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GLOBAL EXISTENCE AND ESTIMATES FOR SOLUTIONS OF CERTAIN HIGHER ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper results on the global existence and estimates for solutions of general higher order differential equations are established. The main tools employed in our analysis are based on the applications of the Leray-Schauder alternative and certain integral inequalities which provide explicit bounds on the unknown functions.

1. Introduction

Let $r_i(t) > 0$, $i = 1, 2, \ldots, n-1$ and $x(t)$ be sufficiently smooth functions on $I = [t_0, T]$, $t_0 \geq 0$, $T > 0$. Then for $x(t)$ the r-derivatives are defined as follows

$$D_r^{(0)} x = x,$$

$$D_r^{(k)} x = r_k (D_r^{(k-1)} x)', \quad k = 1, 2, \ldots, n-1, \quad \left(' = \frac{d}{dt} = D \right),$$

$$D_r^{(n)} x = (D_r^{(n-1)} x)'.$$

In this paper we consider the n-th order ($n > 1$) differential equation of the form

$$(P) \qquad D_r^{(n)} y = f(t, D_r^{(0)} y, D_r^{(1)} y, \ldots, D_r^{(n-1)} y),$$

with the given initial conditions

$$(P_0) \qquad D_r^{(m)} y(t_0) = c_m, \quad m = 0, 1, 2, \ldots, n-1,$$

where $f : I \times R^n \to R$ is a continuous function, R denotes the set of real numbers and c_m are given real constants. We define $B = C^{n-1}(I) = C^{n-1}(I, R)$ to be the Banach space of functions u such that $D_r^{(n-1)}u$ is continuous on I endowed with norm

$$||u|| = \max\{|D_r^{(0)}u|_0, |D_r^{(1)}u|_0, \ldots, |D_r^{(n-1)}u|_0\},$$

where $|u|_0 = \max\{|u(t)| : t \in I\}$. In the past two decades there has been a great deal of interest in the study of oscillatory and asymptotic behavior of the solutions of equations of the form (P) and its various special versions. We choose to refer here the papers by Fink and Kusano [3], Kusano and Trench [4], Pachpatte [7,8], Philos [14,15], Philos and Staikos [16] and Trench [17,18] and the references given therein.

As noted by Kusano and Trench [4,p.381], it seems that very little is known about the global existence and various other properties of the solutions of such equations in the literature. The main purpose of this paper is to establish some results concerning the global existence and estimates for solutions of (P_0) which in turn contains in the special cases a number of higher order differential equations studied by many authors by using different techniques. The main tools employed in our analysis are based on the applications of the Leray-Schauder alternative and certain integral inequalities which provide explicit bounds on the unknown functions.

2. Global existence of solutions

In order to obtain our result on the global existence of solutions of $(P) - (P_0)$, we need the following theorem, which is a version of the topological transversality theorem given by A. Granas in [2,p.61].

Theorem G. *Let B be a convex subset of a normed linear space E and assume $0 \in B$. Let $F : B \to B$ be a completely continuous operator and let*

$$U(F) = \{x \in B : x = \lambda F x \text{ for some } 0 < \lambda < 1\}.$$

Then either $U(F)$ is unbounded or F has a fixed point.

For our purposes, for any integers m and k with $0 \le m \le k \le n-1$, we introduce the function R_{mk} which is defined on I by

$$R_{mk}(t) = \begin{cases} 1, \text{ if } m = k, \\ \displaystyle\int_{t_0}^{t=s_m} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots x \\ \displaystyle x \int_{t_0}^{s_{k-1}} \frac{1}{r_k(x_k)} ds_k \ldots ds_{m+2} ds_{m+1}, \text{ if } m < k. \end{cases} \tag{2.1}$$

In particular, for any integer k with $0 \le k \le n-1$, we put

$$R_k(t) = R_{0k}(t), \quad t \ge t_0.$$

The following theorem constitute the main result of this section.

Theorem 1. *Suppose that there exists a function $a \in C(I, R_+)$, $R_+ = [0, \infty)$ such that*

$$|f(t, D_r^{(0)}y, D_r^{(1)}y, \ldots, D_r^{(n-1)}y)| \le a(t) H\left(\sum_{m=0}^{n-1} |D_r^{(m)}y|\right), \tag{2.2}$$

where $H : R_+ \to (0, \infty)$ is a continuous nondecreasing function. Then the problem $(P) - (P_0)$ has a solution y in B provided that T satisfies

$$int_{t_0}^{T} M(s) ds < \int_c^\infty \frac{ds}{H(s)}, \tag{2.3}$$

where

$$c = \sum_{m=0}^{n-1} \left[|c_m| + \sum_{k=m+1}^{n-1} |c_k| R_{mk}(T) \right], \tag{2.4}$$

in which $R_{mk}(T)$ is defined by taking $t = T$ in (2.1) and

$$M(t) = \sum_{m=0}^{n-1} \frac{1}{r_{m+1}(t)} \int_{t_0}^{t} \frac{1}{r_{m+2}(s_{m+2})} \int_{t_0}^{s_{m+2}} \frac{1}{r_{m+3}(s_{m+3})} \cdots \times \tag{2.5}$$

$$\times \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} a(s) ds ds_{n-1} \cdots \times ds_{m+3} ds_{m+2},$$

for $t \in I$.

Proof. First we establish the a-priori bounds for the problem $(P)_\lambda - (P_0)$, $\lambda \in (0,1)$, where

$$(P)_\lambda \qquad D_r^{(n)} y = \lambda f(t, D_r^{(0)} y, D_r^{(1)} y, \ldots, D_r^{(n-1)} y).$$

Let $y(t)$ be a solution of $(P)_\lambda - (P_0)$. Then $y(t)$ and its derivatives can be written as

$$D_r^{(m)} y(t) = c_m + \sum_{k=m+1}^{n-1} c_k R_{mk}(t) + \lambda \int_{t_0}^{t} \frac{1}{r_{m+1}(s_{m+1})} \times \tag{2.6}$$

$$\times \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} \bar{f}(y(s)) ds ds_{n-1} \ldots ds_{m+2} ds_{m+1},$$

for $0 \le m \le n-1$, where

$$\bar{f}(y(t)) = f(t, D_r^{(0)} y(t), D_r^{(1)} y(t), \ldots, D_r^{(n-1)} y(t)), \tag{2.7}$$

and $R_{mk}(t)$ is defined by (2.1). From (2.6) and using the condition (2.2) we have

$$\sum_{m=0}^{n-1} |D_r^{(m)} y(t)| \le c + \sum_{m=0}^{n-1} \int_{t_0}^{t} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \times \tag{2.8}$$

$$\times \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} a(s) \times H\left(\sum_{m=0}^{n-1} |D_r^{(m)} y(s)|\right) ds ds_{n-1} \ldots ds_{m+2} ds_{m+1},$$

where c is defined by (2.4). Define a function $u(t)$ by the right side of (2.8). Then

$$\sum_{m=0}^{n-1} |D_r^{(m)} y(t)| \le u(t), \quad u(t_0) = c,$$

and

$$u'(t) \le \sum_{m=0}^{n-1} \frac{1}{r_{m+1}(t)} \int_{t_0}^{t} \frac{1}{r_{m+2}(s_{m+2})} \int_{t_0}^{s_{m+2}} \frac{1}{r_{m+3}(s_{m+3})} \times \tag{2.9}$$

$$\times \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} a(s) H(u(s)) ds ds_{n-1} \times ds_{m+3} ds_{m+2} \le M(t) H(u(t)),$$

for $t \in I$. From (2.9) it follows that

$$\frac{u'(t)}{H(u(t))} \le M(t). \tag{2.10}$$

The integration of (2.10) from t_0 to t and the use of the change of variable and the condition (2.3) give

$$\int_c^{u(t)} \frac{ds}{H(s)} = \int_{t_0}^t \frac{u'(s)}{H(u(s))} ds \leq \int_{t_0}^t M(s)ds \leq \int_{t_0}^T M(s)ds < \int_c^\infty \frac{ds}{H(s)}. \qquad (2.11)$$

From (2.11) it follows that $u(t)$ must be bounded on I, i.e. there is a positive constant α independent of $\lambda \in (0,1)$ such that $u(t) \leq \alpha$ and hence $\sum_{m=0}^{n-1} |D_r^{(m)} y(t)| \leq \alpha$ for $t \in I$. Thus we have $|D_r^{(m)} y(t)| \leq \alpha,\ t \in I$ for $0 \leq m \leq n-1$, and consequently $\|y\| \leq \alpha$.

In the next step we rewrite the problem $(P) - (P_0)$ as follows. If $y(t) = z(t) + e(t)$, where $e(t) = c_0 + \sum_{k=1}^{n-1} c_k R_k(t),\ t \in I$, then it is easy to see that $z(t_0) = y(t_0) - e(t_0) = 0$,

$$z(t) = \int_{t_0}^t \frac{1}{r_1(s_1)} \int_{t_0}^{s_1} \frac{1}{r_2(s_2)} \times \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \times \qquad (2.12)$$

$$\times \int_{t_0}^{s_{n-1}} f^*(z(s) + e(s)) ds ds_{n-1} \ldots ds_2 ds_1,$$

if and only if $y(t)$ satisfies $(P) - (P_0)$, where we have used the notation $f^*(z(s) + e(s))$ for

$$f(s, D_r^{(0)}(z(s) + e(s)), D_r^{(1)}(z(s) + e(s)), \ldots, D_r^{(0)}(z(s) + e(s))).$$

Define $F : B_0 \to B_0,\ B_0 = \{z \in B :\ z(t_0) = 0\}$ by

$$Fz(t) = \int_{t_0}^t \frac{1}{r_1(s_1)} \int_{t_0}^{s_1} \frac{1}{r_2(s_2)} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \times \qquad (2.13)$$

$$\times \int_{t_0}^{s_{n-1}} f^*(z(s) + e(s)) ds ds_{n-1} \ldots ds_2 ds_1,$$

for $t \in I$. Then F is clearly continuous. Now we shall prove that F is completely continuous.

Let $\{w_k\}$ be a bounded sequence in B_0, i.e. $\|w_k\| \leq \beta$ for all k, where β is a positive constant. From (2.13) and using condition (2.2) and setting $M^* = \sup\{M(t) : t \in I\}$ and $e^* = \sup\{|D_r^{(m)} e(t)| :\ t \in I,\ 0 \leq m \leq n-1\}$, we have

$$|D_r^{(m)}(Fw_k(t))| \leq \int_{t_0}^t \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \times \qquad (2.14)$$

$$\times \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} |f^*(w_k(s)+e(s))| ds ds_{n-1} \ldots ds_{m+2} ds_{m+1} \leq$$
$$\leq M^*(H(n(\beta+e^*))(T-t_0) = L,$$

for $0 \leq m \leq n-1$, where $L \geq 0$ is a constant. Hence from (2.14) we obtain $\|Fw_k\| \leq L$. This means that $\{Fw_k\}$ is uniformly bounded.

Now we shall that the sequence $\{Fw_k\}$ is equicontinuous. Let $t_0 \leq t_1 \leq t_2 \leq T$. Then from (2.13) and using the condition (2.2), and letting $\{w_k\}$, M^*, e^* as defined above, we observe that

$$|D_r^{(m)}(Fw_k(t_2)) - D_r^{(m)}(Fw_k(t_1))| \leq \tag{2.15}$$
$$\leq \int_{t_1}^{t_2} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \times$$
$$\times \int_{t_0}^{s_{n-1}} |f^*(w_k(s)+e(s))| ds ds_{n-1} \ldots ds_{m+2} ds_{m+1} \leq$$
$$\leq \int_{t_1}^{t_2} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \times$$
$$\times \int_{t_0}^{s_{n-1}} a(s) H\left(\sum_{m=0}^{n-1} [|D_r^{(m)} w_k(s)| + |D_r^{(m)} e(s)|]\right) ds ds_{n-1} \ldots ds_{m+2} ds_{m+1} \leq$$
$$\leq \int_{t_1}^{t_2} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \times$$
$$\times \int_{t_0}^{s_{n-1}} M^* H(n(\beta+e^*)) ds ds_{n-1} \ldots ds_{m+2} ds_{m+1} \leq \int_{t_1}^{t_2} M^* H(n(\beta+e^*)) ds.$$

From (2.15) we conclude that $\{Fw_k\}$ is equicontinuous and hence by Arzela-Ascoli theorem the operator F is completely continuous.

Moreover, the set $U(F) = \{z \in B_0 : z = \lambda F z, \lambda \in (0,1)\}$ is bounded. Since for every z in $U(F)$ the function $y(t) = z(t) + e(t)$ is a solution of $(P)_\lambda - (P_0)$, for which we have proved that $\|y\| \leq \alpha$ and hence $\|z\| \leq \alpha_e^*$. By applying Theorem G, the operator F has a fixed point in B_0. This means that the problem $(P) - (P_0)$ has a solution $y(t)$ in B. The proof is complete.

Remark 1. We note that our Theorem 1 extends the well known theorem of Wintner [20] on the global existence of solution of Cauchy problem for first order differential equation, to higher order differential equations $(P)-(P_0)$. For some recent extensions of Wintner's theorem, see [1,5-10,12,13]. Further we note that out Theorem 1 contains in the special cases the global existence of solutions of the following differential equations

(P_1) $(r(t)y^{(n-1)}(t))' = f(t, r(t)y(t), r(t)y'(t), r(t)y''(t), \ldots, r(t)y^{(n-1)}(t))$,

(P_2) $(r(t)y'(t))^{(n-1)} = f(t, y(t), r(t)y'(t), (r(t)y'(t))', \ldots, (r(t)y'(t))^{(n-2)})$,

(P_3) $y^{(n)}(t) = f(t, y(t), y'(t), \ldots, y^{(n-1)}(t))$,

with some suitable given initial conditions, and studied by many authors in the literature with different viewpoints, see [3,4,14-19].

3. Estimates on the solutions

In this section we obtain estimates on the solutions of $(P)-(P_0)$ which can be used to study the various properties of the solutions of equations $(P)-(P_0)$, by using the integral inequalities given in [11, Theorem 3.3.1, p.222 and Theorem 1.3.2, p.13].

The following theorem deals with the estimates on the solution and their derivatives of the problem $(P)-(P_0)$.

Theorem 2. *Suppose that the function f in (P) satisfies*

$$|f(t, D_r^{(0)}y, D_r^{(1)}y, \ldots, D_r^{(n-1)}y| \leq L\left(t, \sum_{m=0}^{n-1} |D_r^{(m)}y|\right), \tag{3.1}$$

for $t \in I$, where $L : I \times R_+ \to R_+$ be a continuous function such that

$$0 \leq L(t,u) - L(t,v) \leq w(t,v)(u-v), \tag{L}$$

for $t \in I$ and $u \geq v \geq 0$, where $w : I \times R_+ \to R_+$ is a continuous function. If $y(t)$ is a solution of $(P)-(P_0)$ on I, then

$$\sum_{m=0}^{n-1} |D_r^{(m)}y(t)| \leq a(t) + b(t)\int_{t_0}^{t} L(s, a(s)) \exp\left(\int_s^t w(\sigma, a(\sigma))b(\sigma)d\sigma\right) ds, \tag{3.2}$$

where

$$a(t) = \sum_{m=0}^{n-1}\left[|c_m| + \sum_{k=m+1}^{n-1} |c_k R_{mk}(t)|\right], \tag{3.3}$$

$$b(t)=\sum_{m=0}^{n-1}\int_{t_0}^{t}\frac{1}{r_{m+1}(s_{m+1})}\int_{t_0}^{s_{m+1}}\frac{1}{r_{m+2}(s_{m+2})}\cdots\int_{t_0}^{s_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\times \tag{3.4}$$

$$\times ds_{n-1}\dots ds_{m+2}ds_{m+1},$$

for $t\in I$ and $R_{mk}(t)$ is defined by (2.1).

Proof. If $y(t)$ is a solution of $(P)-(P_0)$, then $y(t)$ and its derivatives can be written as

$$D_r^{(m)}y(t)=c_m+\sum_{k=m+1}^{n-1}c_kR_{mk}(t)+\int_{t_0}^{t}\frac{1}{r_{m+1}(s_{m+1})}\times \tag{3.5}$$

$$\times\int_{t_0}^{s_{m+1}}\frac{1}{r_{m+2}(s_{m+2})}\cdots\int_{t_0}^{s_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\int_{t_0}^{s_{n-1}}\overline{f}(y(s))ds\,ds_{n-1}\dots ds_{m+2}ds_{m+1},$$

for $0\le m\le n-1$, where $R_{mk}(t)$ and $\overline{f}(y(t))$ are defined by (2.1) and (2.7) respectively. From (3.5), (3.1), (3.3) and (3.4) we observe that

$$\sum_{m=0}^{n-1}|D_r^{(m)}y(t)|\le a(t)+b(t)\int_{t_0}^{t}L\left(s,\sum_{m=0}^{n-1}|D_r^{(m)}y(s)|\right)ds. \tag{3.6}$$

Now an application of Theorem 3.2.1 given in [11,p.222] to (3.6) yields the desired inequality in (3.2). The proof is complete.

Our next theorem deals with a slight variant of Theorem 2 which can be used more conveniently in certain applications.

Theorem 3. *Suppose that the function f in (P) satisfies the condition (3.1). If $y(t)$ is a solution of $(P)-(P_0)$ existing on I, then*

$$\sum_{m=0}^{n-1}|D_r^{(m)}y(t)-\psi^{(m)}(t)|\le \tag{3.7}$$

$$\le q(t)+b(t)\left(\int_{t_0}^{t}q(s)w\left(s,\sum_{m=0}^{n-1}|\psi^{(m)}(s)|\right)\exp\left(\int_{s}^{t}b(\sigma)w\left(\sigma,\sum_{m=0}^{n-1}|\psi^{(m)}(\sigma)\right)d\sigma\right)ds\right),$$

for $t\in I$, where

$$\psi^{(m)}(t)=c_m+\sum_{k=m+1}^{n-1}c_kR_{mk}(t), \tag{3.8}$$

$$q(t)=b(t)\int_{t_0}^{t}L\left(s,\sum_{m=0}^{n-1}|\psi^{(m)}(s)|\right)ds, \tag{3.9}$$

for $t \in I$, $R_{mk}(t)$ and $b(t)$ are defined by (2.1) and (3.4) respectively.

Proof. If $y(t)$ is a solution of $(P)-(P_0)$, then $y(t)$ and its derivatives can be written as

$$D_r^{(m)}y(t) = \psi^{(m)}(t) + \int_{t_0}^{t} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \times \tag{3.10}$$

$$\times \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} \overline{f}(y(s))ds ds_{n-1} \ldots ds_{m+2} ds_{m+1},$$

where $\overline{f}(y(t))$ is defined by (2.7). From (3.10), (3.1), (3.9), and the condition (L) we observe that

$$\sum_{m=0}^{n-1} |D_r^{(m)}y(t) - \psi^{(m)}(t)| \leq \sum_{m=0}^{n-1} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \times \tag{3.11}$$

$$\times \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} |\overline{f}(y(s))ds ds_{n-1} \ldots ds_{m+2} ds_{m+1} \leq$$

$$\leq b(t) \int_{t_0}^{t} L\left(s, \sum_{m=0}^{n-1} |D_r^{(m)}y(s)|\right) ds \leq$$

$$\leq b(t) \int_{t_0}^{t} \left[L\left(s, \sum_{m=0}^{n-1} |D_r^{(m)}y(s) - \psi^{(m)}(s)| + \sum_{m=0}^{n-1} |\psi^{(m)}(s)|\right) - \right.$$

$$\left. - L\left(s, \sum_{m=0}^{n-1} |\psi^{(m)}(s)|\right) + L\left(s, \sum_{m=0}^{n-1} |\psi^{(m)}(s)|\right) \right] ds \leq$$

$$\leq q(t) + b(t) \int_{t_0}^{t} w\left(s, \sum_{m=0}^{n-1} |\psi^{(m)}(s)|\right) \sum_{m=0}^{n-1} |D_r^{(m)}y(s) - \psi^{(m)}(s)|x ds.$$

Now an application of Theorem 1.3.2 given in [11,p.13] yields the required inequality in (3.7). The proof is complete.

Another useful variant of Theorem 2 which deals with the bounds on the solution $y(t)$ of $(P)-(P_0)$ and its derivatives is given in the following theorem.

Theorem 4. *Suppose that the function f in (P) satisfies the condition*

$$|f(t, D_r^{(0)}y, D_r^{(1)}y, \ldots, D_r^{(n-1)}y)| \leq h(t)\left(\sum_{m=0}^{n-1} |D_r^{(m)}y|\right), \tag{3.12}$$

for $t \in I$, where $h : I \to R_+$ is a continuous function. If $y(t)$ is a solution of $(P)-(P_0)$, then

$$\sum_{m=0}^{n-1} |D_r^{(m)} y(t)| \leq a(t) + b(t) \int_{t_0}^{t} h(s) a(s) \exp\left(\int_s^t h(\sigma) b(\sigma) d\sigma \right) ds, \tag{3.13}$$

for $t \in I$, where $a(t)$ and $b(t)$ are defined by (3.3) and (3.4) respectively.

Proof. Let $y(t)$ be a solution of $(P)-(P_0)$ for $t \in I$, then the solution $y(t)$ and its derivatives can be written as in (3.5). From (3.5), (3.12), (3.3) and (3.4) we have

$$\sum_{m=0}^{n-1} |D_r^{(m)} y(t)| \leq a(t) + b(t) \int_{t_0}^{t} h(s) \left(\sum_{m=0}^{n-1} |D_r^{(m)} y(s)| \right) ds. \tag{3.14}$$

Now an application of Theorem 1.3.2 given in [11,p.13] yields the desired bound in (3.13). The proof is complete.

Our next result deals with the dependency of solutions of equations (P) on initial values.

Theorem 5. *Let $y_1(y)$ and $y_2(t)$ be the solutions of $(P)-(P_0)$ with the given initial conditions*

$$D_r^{(m)} y_1(t_0) = c_m, \tag{3.15}$$

and

$$D_r^{(m)} y_2(t_0) = d_m, \tag{3.16}$$

respectively, for $m = 0, 1, 2, \ldots, n-1$, where c_m, d_m are given real constants. Suppose that the function f in (P) satisfies the condition

$$|f(t, D_r^{(0)} y_1, D_r^{(1)} y_1, \ldots, D_r^{(n-1)} y_1) - f(t, D_r^{(0)} y_2, D_r^{(1)} y_2, \ldots, D_r^{(n-1)} y_2)| \leq \tag{3.17}$$

$$\leq h(t) \left(\sum_{m=0}^{n-1} |D_r^{(m)} y_1 - D_r^{(m)} y_2| \right),$$

for $t \in I$, where $h : I \to R_+$ is a continuous function. Then

$$\sum_{m=0}^{n-1} |D_r^{(m)} y_1(t) - D_r^{(m)} y_2(t)| \leq \tag{3.18}$$

$$\leq A(t) + b(t)\int_{t_0}^{t} h(s)A(s)\exp\left(\int_{s}^{t} h(\sigma)b(\sigma)d\sigma\right)ds,$$

where

$$A(t) = \sum_{m=0}^{n-1}\left[|c_m - d_m| + \sum_{k=m+1}^{n-1}|c_k - d_k|R_{mk}(t)\right], \tag{3.19}$$

for $t \in I$, $R_{mk}(t)$ and $b(t)$ are defined by (2.1) and (3.4) respectively.

Proof. In view of the facts that $y_1(t)$ and $y_2(t)$ are the solutions of (P) with the given initial conditions (3.15) and (3.16) respectively, we have

$$D_r^{(m)}y_1(t) - D_r^{(m)}y_2(t) = (c_m - d_m) + \sum_{k=m+1}^{n-1}(c_k - d_k)R_{mk}(t) + \tag{3.20}$$

$$+\int_{t_0}^{t}\frac{1}{r_{m+1}(s_{m+1})}\int_{t_0}^{s_{m+1}}\frac{1}{r_{m+2}(s_{m+2})}\cdots\int_{t_0}^{s_{n-2}}\frac{1}{r_{n-1}(s_{n-1})}\times$$

$$\times\int_{t_0}^{s_{n-1}}(\overline{f}(y_1(s)) - \overline{f}(y_2(s)))ds\,ds_{n-1}\ldots ds_{m+2}ds_{m+1},$$

where $R_{mk}(t)$ and $\overline{f}(y(t))$ are defined by (2.1) and (2.7) respectively. From (3.20), (3.17), (3.19) and (3.4) we observe that

$$\sum_{m=0}^{n-1}|D_r^{(m)}y_1(t) - D_r^{(m)}y_2(t)| \leq \tag{3.21}$$

$$\leq A(t) + b(t)\int_{t_0}^{t} h(s)\left(\sum_{m=0}^{n-1}|D_r^{(m)}y_1(s) - D_r^{(m)}y_2(s)|\right)ds.$$

Now an application of Theorem 1.3.2 given in [11,p.13] yields the required inequality in (3.18) and hence the proof is complete.

We next consider the following differential equations

$$(Q_1) \qquad D_r^{(n)}y = f(t, D_r^{(0)}y, D_r^{(1)}y, \ldots, D_r^{(n-1)}y, \mu),$$

$$(Q_2) \qquad D_r^{(n)}y = f(t, D_r^{(0)}y, D_r^{(1)}y, \ldots, D_r^{(n-1)}y, \mu_0),$$

with the given initial conditions in (P_0), where $f : I \times R^n \times R \to R$ is a continuous function and μ, μ_0 are real parameters.

The following theorem shows the dependency of solutions of equations $(Q_1) - (P_0)$ and $(Q_2) - (P_0)$ on pure parameters.

Theorem 6. *Suppose that*

$$|f(t, D_r^{(0)}y, D_r^{(1)}y, \ldots, D_r^{(n-1)}y, \mu - f(t, D_r^{(0)}\overline{y}, D_r^{(1)}\overline{y}, \ldots, D_r^{(n-1)}\overline{y}, \mu)| \leq \tag{3.22}$$

$$\leq h(t)\left(\sum_{m=0}^{n-1} |D_r^{(m)}y - D_r^{(m)}\overline{y}|\right),$$

$$|f(t, D_r^{(0)}y, D_r^{(1)}y, \ldots, D_r^{(n-1)}y, \mu) - \tag{3.23}$$

$$-f(t, D_r^{(0)}y, D_r^{(1)}y, \ldots, D_r^{(n-1)}y, \mu_0)| \leq g(t)|\mu - \mu_0|.$$

where $h, g : I \to R_+$ *are continuous functions. If* $y_1(t)$ *and* $y_2(t)$ *are solutions of* $(Q_1) - (P_0)$ *and* $(Q_2) - (P_0)$, *then*

$$\sum_{m=0}^{n-1} |D_r^{(m)}y_1(t) - D_r^{(m)}y_2(t)| \leq \tag{3.24}$$

$$\leq \overline{A}(t) + b(t)\int_{t_0}^{t} h(s)\overline{A}(s)\exp\left(\int_{s}^{t} h(\sigma)b(\sigma)d\sigma\right)ds,$$

for $t \in I$, *where*

$$\overline{A}(t) = |\mu - \mu_0| b(t)\int_{t_0}^{t} g(s)ds, \tag{3.25}$$

for $t \in I$ *and* $b(t)$ *is defined by (3.4).*

Proof. Let $z(t) = y_1(t) - y_2(t)$ for $t \in I$. As in the proof of Theorem 5, from the hypothesis we observe that

$$D_r^{(m)}z(t) = D_r^{(m)}y_1(t) - D_r^{(m)}y_2(t) = \tag{3.26}$$

$$= \int_{t_0}^{t} \frac{1}{r_{m+1}(s_{m+1})}\int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \times$$

$$\times \int_{t_0}^{s_{n-1}} \{f(s, D_r^{(0)}y_1(s), D_r^{(1)}y_1(s), \ldots, D_r^{(n-1)}y_1(s), \mu) -$$

$$-f(s, D_r^{(0)}y_2(s), D_r^{(1)}y_2(s), \ldots, D_r^{(n-1)}y_2(s), \mu) +$$

$$+f(s, D_r^{(0)}y_2(s), D_r^{(1)}y_2(s), \ldots, D_r^{(n-1)}y_2(s), \mu) -$$

$$-f(s, D_r^{(0)}y_2(s), D_r^{(1)}y_2(s), \ldots, D_r^{(n-1)}y_2(s), \mu_0)\} ds ds_{n-1} \ldots ds_{m+2} ds_{m+1}.$$

From (3.26), (3.22), (3.23), (3.25) and (3.4) we observe that

$$\sum_{m=0}^{n-1} |D_r^{(m)} z(t)| \le \int_{t_0}^{t} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \times \tag{3.27}$$

$$\times \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} \left\{ h(s) \left(\sum_{m=0}^{n-1} |D_r^{(m)} y_1(s) - D_r^{(m)} y_2(s)| \right) + \right.$$

$$\left. + g(s)|\mu - \mu_0| \right\} ds ds_{n-1} \ldots ds_{m+2} ds_{m+1} \le$$

$$\le b(t) \int_{t_0}^{t} \left\{ h(s) \left(\sum_{m=0}^{n-1} |D_r^{(m)} z(s)| \right) + g(s)|\mu - \mu_0| \right\} ds =$$

$$= \overline{A}(t) + b(t) \int_{t_0}^{t} h(s) \left(\sum_{m=0}^{n-1} |D_r^{(m)} z(s)| \right) ds$$

Now an application of Theorem 1.3.2 given in [11,p.13] yields the required inequality in (3.24) and the proof is complete.

Remark 2. We note that the results obtained in this paper can be very easily extended to the more general integrodifferential equation of the form

$$(Q) \qquad D_r^{(n)} y = f(t, D_r^{(0)} y, D_r^{(1)} y, \ldots, D_r^{(n-1)} y,$$
$$\int_{t_0}^{t} g(t, s, D_r^{(0)} y(s), D_r^{(1)} y(s), \ldots, D_r^{(n-1)} y(s)) ds),$$

with the given initial conditions in (P_0), under some suitable conditions on the functions involved in (Q) and by using the suitable general versions of the inequalities given in Chapters 1 and 3 in [11]. For similar results, see references [7-10,12,13].

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