\widetilde{x}(\theta(s)),\widetilde{x}'(\theta(s)))\,ds.$$

The mapping A_0 is completely continuous by the Ascoli-Arzelá theorem, while A_1 is a contraction with respect to a suitable norm on $C^1[0,T]$ as shows the following lemma.

Lemma 1.1. *Suppose $0\leq\gamma<1$. Then, for each $\eta>\max\{(\alpha+\beta)/(1-\gamma),\alpha+\beta+\gamma\}$, A_1 is a contraction on K_R with respect to the norm*

$$\|x\|_{1,\eta}=\max\left\{\|x\|_{0,\eta},\|x'\|_{0,\eta}\right\}$$

on $C^1[0,T]$, where

$$\|x\|_{0,\eta}=\max_{[0,T]}(|x(t)|\exp(-\eta t)).$$

Proof. Let $x,y\in K_R$. Using $\theta(t)\leq t$, we obtain

$$|A_1(x)(t)-A_1(y)(t)|\leq\alpha\int_0^t|x(s)-y(s)|\,ds$$

$$+\beta\int_0^t|\widetilde{x}(\theta(s))-\widetilde{y}(\theta(s))|\,ds+\gamma\int_0^t|\widetilde{x}'(\theta(s))-\widetilde{y}'(\theta(s))|\,ds$$

$$\leq\alpha\int_0^t|x(s)-y(s)|\exp(-\eta s)\exp(\eta s)\,ds$$

$$+\beta\int_0^t |\widetilde{x}(\theta(s))-\widetilde{y}(\theta(s))|\exp(-\eta\theta(s))\exp(\eta\theta(s))\,ds$$

$$+\gamma\int_0^t |\widetilde{x}'(\theta(s))-\widetilde{y}'(\theta(s))|\exp(-\eta\theta(s))\exp(\eta\theta(s))\,ds$$

$$\leq\left[(\alpha+\beta)\eta^{-1}\|x-y\|_{0,\eta}+\gamma\eta^{-1}\|x'-y'\|_{0,\eta}\right]\exp(\eta t).$$

It follows that

$$\|A_1(x)-A_1(y)\|_{0,\eta}\leq(\alpha+\beta+\gamma)\eta^{-1}\|x-y\|_{1,\eta}.$$

Similarly

$$\left|A_1(x)'(t)-A_1(y)'(t)\right|\leq\alpha|x(t)-y(t)|$$

$$+\beta|\widetilde{x}(\theta(t))-\widetilde{y}(\theta(t))|+\gamma|\widetilde{x}'(\theta(t))-\widetilde{y}'(\theta(t))|$$

$$\leq\alpha\int_0^t|x'(s)-y'(s)|\,ds+\beta\int_0^{\theta(t)}|\widetilde{x}'(s)-\widetilde{y}'(s)|\,ds$$

$$+\gamma|\widetilde{x}'(\theta(t))-\widetilde{y}'(\theta(t))|\leq(\alpha+\beta)\int_0^t|x'(s)-y'(s)|\,ds$$

$$+\gamma|\widetilde{x}(\theta(t))-\widetilde{y}(\theta(t))|\leq\left[(\alpha+\beta)\eta^{-1}+\gamma\right]\|x'-y'\|_{0,\eta}\exp(\eta t).$$

Hence

$$\left\|A_1(x)'-A_1(y)'\right\|_{0,\eta}\leq\left[(\alpha+\beta)\eta^{-1}+\gamma\right]\|x-y\|_{1,\eta}.$$

Therefore

$$\|A_1(x)-A_1(y)\|_{1,\eta}\leq L\|x-y\|_{1,\eta}, \tag{1.7}$$

where

$$L=\max\left\{(\alpha+\beta+\gamma)\eta^{-1},(\alpha+\beta)\eta^{-1}+\gamma\right\}. \tag{1.8}$$

□

There is a remarkable case when in spite of the neutral variable, we still can work with completely continuous mappings: the case when the *step method* applies. We are in this case if

$$\theta(t) < t \quad \text{on } (0,T] \text{ and } \inf\{t > 0;\ \theta(t) > 0\} > 0. \tag{1.9}$$

By using the step method, the solving of (1.1)-(1.2) is reduced to that of a finite number of Cauchy problems for equations without deviated arguments. To explain this, let $t_0 = 0$ and

$$t_n = \inf\{t \in (t_{n-1}, T];\ \theta(t) > t_{n-1}\}, \quad n = 1, 2, ..., \tag{1.10}$$

where we set $t_n = T$ in case that the infimum is taken over the empty set. Obviously, (t_n) is a bounded nondecreasing sequence and if $t_m = T$ for some m, then $t_n = T$ for all $n \geq m$. In addition, if $t_n < T$, then

$$\theta(t_n) = t_{n-1} \text{ and } \theta(t) \leq t_{n-1} \text{ for } t_{n-1} \leq t \leq t_n. \tag{1.11}$$

The second inequality in (1.9) implies $t_0 < t_1 \leq T$, while the first one assures the strict monotonicity $t_{n-1} < t_n$ whenever $t_{n-1} < T$, and also the existence of a $k \geq 1$ with $t_{k-1} < t_k = T$. Indeed, otherwise, we should have $t_0 < t_1 < ... < t_n < ... < T$. If we denote $t_* = \lim_{n\to\infty} t_n$, then $0 < t_* \leq T$ and $\theta(t_*) = t_*$, which contradicts (1.9). Thus, there exists a finite partition of $[0,T]$, say

$$0 = t_0 < t_1 < ... < t_{k-1} < t_k = T.$$

A solution to (1.1)-(1.2) will be defined step by step, on each subinterval $[-\tau, t_n]$, $n = 1, 2, ..., k$. Denote $x_0 = \phi$ and let $x_{n+1} \in C^1[-\tau, t_{n+1}]$ be a prolongation of $x_n \in C^1[-\tau, t_n]$ by a solution of the following problem

$$\begin{cases} x'(t) = f(t, x(t), x_n(\theta(t)), x_n'(\theta(t))), & t_n \leq t \leq t_{n+1}, \\ x(t_n) = a_n, \end{cases} \tag{1.12}$$

where $a_n = x_n(t_n)$, $n = 0, 1, ..., k-1$. It is clear that x_k will represent a solution of (1.1)-(1.2). Thus, at each step n, we have to solve (1.12), or equivalently, to find a fixed point of the completely continuous mapping $A_n : C[t_n, t_{n+1}] \to C[t_n, t_{n+1}]$,

$$A_n(x)(t) = a_n + \int_{t_n}^{t} f(s, x(s), x_n(\theta(s)), x_n'(\theta(t)))\, ds. \qquad (1.13)$$

Organization of the paper

In Section 2, we discuss the initial value problem for (1.1) in case that the step method applies. In Section 3, the same problem is studied when the step method does not apply. In Section 4, we obtain minimal and maximal solutions to the Cauchy problem. We use fixed point theorems (Schauder, Krasnoselskii, Leray-Schauder) and monotone iterative techniques.

Notice that by a somewhat similar approach, we discussed in [6] the initial value problem for a delay integral equation modelling infectious disease (see also [11]). The results are new and they improve and complement the existing literature (see [10] for example, for related topics).

We finish this introductory section by some abstract existence principles.

Fixed point theory

Theorem 1.2. *(Schauder) Let X be a Banach space and $D \subset X$ nonempty bounded closed convex. Suppose $A : D \to D$ is compact (i.e. continuous with $A(D)$ relatively compact). Then A has at least one fixed point.*

Theorem 1.3. *(Krasnoselskii) Let X be a Banach space and $D \subset X$ nonempty bounded closed convex. Suppose $A_0 : D \to X$ is compact, $A_1 : D \to X$ is a contraction and that $A_0(x) + A_1(y) \in D$ for all $x, y \in D$. Then $A_0 + A_1$ has at least one fixed point.*

Theorem 1.4. *(Leray-Schauder) Let X be a Banach space, $K \subset X$ closed convex and $U \subset K$ bounded open in K. Suppose $A : \bar{U} \to K$ is compact and*

$$(1-\lambda)x_0 + \lambda A(x) \neq x \quad \text{for all } x \in \partial U \text{ and } \lambda \in [0,1],$$

for some $x_0 \in U$. Then A has at least one fixed point in U.

2. Existence via the step method

Let us list our assumptions:

(a1) $\theta \in C[0,T]$, $-\tau \leq \theta$ and (1.9) (*step condition*) holds.

(a2) $\phi \in C^1[-\tau, 0]$ and (1.3) (*sewing condition*) is satisfied.

(a3) $f(t,x,y,z)$ is nonnegative and continuous on $D = [0,T]\times[a,R]\times[m,M]\times[m',\infty)$, where $a < R \leq \infty$, $m = \min_{[-\tau,0]}\phi(t)$, $M = \max\{R, \max_{[-\tau,0]}\phi(t)\}$ and $m' = \min\{0, \min_{[-\tau,0]}\phi'(t)\}$.

(a4) $f(t,x,y,z) \leq \alpha(t)\beta(x)\gamma(y,z)$ on D, where α, β, γ are continuous, $\alpha \geq 0$, $\beta > 0$, $\gamma \geq 0$ and

$$\sup_{[m,M]\times[m',\infty)} \gamma(y,z) \cdot \int_0^T \alpha(t)\,dt < \int_0^R \frac{du}{\beta(u)} \tag{2.1}$$

(*Wintner type condition*).

We make the convention that when the left side in (2.1) equals ∞, then the right side is ∞ too.

Theorem 2.1. *Suppose (a1)-(a4) are satisfied. Then (1.1)-(1.2) has at least one solution $x \in C^1[-\tau,T]$ with $a \leq x < R$ and $x' \geq 0$ on $[0,T]$.*

Proof. First we prove that for each $x \in C^1[-\tau, t_n]$ with $a \leq x \leq R$ and satisfying (1.1) and (1.2) on $[0,t_n]$, there exists $R_n \in [a,R)$ depending only on the restriction of x to $[-\tau, t_{n-1}]$, such that $x \leq R_n$ on $[0,t_n]$.

Indeed, by (a4), we have

$$x'(t) \leq \alpha(t)\beta(x(t))\gamma(x(\theta(t)), x'(\theta(t))), \quad 0 \leq t \leq t_n.$$

Divide by $\beta(x(t))$ and integrate from 0 to t_n to obtain

$$\int_a^{x(t_n)} \frac{du}{\beta(u)} = \int_0^{t_n} \frac{x'(t)}{\beta(x(t))}dt \leq M_n \int_0^{t_n} \alpha(t)\,dt,$$

where $M_n = \max_{[0,t_{n-1}]} \gamma(x(\theta(t)), x'(\theta(t)))$. By (2.1), this implies

$$\int_a^{x(t_n)} \frac{du}{\beta(u)} \leq M_n \int_0^{t_n} \alpha(t)\,dt < \int_a^R \frac{du}{\beta(u)}.$$

Thus $x(t_n) \leq R_n < R$, where

$$M_n \int_0^{t_n} \alpha(t)\, dt = \int_a^{R_n} \frac{du}{\beta(u)}. \tag{2.2}$$

Since x is nondecreasing on $[0, t_n]$, we have $x(t) \leq x(t_n) \leq R_n$ for all $t \in [0, t_n]$, as claimed.

Now suppose we have already defined $x_n \in C^1[-\tau, t_n]$, a solution of (1.1)-(1.2) on $[-\tau, t_n]$, with $a \leq x_n \leq R$ and $x_n' \geq 0$ on $[0, t_n]$. Then $x_n \leq R_n < R$ and

$$\int_a^{a_n} \frac{du}{\beta(u)} \leq M_n \int_0^{t_n} \alpha(t)\, dt, \tag{2.3}$$

where R_n is given by (2.2), with $M_n = \max_{[0,t_{n-1}]} \gamma(x_n(\theta(t)), x_n'(\theta(t)))$.

Next we try to extend x_n to a solution $x_{n+1} \in C^1[-\tau, t_{n+1}]$ satisfying $a \leq x_{n+1} \leq R$ and $x_{n+1}' \geq 0$ on $[0, t_{n+1}]$. Let R_{n+1} be given by (2.2), for $M_{n+1} = \max_{[0,t_n]} \gamma(x_n(\theta(t)), x_n'(\theta(t)))$. It is clear that $M_n \leq M_{n+1}$ and $R_n \leq R_{n+1} < R$. Choose a finite $R' \in (R_{n+1}, R]$ and define

$$K_n = \{x \in C[t_n, t_{n+1}];\ a \leq x\}, \quad U_n = \{x \in K_n;\ x < R'\}$$

and

$$A_n : \overline{U}_n \to K_n, \quad A_n(x)(t) = a_n + \int_{t_n}^{t} f(s, x(s), x_n(\theta(s)), x_n'(\theta(s)))\, ds.$$

Obviously, $K_n \subset C[t_n, t_{n+1}]$ is closed and convex, $U_n \subset K_n$ is bounded and open in K_n, the constant function a_n belongs to U_n (because $a_n \leq R_n < R'$) and A_n is completely continuous. Also, if x is a fixed point of A_n, then $x(t_n) = a_n$, $x'(t_n) = x_n'(t_n)$ and the prolongation x_{n+1} of x_n by x will represent a solution of (1.1)-(1.2) on $[-\tau, t_{n+1}]$ satisfying $a \leq x_{n+1} \leq R$ and $x_{n+1}' \geq 0$ on $[0, t_{n+1}]$.

The existence of a fixed point of A_n will follow by the Leray-Schauder principle if the boundary condition

$$x \neq (1-\lambda) a_n + \lambda A_n(x) \quad \text{for all } x \in \partial U_n,\ \lambda \in (0,1) \tag{2.4}$$

holds. To check it, suppose $x \in \overline{U}_n$ satisfies $x = (1-\lambda)\, a_n + \lambda A_n(x)$ for some $\lambda \in (0,1)$. Then, $x(t_n) = a_n$ and

$$x'(t) = \lambda f\left(t, x(t), x_n(\theta(t)), x_n'(\theta(t))\right) \quad \text{for all } t \in [t_n, t_{n+1}].$$

As above, we obtain

$$\int_{a_n}^{x(t_{n+1})} \frac{du}{\beta(u)} \le M_{n+1} \int_{t_n}^{t_{n+1}} \alpha(t)\, dt.$$

Taking into account (2.3) and $M_n \le M_{n+1}$, we deduce

$$\int_{a}^{x(t_{n+1})} \frac{du}{\beta(u)} \le M_{n+1} \int_{0}^{t_{n+1}} \alpha(t)\, dt.$$

Hence $x(t_{n+1}) \le R_{n+1}$ and consequently, $x \le R_{n+1} < R'$ on $[t_n, t_{n+1}]$. Thus, $x \notin \partial U_n$ and (2.4) is proved. □

Remark 2.1. *The conclusion of Theorem 2.1 remains true if instead of (a4) the following condition is satisfied:*
(a4') $f(t,x,y,z) \le \beta(x)\, \delta(t,y,z)$ on D, where $\beta > 0, \delta \ge 0$ and

$$T \cdot \sup_{[0,T]\times[m,M]\times[m',\infty)} \delta(t,y,z) < \int_a^R \frac{du}{\beta(u)}. \tag{2.5}$$

Remark 2.2. *Suppose $R = \infty$ and that in (a4'), $\beta(u) = u + c$, where $c \ge 0$. In this case, (2.5) trivially holds since its right side equals to infinity. Moreover, a fixed point of A_n follows directly by Schauder's fixed point theorem. Indeed, if $R = \infty$, the map A_n can be defined on the entire K_n and $A_n(K_n) \subset K_n$. In addition, for $\eta > 0$ and $x \in K_n$, we have*

$$0 \le A_n(x)(t) \le a_n + \widetilde{M}_n \int_{t_n}^{t} (x(s) + c)\, ds$$

$$= a_n + c\widetilde{M}_n (t - t_n) + \widetilde{M}_n \int_{t_n}^{t} x(s)\, ds$$

$$\le \tilde{a}_n + \widetilde{M}_n \eta^{-1} \|x\|_{0,\eta} \exp(-\eta t), \quad t_n \le t \le t_{n+1},$$

where $\widetilde{M}_n = \max_{[t_n,t_{n+1}]} \delta(t, x_n(\theta(t)), x_n'(\theta(t)))$ *and* $\widetilde{a}_n = a_n + c\widetilde{M}_n(t_{n+1} - t_n)$. *In consequence,*

$$\|A_n(x)\|_{0,\eta} \le \widetilde{a}_n + \widetilde{M}_n \eta^{-1} \|x\|_{0,\eta} \quad (x \in K_n).$$

Thus, if we choose $\eta > \widetilde{M}_n$ *and* $R' \ge \widetilde{a}_n / \left(1 - \widetilde{M}_n \eta^{-1}\right)$, *then Schauder's theorem applies on* $\{x \in K_n;\ \|x\|_{0,\eta} \le R'\}$.

Remark 2.3. *Suppose* $R = \infty$ *and that a more restrictive condition than (a4') holds, namely*
(a4") $|f(t,x,y,z) - f(t,\bar{x},y,z)| \le L(t,y,z)|x - \bar{x}|$ *on* D,
where L *is continuous and nonnegative.*

From (a4"),

$$f(t,x,y,z) \le L(t,y,z)(x-a) + f(t,a,y,z) \le \delta(t,y,z)\,x,$$

where $\delta(t,y,z) = \max\{L(t,y,z), f(t,a,y,z)/a\}$. *Hence we are in the frame of Remark 2.2. In addition, the initial value problem has a unique solution and at each step, the unique fixed point of* A_n *can be obtained by means of the contraction principle. Indeed, for* $\eta > 0$ *and* $x, y \in K_n$, *we have*

$$|A_n(x)(t) - A_n(y)(t)| \le \int_{t_n}^{t} L(s, x_n(\theta(s)), x_n'(\theta(s)))\,|x(s) - y(s)|\,ds$$

$$\le \overline{M}_n \int_{t_n}^{t} |x(s) - y(s)|\,ds \le \overline{M}_n \eta^{-1} \|x - y\|_{0,\eta} \exp(-\eta t),$$

where $\overline{M}_n = \max_{[t_n,t_{n+1}]} L(t, x_n(\theta(t)), x_n'(\theta(t)))$. *Now our claim follows if we choose* $\eta > \overline{M}_n$.

3. Existence without the step condition

The assumptions for this section are as follows:

(A1) $\theta \in C[0,T]$ and $-\tau \le \theta(t) \le t$.

(A2) = (a2).

(A3) $f(t,x,y,z)$ is nonnegative on D and admits the decomposition (1.5), where f_0, f_1 are continuous and f_1 satisfies the Lipschitz condition (1.6) for some $\alpha, \beta \geq 0$ and $\gamma \in [0,1)$.

(A4) $f(t,x,y,z) \leq \alpha(t)\beta(x)$ on D, where α, β are continuous, $\alpha \geq 0$, $\beta > 0$ and

$$\int_0^T \alpha(t)\,dt < \int_a^R \frac{du}{\beta(u)}. \tag{3.1}$$

Theorem 3.1. *Suppose (A1)-(A4) are satisfied. Then (1.1)-(1.2) has at least one solution $x \in C^1[-\tau, T]$ with $a \leq x \leq R$ and $x' \geq 0$ on $[0,T]$. Moreover, any such solution satisfies*

$$x(t) \leq R_*, \quad 0 \leq t \leq T, \tag{3.2}$$

where $R_* < R$ is so that

$$\int_0^T \alpha(t)\,dt = \int_a^{R_*} \frac{du}{\beta(u)}. \tag{3.3}$$

Proof. With the notations of Section 1, the mapping $A : K_R \to K$ is the sum $A_0 + A_1$, where A_0 is completely continuous and A_1 is a contraction with respect to a suitable norm on $C^1[0,T]$.

We claim that (3.2) holds for each solution $x \in K_R$ to

$$x = (1-\lambda)a + \lambda A(x) \quad (\lambda \in [0,1]). \tag{3.4}$$

Once the claim is satisfied the result follows from the Leray-Schauder principle applied to $A : \overline{U} \to K$, where $U = \{x \in K;\ x < R' \text{ on } [0,T]\}$ and R' is any number such that $R_* < R' \leq R$.

To prove the claim, let $x \in K_R$ be any solution of (3.4). Then

$$x'(t) = \lambda f(t, x(t), x(\theta(t)), x'(\theta(t))) \leq \alpha(t)\beta(x(t)) \quad \text{on } [0,T].$$

It follows that

$$\int_a^{x(t)} \frac{du}{\beta(u)} = \int_0^t \frac{x'(s)}{\beta(x(s))} ds \leq \int_0^t \alpha(s)\,ds.$$

This together with (3.3) implies (3.2). $\square$

Suppose now that instead of (A4), the following condition holds.

(A4') $|f_0(t,x,y)| \leq \alpha_0 x + \beta_0 |y| + \delta$ on D, where α_0, β_0 and δ are nonnegative constants.

Theorem 3.2. *Suppose (A1)-(A3), (A4') are satisfied and $R = \infty$. Then (1.1)-(1.2) has at least one solution $x \in C^1[-\tau, T]$ such that $a \leq x$ and $x' \geq 0$ on $[0,T]$.*

Proof. Since $R = \infty$, we may define $A : K \to K$ and, as above, $A = A_0 + A_1$, where A_0 is completely continuous and A_1 is a contraction with respect to the norm $\|.\|_{1,\eta}$ on $C^1[0,T]$, for $\eta > \max\{(\alpha+\beta)/(1-\gamma), \alpha+\beta+\gamma\}$.

We claim that there exists η sufficiently large and a finite $R' > 0$ such that

$$x, y \in K, \ \|x\|_{1,\eta} \leq R', \ \|y\|_{1,\eta} \leq R' \text{ imply } \|A_0(x) + A_1(y)\|_{1,\eta} \leq R'. \tag{3.5}$$

Once the claim is proved the result is a consequence of the Krasnoselskii fixed point theorem.

To establish (3.5) we need the following estimates:

$$|A_0(x)(t)| \leq a + \alpha_0 \int_0^t x(s)\,ds + \beta_0 \int_0^t \widetilde{x}(\theta(s))\,ds + \delta t$$

$$= a + \alpha_0 \int_0^t x(s)\,ds + \beta_0 \int_{(0<\theta(s))} x(\theta(s))\,ds + \beta_0 \int_{(\theta(s)<0)} \phi(\theta(s))\,ds + \delta t$$

$$\leq a + (\alpha_0 + \beta_0)\eta^{-1} \|x\|_{0,\eta} \exp(\eta t) + \beta_0 \|\phi\|_0 T + \delta T$$

$$= (\alpha_0 + \beta_0)\eta^{-1} \|x\|_{0,\eta} \exp(\eta t) + c_0'.$$

Also

$$\left|A_0(x)'(t)\right| \leq \alpha_0 x(t) + \beta_0 \widetilde{x}(\theta(t)) + \delta$$

$$= \alpha_0 \int_0^t x'(s)\,ds + \beta_0 \int_0^{\theta(t)} \widetilde{x}'(s)\,ds + (\alpha_0 + \beta_0)a + \delta$$

$$\leq (\alpha_0 + \beta_0)\eta^{-1} \|x'\|_{0,\eta} \exp(\eta t) + \beta_0 \|\phi'\|_0 T + (\alpha_0 + \beta_0)a + \delta$$

$$= (\alpha_0 + \beta_0)\eta^{-1} \|x'\|_{0,\eta} \exp(\eta t) + c_0''.$$

Thus

$$\|A_0(x)\|_{1,\eta} \le (\alpha_0 + \beta_0)\,\eta^{-1}\,\|x\|_{1,\eta} + c_0.$$

This together with (1.7) yields

$$\|A_0(x) + A_1(y)\|_{1,\eta} \le (\alpha_0 + \beta_0)\,\eta^{-1}\,\|x\|_{1,\eta} + L\,\|y\|_{1,\eta} + c,$$

where L is given by (1.8). It is clear that if η is sufficiently large, then $(\alpha_0 + \beta_0)\,\eta^{-1} + L < 1$ and we may find $R' > 0$ such that (3.5) holds. □

4. Minimal and maximal solutions

Theorem 4.1. *Suppose (a1)-(a3) are satisfied and $w \in C^1[0,T]$, $a \le w \le R$, is an upper solution, i.e.*

$$w'(t) \ge f\left(t, w(t), \tilde{w}(\theta(t)), \tilde{w}'(\theta(t))\right), \quad 0 \le t \le T. \tag{4.1}$$

In addition assume that

$$f(t, x_1, y_1, z_1) \le f(t, x_2, y_2, z_2) \tag{4.2}$$

for $x_1 \le x_2 \le w(t)$, $y_1 \le y_2 \le \tilde{w}(\theta(t))$ and $z_1 \le z_2 \le \tilde{w}'(\theta(t))$. Then we may define $\underline{x}_0 = \bar{x}_0 = \phi$,

$$\underline{x}_{n+1}(t) = \begin{cases} \underline{x}_n(t) & \text{on } [-\tau, t_n] \\ \lim\limits_{j\to\infty} u_{nj}(t) & \text{on } [t_n, t_{n+1}] \end{cases} \tag{4.3}$$

and

$$\bar{x}_{n+1}(t) = \begin{cases} \bar{x}_n(t) & \text{on } [-\tau, t_n] \\ \lim\limits_{j\to\infty} v_{nj}(t) & \text{on } [t_n, t_{n+1}], \end{cases} \tag{4.4}$$

where $u_{n0}(t) \equiv a$, $v_{n0}(t) = w(t)$, $u_{nj} = \underline{A}_n(u_{nj-1})$, $v_{nj} = \bar{A}_n(v_{nj-1})$,

$$\underline{A}_n(x)(t) = \underline{x}_n(t_n) + \int_{t_n}^{t} f\left(s, x(s), \underline{x}_n(\theta(s)), \underline{x}_n'(\theta(s))\right) ds,$$

$$\bar{A}_n(x)(t) = \bar{x}_n(t_n) + \int_{t_n}^{t} f\left(s, x(s), \bar{x}_n(\theta(s)), \bar{x}_n(\theta(s))\right) ds$$

$(t \in [t_n, t_{n+1}])$, $j = 1, 2, ..., n = 0, 1, ..., k-1$. *Moreover,* $\underline{x} = \underline{x}_k$ *and* $\bar{x} = \bar{x}_k$ *are the minimal and maximal solutions of (1.1)-(1.2) satisfying* $a \leq x \leq w$ *on* $[0, T]$,

$$a \leq \underline{x} \leq \bar{x} \leq w, \quad 0 \leq \underline{x}' \leq \bar{x}' \leq w' \quad on \ [0, T],$$

$$u_{n0} \leq u_{n1} \leq ... \leq u_{nj} \leq ... \quad on \ [t_n, t_{n+1}]$$

$$v_{n0} \geq v_{n1} \geq ... \geq v_{nj} \geq ... \quad on \ [t_n, t_{n+1}]$$

and

$$u_{nj}(t) \to \underline{x}(t), \quad v_{nj}(t) \to \bar{x}(t) \quad as \ j \to \infty$$

uniformly on $[t_n, t_{n+1}]$ $(n = 0, 1, ..., k-1)$.

Proof. Suppose we have already defined $\underline{x}_n$ and $\bar{x}_n$ such that

$$a \leq \underline{x}_n \leq \bar{x}_n \leq w \text{ and } 0 \leq \underline{x}_n' \leq \bar{x}_n' \leq w' \text{ on } [0, t_n]. \tag{4.5}$$

First we prove that

$$a \leq u_{nj} \leq v_{nj} \leq w \quad \text{on } [t_n, t_{n+1}], \tag{4.6}$$

by induction after j. For $j = 0$, (4.6) trivially holds. Assume (4.6) is true for some j. Then, using also (4.2), we easily see that

$$a \leq \underline{A}_n(a) \leq \underline{A}_n(u_{nj}) \leq \underline{A}_n(v_{nj}) \leq \bar{A}_n(v_{nj}) \leq \bar{A}_n(w) \leq w,$$

which shows that (4.6) also holds for $j+1$. Thus (4.6) is true for all $j \geq 0$. Since $\underline{A}_n$ and $\bar{A}_n$ are completely continuous, the sequences $(u_{nj})_{j\geq 0}$ and $(v_{nj})_{j\geq 0}$ will contain convergent subsequences. Due to their monotonicity, the entire sequences will converge on $[t_n, t_{n+1}]$, which justifies (4.3) and (4.4). Also, by (4.6), $a \leq \underline{x}_{n+1} \leq \bar{x}_{n+1} \leq w$ on $[0, t_{n+1}]$. Then

$$0 \leq \underline{x}_{n+1}'(t) = f(t, \underline{x}_{n+1}(t), \underline{x}_n(\theta(t)), \underline{x}_n'(\theta(t)))$$

$$\leq f(t, \bar{x}_{n+1}(t), \bar{x}_n(\theta(t)), \bar{x}_n'(\theta(t))) = \bar{x}_{n+1}'(t)$$

$$\leq f(t, w(t), \widetilde{w}(\theta(t)), \widetilde{w}'(\theta(t))) \leq w'(t).$$

Hence (4.5) also holds for $j+1$. $\square$

The next result is about the equality $\underline{x} = \bar{x}$ in Theorem 3.1.

Theorem 4.2. *Suppose the assumptions of Theorem 3.1 are satisfied. In addition assume $a > 0$ and that there exists a function $\chi : [a_w, 1) \to \mathbf{R}$, where $a_w = a \min_{[0,T]} 1/w(t)$, such that for all $\rho \in [a_w, 1)$, $t \in [0,T]$, $x \in [a, w(t)]$, $y \in [m, M]$ and $z \in \{m', \infty)$, one has*

$$1 \geq \chi(\rho) > \rho \quad \text{and} \quad f(t, \rho x, y, z) \geq \chi(\rho) f(t, x, y, z). \tag{4.7}$$

Then $\underline{x} = \bar{x}$ is the unique solution of (1.1)-(1.2) satisfying $a \leq x \leq w$ on $[0,T]$.

Proof. We show succesively that $\underline{x}_n = \bar{x}_n$ for $n = 0, 1, \ldots, k$. For $n = 0$, this trivially holds. Assume $\underline{x}_n = \bar{x}_n$ for some n. Then $\underline{A}_n = \bar{A}_n$. Clearly, the restrictions of $\underline{x}_{n+1}$, $\bar{x}_{n+1}$to $[t_n, t_{n+1}]$ represent the minimal and maximal fixed point of $B_n := \bar{A}_n$ satisfying $a \leq x \leq w$ on $[t_n, t_{n+1}]$. To show that $\underline{x}_{n+1} = \bar{x}_{n+1}$ on $[t_n, t_{n+1}]$, let $\rho_0 = \min_{[t_n, t_{n+1}]}(\underline{x}_{n+1}(t) / \bar{x}_{n+1}(t))$. We have $\rho_0 \in [a_w, 1]$. We claim that $\rho_0 = 1$. Assume $\rho_0 < 1$. Since $\underline{x}_{n+1}(t) \geq \max\{a, \rho_0 \bar{x}_{n+1}(t)\} = \rho_0 \max\{a/\rho_0, \bar{x}_{n+1}(t)\} \geq a$ on $[t_n, t_{n+1}]$, we get

$$\underline{x}_{n+1} = B_n(\underline{x}_{n+1}) \geq B_n(\rho_0 \max\{a/\rho_0, \bar{x}_{n+1}\})$$

$$\geq \chi(\rho_0) B_n(\max\{a/\rho_0, \bar{x}_{n+1}\}) \geq \chi(\rho_0) B_n(\bar{x}_{n+1}) = \chi(\rho_0) \bar{x}_{n+1},$$

on $[t_n, t_{n+1}]$. It follows $\rho_0 \geq \chi(\rho_0)$, a contradiction. Therefore $\rho_0 = 1$ and so $\underline{x}_{n+1} = \bar{x}_{n+1}$ on $[t_n, t_{n+1}]$. $\square$

Remark 4.1. *For example, we may take $\chi(\rho) = \rho^\alpha$, where $\alpha \in (0,1)$, in case that $f(t,x,y,z)$ is of the form $x^\alpha g(t,y,z)$. Also, $\chi(\rho) = \log(1 + a\rho) / \log(1+a)$ for $f(t,x,y,z) = g(t,y,z)\log(1+x)$.*

Theorem 4.3. *Suppose (A1)-(A3) are satisfied and that $w \in C^1[0,T]$, $a \leq w \leq R$, is an upper solution. In addition assume that (4.2) holds. Denote*

$$U_0(t) \equiv a, \; V_0(t) = w(t), \; U_{n+1} = A(U_n) \;\; \text{and} \;\; V_{n+1} = A(V_n)$$

($t \in [0,T]$), $n = 0, 1, \ldots$. Then

$$a = U_0 \le U_1 \le \ldots \le U_n \le \ldots \le V_n \le \ldots \le V_1 \le V_0 = w, \tag{4.8}$$

$$0 \le U_1' \le \ldots \le U_n' \le \ldots \le V_n' \le \ldots \le V_1' \le w' \tag{4.9}$$

on $[0,T]$. Also, the following limits exist

$$\underline{x}(t) = \lim_{n\to\infty} U_n(t), \quad \bar{x}(t) = \lim_{n\to\infty} V_n(t) \tag{4.10}$$

uniformly on $[0,T]$. Moreover, $\underline{x}$, $\bar{x}$ are the minimal and maximal solutions to (1.1)-(1.2) in K_R satisfying $x \le w$ on $[0,T]$.

Proof. From $a \le w$ we see that $a \le A(a) \le A(w) \le w$ and $0 \le A(a)' \le A(w)' \le w'$, i.e. $U_0 \le U_1 \le V_1 \le V_0$ and $0 \le U_1' \le V_1' \le w'$. Further, (4.8) and (4.9) follow succesively. Let α be the Kuratowski measure of noncompactness on the space $C^1[0,T]$ endowed with the norm $\|.\|_{1,\eta}$. Since

$$(U_n)_{n\ge 1} = A\left((U_n)_{n\ge 0}\right),$$

A_0 is completely continuous and A_1 is a contraction, we have

$$\alpha\left((U_n)_{n\ge 1}\right) = \alpha\left(A\left((U_n)_{n\ge 0}\right)\right) \le \alpha\left(A_0\left((U_n)_{n\ge 0}\right)\right)$$

$$+\alpha\left(A_1\left((U_n)_{n\ge 0}\right)\right) = \alpha\left(A_1\left((U_n)_{n\ge 0}\right)\right) \le L\alpha\left((U_n)_{n\ge 0}\right),$$

where L is given by (1.8). Recall that $L < 1$. In consequence, $\alpha\left((U_n)_{n\ge 0}\right) = 0$. Thus $(U_n)_{n\ge 0}$ contains a convergent subsequence. By the monotonicity, the entire sequence (U_n) will converge. Similarly, (V_n) is convergent. □

Remark 4.2. *Let (A1)-(A2) be satisfied. In addition assume that the following condition holds instead of (A3):*

(A3') $f(t,x,y,z)$ is nonnegative and continuous on D.

Then Theorem 3.2 is still true with the meaning that $\underline{x}$ and $\bar{x}$ are weak solutions of (1.1), i.e. $\underline{x}$, $\bar{x} \in AC[0,T]$ (are absolutely continuous) and satisfy (1.1) *almost everywhere on* $[0,T]$. *Indeed, by (4.8), (4.9) and the Beppo-Levi theorem, there exist* $\underline{x}, \underline{y} \in L^1(0,T)$ *with*

$$U_n(t) \to \underline{x}(t), \quad U_n'(t) \to \underline{y}(t) \quad \text{on } [0,T],$$

$$U_n \to \underline{x} \quad \text{and} \quad U_n' \to \underline{y} \quad \text{in } L^1(0,T).$$

From

$$U_n(t) = a + \int_0^t U_n'(s)\,ds$$

we then derive

$$\underline{x}(t) = a + \int_0^t \underline{y}(s)\,ds,$$

which shows that $\underline{x} \in AC[0,T]$ *and* $\underline{x}'(t) = \underline{y}(t)$ *for a.e.* $t \in [0,T]$. *Letting* $n \to \infty$ *in*

$$U_{n+1}'(t) = f\left(t, U_n(t), \widetilde{U}_n(\theta(t)), \widetilde{U}_n'(\theta(t))\right),$$

we obtain

$$\underline{y}(t) = f\left(t, \underline{x}(t), \widetilde{\underline{x}}(\theta(t)), \widetilde{\underline{y}}(\theta(t))\right) \quad \text{for all } t \in [0,T],$$

i.e.

$$\underline{x}'(t) = f\left(t, \underline{x}(t), \widetilde{\underline{x}}(\theta(t)), \widetilde{\underline{x}}'(\theta(t))\right) \quad \text{a.e. } t \in [0,T].$$

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