STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLIV, Number 3, September 1999

ON THE GOURSAT PROBLEM FOR HYPERBOLIC FUNCTIONAL-DIFFERENTIAL EQUATIONS

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It is known that in many problems of nonlinear fields theory, plasma physics and etc. (cf. [1]) arise hyperbolic functional-differential equations with so-called 'distributed' deviations (cf. [2]). The main purpose of the present paper is to formulate conditions under which there exist solutions of the Goursat problem, characteristic of functional-differential equations with 'concentrated' deviations (particular case of distributed deviations), using the fixed point theorems, proved by Angelov [3].

Typical in this respect is the following simple example, where the discontinuity of the initial function generates the discontinuity of the solution:

$$\begin{cases} u_{xy}(x,y) = k(x,y)u_{xy}(x-1,y-1), & (x,y) \in \mathbb{R}^2_+ = \{(x,y):\ x>0,\ y>0\} \\ u(x,y) = \varphi(x,y), & (x,y) \in A_0 \cup B_0, \end{cases} \tag{1}$$

where

$$u_{xy} = \frac{\partial^2 u}{\partial x \partial y}; \quad A_0 = \{(x,y):\ x \ge -1,\ -1 \le y \le 0\},$$

$$B_0 = \{(x,y):\ -1 \le x \le 0,\ y \ge -1\};$$

$$\varphi(x,y) = \begin{cases} 1, & (x,y) \in \mathbb{R}^x_+ \cup \mathbb{R}^y_+, \\ & \mathbb{R}^x_+ = \{(x,y) : x \ge 0,\ y = 0\},\ \mathbb{R}^y_+ = \{(x,y):\ x = 0,\ y \ge 0\} \\ 0, & (x,y) \in A_0 \cup B_0 \setminus (\mathbb{R}^x_+ \cup \mathbb{R}^y_+) \end{cases}$$

$$k(x,y) = 1 - \frac{1}{1+n},\ (x,y) \in A_n \cup B_n\ (n = 1,2,\ldots),$$
$$A_n = \{(x,y):\ x \ge n-1,\ n-1 \le y \le n\},\ B_n = \{(x,y):\ n-1 \le x \le n,\ y \ge n-1\}.$$

Integrating the above equation we have

$$u(x,y) - k(x,y)u(x-1,y-1) = C_1(x) + C_2(y).$$

Then the conditions

$$u(0,0) - k(0,0)u(-1,-1) = C_1(0) + C_2(0),$$

$$u(x,0) - k(x,0)u(x-1,-1) = C_1(x) + C_2(0),$$

$$u(0,y) - k(0,y)u(-1,y-1) = C_1(0) + C_2(0)$$

imply $C_1(x) + C_2(y) = 1$, so that we obtain the problem

$$u(x,y) = \begin{cases} k(x,y)u(x-1,y-1) + 1, & (x,y) \in \mathbb{R}^2_+ \\ \varphi(x,y), & (x,y) \in A_0 \cup B_0 \end{cases}$$

It is quite obvious that when $(x,y) \in A_n \cup B_n$ then $(x-1,y-1) \in A_{n-1} \cup B_{n-1}$ and we can construct immediately the following solution

$$u(x,y) = \begin{cases} 0, \quad (x,y) \in (A_0 \cup B_0) \setminus (\mathbb{R}^x_+ \cup \mathbb{R}^y_+) \\ 1, \quad (x,y) \in (A_1 \cup B_1) \setminus (\mathbb{R}^{x+1}_+ \cup \mathbb{R}^{y+1}_+) \\ \left(1 - \frac{1}{3}\right) + 1, \quad (x,y) \in (A_2 \cup B_2) \setminus (\mathbb{R}^{x+2}_+ \cup \mathbb{R}^{y+2}_+) \\ \dots \\ \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) \dots \left(1 - \frac{1}{n+1}\right) + \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{5}\right) \dots \left(1 - \frac{1}{n+1}\right) + \dots + \\ \quad + \left(1 - \frac{1}{n+1}\right) + 1 = \frac{n(n+3)}{2(n+1)}, \quad (x,y) \in (A_n \cup B_n) \setminus (\mathbb{R}^{x+n}_+ \cup \mathbb{R}^{y+n}_+) \\ \dots \end{cases}$$

where $\mathbb{R}^{x+n}_+ = \{(x,y) : x \geq n, \ y = n\}$, $\mathbb{R}^{y+n}_+ = \{(x,y) : x = n, \ y \geq n\}$, $n = 0, 1, 2, \dots$

The fixed point technique for operators in metric spaces has been very well developed (cf. [4]), but the above example shows that the hyperbolic functional-differential equations of neutral type (following the terminology introduced in [5]) possesses solutions with locally essentially bounded mixed derivative u_{xy}. (We note the known results [6]-[8], where only continuous solutions have been obtained with restrictions on the deviations of retarded type.) Moreover the example shows:

1. the Goursat problem allows L^∞_{loc}-solutions so that it cannot formulate as an operator equation in Banach or metric space.

2. the operator defined by the right-hand side (even in the linear case) will be not a global contraction because of $esssup\{k(x,y) : x \geq 0, \ y \geq 0\} = 1$.

That is why, we shall use the fixed point theorems from [3].

Let X be a Hausdorff sequentially complete uniform space with uniformity defined by a saturated family of pseudometrics $\{\rho_\alpha(x,y)\}_{\alpha\in\mathcal{A}}$, $\mathcal{A}$ being an index set.

Let $\Phi = \{\Phi_\alpha(t) : \alpha \in \mathcal{A}\}$ be a family of functions $\Phi_\alpha(t) : [0,\infty) \to [0,\infty)$ with the properties

1) $\Phi_\alpha(t)$ is monotone non-decreasing and continuous from the right on $[0,\infty)$;

2) $\Phi_\alpha(t) < t$, $\forall\, t > 0$,

and $j : \mathcal{A} \to \mathcal{A}$ is a mapping on the index set $\mathcal{A}$ into itself, where $j^0(\alpha) = \alpha$, $j^k(\alpha) = j(j^{k-1}(\alpha))$, $k \in \mathbb{N}$.

Definition. The map $T : Y \to Y$ is said to be Φ-*contraction* on Y if

$$\rho_\alpha(Tx,Ty) \le \Phi_\alpha(\rho_{j(\alpha)}(x,y))$$

for every $x,y \in Y$ and $\alpha \in \mathcal{A}$, $Y \subset X$.

Theorem 1. (theorem 2 from [3]) *Let us suppose*

1. the operator $T : X \to X$ is a Φ-contraction;

2. for each $\alpha \in \mathcal{A}$ there exists a Φ-function $\overline{\Phi}_\alpha(t)$ such that

$$\sup\{\Phi_{j^k(\alpha)}(t) :\ k = 0,1,2,\dots\} \le \overline{\Phi}_\alpha(t)$$

and $\overline{\Phi}_\alpha(t)/t$ is non-decreasing;

3. there exists an element $x_0 \in X$ such that $\rho_{j^k(\alpha)}(x_0,Tx_0) \le p(\alpha) < \infty$ $(k = 0,1,2,\dots)$.

Then T has at least one fixed point in X.

Theorem 2. (theorem 3 from [3]) *If, in addition, we suppose that*

4. the sequence $\{\rho_{j^k(\alpha)}(x,y)\}_{k=0}^{\infty}$ is bounded for each $\alpha \in \mathcal{A}$ and $x,y \in X$, i.e.

$$\rho_{j^k(\alpha)}(x,y) \le q(x,y,\alpha) < \infty \quad (k = 0,1,2,\dots).$$

Then the fixed point of T is unique.

Consider the general Goursat problem for hyperbolic functional-differential equation:

$$\begin{aligned} &u_{xy}(x,y) = F(x,y,u(\Delta,\tau),u_x(\alpha,\beta),u_y(\theta,\kappa),u_{xy}(\mu,\nu)),\ (x,y)\in\mathbb{R}^2_+ \\ &u(x,y)=\psi(x,y),\ u_x(x,y)=\psi_x(x,y),\ u_y(x,y)=\psi_y(x,y), \\ &u_{xy}(x,y)=\psi_{xy}(x,y),\ (x,y)\in\mathbb{R}^2\setminus\mathbb{R}^2_+, \end{aligned} \tag{2}$$

where $F(x,y,z_1,z_2,z_3,z_4)$, $\Delta=\Delta(x,y)$, $\tau=\tau(x,y)$, $\alpha=\alpha(x,y)$, $\beta=\beta(x,y)$, $\theta=\theta(x,y)$, $\kappa=\kappa(x,y)$, $\mu=\mu(x,y)$, $\nu=\nu(x,y)$ and $\psi(x,y)$ are given functions.

We set

$v(x,y)=u_{xy}(x,y)$, when $(x,y)\in\mathbb{R}^2_+$,

$\varphi(x,y)=\psi_{xy}(x,y)$, when $(x,y)\in\mathbb{R}^2\setminus\mathbb{R}^2_+$

and after standard calculations, we obtain

$$u(x,y)=\varphi_0(x,y)+\int_0^x\int_0^y v(\xi,\eta)d\eta d\xi,$$

$$u_x(x,y)=\varphi_1(x)+\int_0^y v(x,\eta)d\eta,$$

$$u_y(x,y)=\varphi_2(y)+\int_0^x v(\xi,y)d\xi,$$

where

$$\varphi_0(x,y)=\psi(0,y)+\psi(x,0)-\psi(0,0),$$

$$\varphi_1(x)=\psi_x(x,0),\quad \varphi_2(y)=\psi_y(0,y),$$

so that the problem (2) corresponds the following problem

$$v(x,y)=\begin{cases} F(x,y,\overline{\varphi}_0+\int_0^\Delta\int_0^\tau v(\xi,\eta)d\eta d\xi,\overline{\varphi}_1+\int_0^\beta v(\alpha,\eta)d\eta,\overline{\varphi}_2+ \\ \quad +\int_0^\theta v(\xi,\kappa)d\xi, v(\mu,\nu)),(x,y)\in\mathbb{R}^2_+ \\ \varphi(x,y),(x,y)\in\mathbb{R}^2\setminus\mathbb{R}^2_+, \end{cases} \tag{3}$$

where $\overline{\varphi}_0=\varphi_0(\Delta(x,y),\tau(x,y))$, $\overline{\varphi}_1=\varphi_1(\alpha(x,y))$, $\overline{\varphi}_2=\varphi_2(\kappa(x,y))$.

Definition. The function $u(x,y)$ is said to be a solution (in generalized sense) of problem (2) if the function $v(x,y)$ is a solution of problem (3).

In what follows, we look for a solution of (3), belonging to L^∞_{loc}.

We say that the function $G : \mathbb{R}^2 \to \mathbb{R}^2$ has the property (M) if inverse image of every set with null measure is measurable.

Let us suppose:

(A1) ψ is absolutely continuous;

$\psi(x,0), \psi(0,y), \psi_x(x,0), \psi_y(0,y)$ are continuous and $\varphi = \psi_{xy} \in L^{\infty}_{loc}(\mathbb{R}^2 \backslash \mathbb{R}^2_+)$.

(A2) The functions $\Delta, \tau, \alpha, \beta, \theta, \kappa, \mu, \nu : \mathbb{R}^2_+ \to \mathbb{R}$ are measurable, have the property (M) (without Δ and τ) and map bounded sets into bounded sets.

(A3) $\forall\ (x,y) \in \mathbb{R}^2_+$ for which $(\Delta(x,y), \tau(x,y)) \in \mathbb{R}^2_+$ (or $(\alpha(x,y), \beta(x,y)) \in \mathbb{R}^2_+$, or $(\theta(x,y), \kappa(x,y)) \in \mathbb{R}^2_+$ is fulfilled $\Delta(x,y) + \tau(x,y) \leq x + y$ (respectively $\alpha(x,y) + \beta(x,y) \leq x + y$, or $\theta(x,y) + \kappa(x,y) \leq x + y$);

$\exists \delta_0 > 0$ such that $\forall\ (x,y) \in \mathbb{R}^2_+ :\ (\mu(x,y), \nu(x,y)) \in \mathbb{R}^2_+$ is fulfilled $\mu(x,y) + \nu(x,y) \leq x + y - \delta_0$.

(A4) The function $F(x,y,z_1,z_2,z_3,z_4)\ :\ \mathbb{R}^2_+ \times \mathbb{R}^4 \to \mathbb{R}$ satisfies the Caratheodory condition (measurable in x and y and continuous in $z_1, \ldots, z_4$) and the conditions:

$$|F(x,y,z_1,z_2,z_3,z_4)| \leq \Omega_1(x,y,|z_1|,|z_2|,|z_3|,|z_4|)$$

$$|F(x,y,z_1,z_2,z_3,z_4) - F(x,y,\overline{z}_1,\overline{z}_2,\overline{z}_3,\overline{z}_4)| \leq$$
$$\leq \Omega_2(x,y,|z_1 - \overline{z}_1|,|z_2 - \overline{z}_2|,|z_3 - \overline{z}_3|,|z_4 - \overline{z}_4|),$$

where the functions $\Omega_{1,2}(x,y,t_1,\ldots,t_4) : \mathbb{R}^2_+ \times \overline{\mathbb{R}}^4_+ \to [0,\infty)$ ($\overline{\mathbb{R}}^n_+ = [0,\infty) \times \cdots \times [0,\infty)$ - n times) satisfy the Caratheodory condition, $\Omega_1(\cdot,\cdot,t_1,\ldots,t_4) \in L^{\infty}_{loc}(\mathbb{R}^2_+)$, $\Omega_2(x,y,t_1,\ldots,t_4)$ is non-decreasing in $t_1,\ldots,t_4$ and

$\exists\ \omega \in L^{\infty}(\mathbb{R}^2_+)$ such that $\forall\ t \geq 0\ \Omega_2(\cdot,\cdot,t,t,t,t) \leq t\omega(\cdot,\cdot)$ a.e. in $\mathbb{R}^2_+$.

Let $\mathcal{A}$ be the set of all compact sets $K \subset \mathbb{R}^2$. Denote by $K_+ = K \cap \mathbb{R}^2_+$, we define the map $j : \mathcal{A} \to \mathcal{A}$:

$$j(K) = \begin{cases} K, & K_+ = \emptyset \\ K_{\Delta\tau} \cup K_{\alpha\beta} \cup K_{\theta\kappa} \cup K_{\mu\nu}, & K_+ \neq \emptyset \end{cases}$$

where $K_{\Delta\tau} = K_\Delta \times K_\tau$, $K_{\alpha\beta} = K_\alpha \times K_\beta$, $K_{\theta\kappa} = K_\theta \times K_\kappa$,

$$K_{\mu\nu} = \overline{\{(\mu(x,y), \nu(x,y)) :\ (x,y) \in K\}}, \quad (\overline{A} \stackrel{def}{=} clA),$$

$$K_\Delta = \begin{cases} [\Delta_{inf}, \Delta_{sup}], & \Delta_{inf} < 0 < \Delta_{sup} \\ [0, \Delta_{sup}], & \Delta_{inf} \geq 0 \\ [\Delta_{inf}, 0], & \Delta_{sup} \leq 0 \end{cases}$$

$$K_\tau = \begin{cases} [\tau_{inf}, \tau_{sup}], & \tau_{inf} < 0 < \tau_{sup} \\ [0, \tau_{sup}], & \tau_{inf} \geq 0 \\ [\tau_{inf}, 0], & \tau_{sup} \leq 0 \end{cases}$$

$$K_\beta = \begin{cases} [\beta_{inf}, \beta_{sup}], & \beta_{inf} < 0 < \beta_{sup} \\ [0, \beta_{sup}], & \beta_{inf} \geq 0 \\ [\beta_{inf}, 0], & \beta_{sup} \leq 0 \end{cases}$$

$$K_\theta = \begin{cases} [\theta_{inf}, \theta_{sup}], & \theta_{inf} < 0 < \theta_{sup} \\ [0, \theta_{sup}], & \theta_{inf} \geq 0 \\ [\theta_{inf}, 0], & \theta_{sup} \leq 0 \end{cases}$$

$$K_\alpha = \overline{\alpha(K)}, \quad K_\kappa = \overline{\kappa(K)}$$

$$(\Delta_{inf} = \inf\{\Delta(x,y) : (x,y) \in K_+\}, \ldots, \theta_{sup} = \sup\{\theta(x,y) : (x,y) \in K_+\}).$$

It is obvious that $j(K)$ is compact set and $j^l(K)$ can be defined inductively: $j^l(K) = j(j^{l-1}(K))$ for all $l \in \mathbb{N}$.

Now we assume:

(A5) $\forall\ K \in \mathcal{A}\ \exists\ \widehat{K} \in \mathcal{A} :\ j^l(K) \subset \widehat{K}\ \forall\ l = 0, 1, 2, \ldots$

We prove the following existence-uniqueness result:

Theorem 3. *If conditions (A1)-(A5) hold true, then there exists a unique solution $v(x,y) \in L^\infty_{loc}(\mathbb{R}^2)$ of problem (3).*

Proof. Let X be the uniform sequentially complete Hausdorff space consisting of all functions, belonging to $L^\infty_{loc}(\mathbb{R}^2)$, which equal $\varphi(x,y)$ for a.e. $(x,y) \in \mathbb{R}^2 \backslash \mathbb{R}^2_+$, with a saturated family $P = \{\rho_K :\ K \in \mathcal{A}\}$ of pseudometrics

$$\rho_K(f,g) = esssup\{e^{-\lambda(|x|+|y|)}|f(x,y) - g(x,y)| :\ (x,y) \in K\},$$

where K runs over all compact subsets of $\mathbb{R}^2$ (with some $\lambda > 0$).

The operator $T : X \to X$ is defined by the formula:

$$T(f)(x,y) = \begin{cases} F(x,y,\overline{\varphi}_0 + \int_0^{\Delta}\int_0^{\tau} f(\xi,\eta)d\eta d\xi, \overline{\varphi}_1 + \int_0^{\beta} f(\alpha,\eta)d\eta, \overline{\varphi}_2 + \\ \quad + \int_0^{\theta} f(\xi,\kappa)d\xi, f(\mu,\nu)), (x,y) \in \mathbb{R}^2_+ \\ \varphi(x,y), (x,y) \in \mathbb{R}^2 \setminus \mathbb{R}^2_+, \end{cases}$$

The measurability of $T(f)(x,y)$ follows from the fact that $\alpha, \beta, \theta, \kappa, \mu, \nu$ have the property (M).

$T(f) \in L^{\infty}_{loc}(\mathbb{R}^2)$ because of conditions A1, A4. Consequently $T(f) \in X$.

Let $K \subset \mathbb{R}^2$ be any fixed compact set. Of $K_+ = \emptyset$ then $T(f) - T(g) = 0$ for all $f, g \in X$ a.e. in K. Let $K_+ \neq \emptyset$. For a.e. $(x,y) \in K \cap (\mathbb{R}^2 \setminus \mathbb{R}^2_+)$ we have $T(f) - T(g) = 0$.

For a.e. $(x,y) \in K_+$ we obtain (by means of (A4)):

$$|T(f)(x,y) - T(g)(x,y)| \leq$$
$$\leq \Omega_2(x,y, |\int_0^{\Delta}\int_0^{\tau} (f(\xi,\eta) - g(\xi,\eta))d\eta d\xi|, |\int_0^{\beta} (f(\alpha,\eta) - g(\alpha,\eta))d\eta|,$$
$$|\int_0^{\theta} (f(\xi,\kappa) - g(\xi,\kappa))d\xi|, |f(\mu,\nu) - g(\mu,\nu)|)$$

If $(\Delta(x,y), \tau(x,y)) \notin \mathbb{R}^2_+$ then

$$\int_0^{\Delta}\int_0^{\tau} (f(\xi,\eta) - g(\xi,\eta))d\eta d\xi = 0$$

and respectively if $(\alpha(x,y), \beta(x,y)) \notin \mathbb{R}^2_+$ then

$$\int_0^{\beta} (f(\alpha,\eta) - g(\alpha,\eta))d\eta = 0,$$

if $(\theta(x,y), \kappa(x,y)) \notin \mathbb{R}^2_+$ then

$$\int_0^{\theta} (f(\xi,\kappa) - g(\xi,\kappa))d\xi = 0,$$

if $(\mu(x,y), \nu(x,y)) \notin \mathbb{R}^2_+$ then $f(\mu,\nu) - g(\mu,\nu) = 0$.

For positive values of $\Delta(x,y), \tau(x,y)$; $\alpha(x,y), \beta(x,y)$; $\theta(x,y), \kappa(x,y)$; $\mu(x,y)$, $\nu(x,y)$ we obtain as follows:

$$|\int_0^{\Delta}\int_0^{\tau} (f(\xi,\eta) - g(\xi,\eta))d\eta d\xi| \leq \int_0^{\Delta}\int_0^{\tau} |f(\xi,\eta) - g(\xi,\eta)|d\eta d\xi \leq$$

$$\leq esssup\{e^{-\lambda(\xi+\eta)}|f(\xi,\eta)-g(\xi,\eta)|: 0\leq\xi\leq\Delta_{sup}, 0\leq\eta\leq\tau_{sup}\}\int_0^{\Delta}\int_0^{\tau}e^{\lambda(\xi+\eta)}d\eta d\xi=$$

$$=\lambda^{-2}\rho_{K_{\Delta\tau}}(f,g)(e^{\lambda\Delta}-1)(e^{\lambda\tau}-1)\leq\lambda^{-2}e^{\lambda(\Delta+\tau)}\rho_{K_{\Delta\tau}}\leq\lambda^{-2}e^{\lambda(x+y)}\rho_{K_{\Delta\tau}}(f,g)(\text{cf.}(A3)).$$

$$|\int_0^{\beta}(f(\alpha,\eta)-g(\alpha,\eta))d\eta|\leq\int_0^{\beta}|f(\alpha,\eta)-g(\alpha,\eta)|d\eta\leq$$

$$\leq esssup\{e^{-\lambda(\alpha+\eta)}|f(\alpha,\eta)-g(\alpha,\eta)|:\ 0\leq\eta\leq\beta_{sup}\}e^{\lambda\alpha}\int_0^{\beta}e^{\lambda\eta}d\eta\leq$$

$$\leq\lambda^{-1}\rho_{K_{\alpha\beta}}(f,g)e^{\lambda\alpha}(e^{\lambda\beta}-1)\leq\lambda^{-1}e^{\lambda(\alpha+\beta)}\rho_{K_{\alpha\beta}}(f,g)\leq\lambda^{-1}e^{\lambda(x+y)}\rho_{K_{\alpha\beta}}(f,g)\ (\text{cf.}(A3)).$$

In the same way we prove (by means of (A3)) that

$$|\int_0^{\theta}(f(\xi,\kappa)-g(\xi,\kappa))d\xi|\leq\lambda^{-1}e^{\lambda(x+y)}\rho_{K_{\theta\kappa}}(f,g).$$

$$|f(\mu,\nu)-g(\mu,\nu)|\leq e^{\lambda(\mu+\nu)}esssup\{e^{-\lambda(p+q)}|f(p,q)-g(p,q)|:(p,q)\leq K_{\mu\nu}\}\leq$$

$$\leq e^{\lambda(x+y-\delta_0)}\rho_{K_{\mu\nu}}(f,g)\leq\lambda^{-\delta_0}e^{\lambda(x+y)}\rho_{K_{\mu\nu}}(f,g)\ (\text{cf.}(A3)).$$

Let $\lambda>1$. Chosing γ so that

$$\lambda^{-\gamma}=\max\{\lambda^{-1},\lambda^{-\delta_0}\}=\begin{cases}\lambda^{-1}, & \delta_0\geq 1\\ \lambda^{-\delta_0}, & 0<\delta_0\leq 1\end{cases}$$

we obtain (since $\Omega_2(x,y,t_1,\ldots,t_4)$ is non-decreasing in $t_1,\ldots,t_4$)

$$|T(f)(x,y)-T(g)(x,y)|\leq$$

$$\leq\Omega_2(x,y,\lambda^{-\gamma}e^{\lambda(x+y)}\rho_{j(K)}(f,g),\lambda^{-\gamma}e^{\lambda(x+y)}\rho_{j(K)}(f,g),$$

$$\lambda^{-\gamma}e^{\lambda(x+y)}\rho_{j(K)}(f,g),\lambda^{-\gamma}e^{\lambda(x+y)}\rho_{j(K)}(f,g))\leq$$

$$\leq\lambda^{-\gamma}e^{\lambda(x+y)}\rho_{j(K)}(f,g)\omega(x,y)\quad(\text{cf.}(A4)).$$

Define (for $t\geq 0$)

$$\Phi_K(t)=\begin{cases}0, & \text{if}\quad K_+=\emptyset\\ t\lambda^{-\gamma}||\omega||_{L^\infty(K_+)}, & \text{if}\quad K_+\neq\emptyset\end{cases}$$

We can find and fix λ so that $\lambda^{\gamma}>||\omega||_{L^\infty(\mathbb{R}^2_+)}$. Consequently $\Phi_K(t)<t$ $\forall\ t>0$, $\forall\ K\in\mathcal{A}$ and $\Phi_K(t)$ is continuous non-decreasing in $[0,\infty)$. On the other hand

$$\rho_K(T(f),T(g))=\rho_{\overline{K}_+}(T(f),T(g))\leq\Phi_k(\rho_{j(K)}(f,g)),$$

i.e. $T:X\to X$ is a Φ-contraction.

$\forall\ K \in \mathcal{A}$ we set $\overline{\Phi}_K = \Phi_{\widehat{K}}$ (recall that (A5) assures an existence of such a $\widehat{K}$ that $j^l(K) \subset \widehat{K}$ $(l = 0, 1, 2, \ldots)$) and so $\sup\{\Phi_{j^l(K)}(t) : l = 0, 1, 2, \ldots\} \leq \overline{\Phi}_K(t)$, $\dfrac{\overline{\Phi}_K(t)}{t} = const \Rightarrow$ non-decreasing.

Hence condition 1, 2 of theorem 1 is fulfilled.

We choose the element $f_0 \in X$:

$$f_0(x,y) = \begin{cases} 0, & \text{a.e. on } \mathbb{R}^2_+ \\ \varphi(x,y), & (x,y) \in \mathbb{R}^2 \setminus \mathbb{R}^2_+. \end{cases}$$

Then for any integer $l \geq 0$ we have

$$\rho_{j^l(K)}(f_0, T(f_0)) \leq \rho_{\widehat{K}}(f_0, T(f_0)) = \rho_{\overline{\widehat{K}_+}}(f_0, T(f_0)) =$$

$$= esssup\{e^{-\lambda(x+y)}|F(x, y, \overline{\varphi}_0, \overline{\varphi}_1, \overline{\varphi}_2, 0)| : (x,y) \in \overline{\widehat{K}_+}\} < \infty$$

(i.e. condition 3 of theorem 1 is fulfilled).

Besides $\rho_{j^l(K)}(f,g) \leq \rho_{\overline{\widehat{K}_+}}(f,g)$ for abitrary $f, g \in X$, i.e. condiyion 4 of theorem 2 is also fulfilled.

All conditions of theorems 1 and 2 are satisfied. Therefore the problem (3) has a unique solution $v \in L^{\infty}_{loc}(\mathbb{R}^2)$.

We are going to formulate conditions for the existence and uniqueness of a solution of (3) belonging to $L^p_{loc}(\mathbb{R}^2)$ for some $p \in (1, \infty)$:

(A1') The initial function ψ is absolutely continuous;

$\psi(x,0), \psi(0,y), \psi_x(x,0), \psi_y(0,y)$ are continuous and $\varphi = \psi_{xy} \in L^p_{loc}(\mathbb{R}^2 \backslash \mathbb{R}^2_+)$.

(A4') The function $F(x, y, z_1, z_2, z_3, z_4) : \mathbb{R}^2_+ \times \mathbb{R}^4 \to \mathbb{R}$ satisfies the Caratheodory condition (measurable in x and y and continuous in $z_1, \ldots, z_4$) and the conditions:

$$|F(x, y, z_1, z_2, z_3, z_4)| \leq a(x,y) + b(|z_1| + |z_2| + |z_3| + |z_4|)$$

$$|F(x, y, z_1, z_2, z_3, z_4) - F(x, y, \overline{z}_1, \overline{z}_2, \overline{z}_3, \overline{z}_4)| \leq$$

$$\leq \omega_1(x,y)|z_1 - \overline{z}_1| + \omega_2|z_2 - \overline{z}_2| + \omega_3|z_3 - \overline{z}_3| + \omega_4|z_4 - \overline{z}_4|,$$

where $a(\cdot,\cdot) \in L^p_{loc}(\mathbb{R}^2_+)$, $b = const \geq 0$, $\omega_1(\cdot,\cdot) \in L^p(\mathbb{R}^2_+)$, $\omega_{2,3}(\cdot) \in L^p(\mathbb{R}^1_+)$, $\omega_4 = const \geq 0$

(A6) The transformations

$$\left| \begin{array}{l} u=\alpha(x,y) \\ v=y \end{array} \right. \qquad \left| \begin{array}{l} u=x \\ v=\kappa(x,y) \end{array} \right. \qquad \left| \begin{array}{l} u=\mu(x,y) \\ v=\nu(x,y) \end{array} \right.$$

are admissible, sufficiently smooth and $\alpha_u^*, \kappa_v^*, \dfrac{D(\mu^*,\nu^*)}{D(u,v)} \in L^\infty(\mathbb{R}_+^2)$, where

$$(\alpha^*(\alpha(x,y),y),y)=(x,y), \quad (x,\kappa^*(x,\kappa(x,y)))=(x,y),$$

$$(\mu^*(\mu(x,y),\nu(x,y)),\nu^*(\mu(x,y),\nu(x,y)))=(x,y).$$

Theorem 4. *If conditions (A1'), (A2), (A3), (A4'), (A5), (A6) hold true, then there exists a unique solution $v(x,y)\in L^p_{loc}(\mathbb{R}^2)$ of problem (3).*

Proof. Let X be the space consisting of all functions, belonging to $L^p_{loc}(\mathbb{R}^2)$, which equal $\varphi(x,y)$ a.e. $(x,y)\in\mathbb{R}^2\setminus\mathbb{R}^2_+$, with a saturated family P of pseudometrics

$$\rho_K(f,g)=\left(\int_K\int e^{-\lambda(|x|+|y|)}|f(x,y)-g(x,y)|^p\,dx\,dy\right)^{\frac{1}{p}} \qquad (K\in\mathcal{A}),$$

where $\mathcal{A}$ is the family of all compact sets in $\mathbb{R}^2$, $\lambda>0$.

The map $j:\mathcal{A}\to\mathcal{A}$ and the operator $T:X\to X$ are defined as in the proof of theorem 3.

For any $K\in\mathcal{A}$, $f,g\in X$ we have $T(f)(x,y)-T(g)(x,y)=0$, for a.e. $(x,y)\in K\setminus K_+$;

If $(x,y)\in K_+\neq\emptyset$, then

$$|T(f)(x,y)-T(g)(x,y)|^p\le$$

$$\le\Bigg(\omega_1(x,y)\left|\int_0^\Delta\int_0^\tau(f(\xi,\eta)-g(\xi,\eta))d\eta d\xi\right|+\omega_2(y)\left|\int_0^\beta(f(\alpha,\eta)-g(\alpha,\eta))d\eta\right|+$$

$$+\omega_3(x)\left|\int_0^\theta(f(\xi,\kappa)-g(\xi,\kappa))d\xi\right|+\omega_4(f(\mu,\nu)-g(\mu,\nu)|\Bigg)^p\le$$

$$\le 4^{p-1}\Bigg(\omega_1^p(x,y)\left|\int_0^\Delta\int_0^\tau(f(\xi,\eta)-g(\xi,\eta))d\eta d\xi\right|^p+\omega_2^p(y)\left|\int_0^\beta(f(\alpha,\eta)-g(\alpha,\eta))d\eta\right|^p+$$

$$+\omega_3^p(x)\left|\int_0^\theta(f(\xi,\kappa)-g(\xi,\kappa))d\xi\right|^p+\omega_4^p|f(\mu,\nu)-g(\mu,\nu)|^p\Bigg).$$

If $(\Delta,\tau),(\alpha,\beta),(\theta,\kappa),(\mu,\nu)\notin\mathbb{R}^2_+$, then $T(f)(x,y)-T(g)(x,y)=0$.

If $(\Delta,\tau)\in\mathbb{R}^2_+$, then (with $\frac{1}{p}+\frac{1}{q}=1$)

$$\left|\int_0^\Delta\int_0^\tau (f(\xi,\eta)-g(\xi,\eta))d\eta d\xi\right|^p \le$$

$$\le \left(\int_0^\Delta\int_0^\tau |f(\xi,\eta)-g(\xi,\eta)|d\eta d\xi\right)^p =$$

$$= \left(\int_0^\Delta\int_0^\tau e^{\frac{\lambda}{p}(\xi+\eta-\xi-\eta)}|f(\xi,\eta)-g(\xi,\eta)|d\eta d\xi\right)^p \le$$

$$\le \left(\int_0^\Delta\int_0^\tau e^{\frac{q\lambda}{p}(\xi+\eta)}d\eta d\xi\right)^{\frac{p}{q}}\int_0^\Delta\int_0^\tau e^{-\lambda(\xi+\eta)}|f(\xi,\eta)-g(\xi,\eta)|^p d\eta d\xi \le$$

$$\le \left(\frac{p-1}{\lambda}\right)^{2(p-1)} e^{\lambda(\Delta+\tau)}\rho^p_{K_{\Delta\tau}}(f,g) \le$$

$$\le \left(\frac{p-1}{\lambda}\right)^{2(p-1)} e^{\lambda(x+y)}\rho^p_{K_{\Delta\tau}}(f,g) \quad \text{(cf.(A3))}.$$

If $(\alpha,\beta)\in\mathbb{R}^2_+$, then

$$\left|\int_0^\beta (f(\alpha,\eta)-g(\alpha,\eta))d\eta\right|^p \le$$

$$\le \left(\int_0^\beta |f(\alpha,\eta)-g(\alpha,\eta)|d\eta\right)^p = \left(\int_0^\beta e^{\frac{\lambda}{p}(\alpha+\eta-\alpha-\eta)}|f(\alpha,\eta)-g(\alpha,\eta)|d\eta\right)^p \le$$

$$\le \left(\int_0^\beta e^{\frac{q\lambda}{p}(\alpha+\eta)}d\eta\right)^{\frac{p}{q}}\int_0^\beta e^{-\lambda(\alpha+\eta)}|f(\alpha,\eta)-g(\alpha,\eta)|^p d\eta \le$$

$$\le \left(\frac{p-1}{\lambda}\right)^{p-1} e^{\lambda(\alpha+\beta)}\int_0^\beta e^{-\lambda(\alpha+\eta)}|f(\alpha,\eta)-g(\alpha,\eta)|^p d\eta \le$$

$$\le \left(\frac{p-1}{\lambda}\right)^{p-1} e^{\lambda(x+y)}\int_0^\beta e^{-\lambda(\alpha+\eta)}|f(\alpha,\eta)-g(\alpha,\eta)|^p d\eta \quad \text{(cf.(A3))}.$$

In the same way (by means of (A3)) we obtain: if $(\theta,\kappa)\in\mathbb{R}^2_+$, then

$$\left|\int_0^\theta (f(\xi,\kappa)-g(\xi,\kappa))d\xi\right|^p \le \left(\frac{p-1}{\lambda}\right)^{p-1} e^{\lambda(x+y)}\int_0^\theta e^{-\lambda(\xi+\kappa)}|f(\xi,\kappa)-g(\xi,\kappa)|^p d\xi.$$

Hence

$$\int_K\int e^{-\lambda(|x|+|y|)}|T(f)(x,y)-T(g)(x,y)|^p dxdy \le$$

$$\leq 4^{p-1}\Bigg(\left(\frac{p-1}{\lambda}\right)^{2(p-1)}\rho^p_{K_{\Delta r}}(f,g)\int_{K_+}\int\omega_1^p(x,y)dxdy+$$

$$+\left(\frac{p-1}{\lambda}\right)^{p-1}\int_{K_+}\int\omega_2^p(y)\int_0^{\beta}e^{-\lambda(\alpha+\eta)}|f(\alpha,\eta)-g(\alpha,\eta)|^p d\eta dxdy+$$

$$+\left(\frac{p-1}{\lambda}\right)^{p-1}\int_{K_+}\int\omega_3^p(x)\int_0^{\theta}e^{-\lambda(\xi+\kappa)}|f(\xi,\kappa)-g(\xi,\kappa)|^p d\xi dxdy+$$

$$+\omega_4^p e^{-\lambda\delta_0}\int_{K_+}\int e^{-\lambda(\mu+\nu)}|f(\mu,\nu)-g(\mu,\nu)|^p dxdy\Bigg).$$

Denote $K_y=\{y:(x,y)\in K_+\}$, $K_x=\{x:(x,y)\in K_+\}$. Consequently

$$\int_{K_+}\int\omega_2^p(y)\int_0^{\beta}e^{-\lambda(\alpha+\eta)}|f(\alpha,\eta)-g(\alpha,\eta)|^p d\eta dxdy\leq$$

$$\leq\int_{K_y}\omega_2^p(v)\int_{K_\alpha}\int_{K_\beta}|\alpha_u^*(u,v)|e^{-\lambda(u+\eta)}|f(u,\eta)-g(u,\eta)|^p d\eta dudv\leq$$

$$\leq\|\alpha_u^*\|_{L^\infty(\mathbb{R}^2_+)}\rho^p_{K_{\alpha\beta}}(f,g)\int_{K_y}\omega_2^p(v)dv$$

and similarly

$$\int_{K_+}\int\omega_3^p(x)\int_0^{\theta}e^{-\lambda(\xi+\kappa)}|f(\xi,\kappa)-g(\xi,\kappa)|^p d\xi dxdy\leq$$

$$\leq\|\kappa_v^*\|_{L^\infty(\mathbb{R}^2_+)}\rho^p_{K_{\theta\kappa}}(f,g)\int_{K_x}\omega_3^p(u)du.$$

$$\int_{K_+}\int e^{-\lambda(\mu+\nu)}|f(\mu,\nu)-g(\mu,\nu)|^p dxdy=$$

$$=\int_{K_{\mu\nu}}\int\left|\frac{D(\mu^*,\nu^*)}{D(u,v)}\right|e^{-\lambda(u+v)}|f(u,v)-g(u,v)|^p dudv\leq$$

$$\leq\left\|\frac{D(\mu^*,\nu^*)}{D(u,v)}\right\|_{L^\infty(\mathbb{R}^2_+)}\rho^p_{K_{\mu\nu}}(f,g).$$

Thus we receive the estimate

$$\rho^p_K(T(f),T(g))\leq 4^{p-1}\rho^p_{j(K)}(f,g)\Bigg(\left(\frac{p-1}{\lambda}\right)^{2p-2}\|\omega_1\|^p_{L^p(K_{\Delta r})}+$$

$$+\left(\frac{p-1}{\lambda}\right)^{p-1}C_\alpha\|\omega_2\|^p_{L^p(K_y)}+\left(\frac{p-1}{\lambda}\right)^{p-1}C_\kappa\|\omega_3\|^p_{L^p(K_x)}+\lambda^{-\delta_0}C_{\mu\nu}\omega_4^p\Bigg),$$

where $C_\alpha=\|\alpha_u^*\|_{L^\infty(\mathbb{R}^2_+)}$, $C_\kappa=\|\kappa_v^*\|_{L^\infty(\mathbb{R}^2_+)}$, $C_{\mu\nu}=\left\|\frac{D(\mu^*,\nu^*)}{D(u,v)}\right\|_{L^\infty(\mathbb{R}^2_+)}$.

Define

$$\Phi_K(t) =$$

$$\begin{cases} 0, & K_+ = \emptyset \\ t\sqrt[p]{\left(\frac{2p-2}{\lambda}\right)^{2p-2}\|\omega_1\|^p_{L^p(K_{\Delta\tau})} + \left(\frac{4p-4}{\lambda}\right)^{p-1} C_\alpha \|\omega_2\|^p_{L^p(K_y)} + \left(\frac{4p-4}{\lambda}\right)^{p-1} C_\kappa \|\omega_3\|^p_{L^p(K_x)} + \frac{(4\omega_4)^p C_{\mu\nu}}{4\lambda^{\delta_0}}}, & K_+ \neq \emptyset \end{cases}$$

Then $\rho_K(T(f), T(g)) \le \Phi_K(\rho_{j(K)}(f,g))\ \forall\ K \in \mathcal{A},\ \forall\ f, g \in X$.

We can find and fix $\lambda > 1$ so that

$$\left(\frac{2p-2}{\lambda}\right)^{2p-2} \|\omega_1\|^p_{L^p(\mathbb{R}^2_+)} + \left(\frac{4p-4}{\lambda}\right)^{p-1} C_\alpha \|\omega_2\|^p_{L^p(\mathbb{R}^1_+)} +$$

$$+\left(\frac{4p-4}{\lambda}\right)^{p-1} C_\kappa \|\omega_3\|^p_{L^p(\mathbb{R}^1_+)} + \frac{(4\omega_4)^p C_{\mu\nu}}{4\lambda^{\delta_0}} < 1$$

for example

$$\lambda > \max\{2^q(p-1)\|\omega_1\|^{q/2}_{L^p(\mathbb{R}^2_+)}, 4^q C_\alpha^{q/p}(p-1)\|\omega_2\|^q_{L^p(\mathbb{R}^1_+)},$$

$$4^q C_\kappa^{q/p}(p-1)\|\omega_3\|^q_{L^p(\mathbb{R}^1_+)}, C_{\mu\nu}^{1/\delta_0}(4\omega_4)^{p/\delta_0}\}.$$

Consequently $\Phi_K(t) < t$, $\Phi_K(t)/t = const$ and T is a Φ-contraction.

K_+ is bounded set $\Rightarrow$ $\Delta(K_+), \tau(K_+), \alpha(K_+), \kappa(K_+)$ are bounded sets too, so (A1') implies $\exists\ C_K = const \ge 0$:

$$|F(x, y, \varphi_0(\Delta, \tau), \varphi_1(\alpha), \varphi_2(\kappa), 0)| \le a(x,y) + bC_K \in L^p_{loc}(\mathbb{R}^2_+) \text{ (cf.(A1'))}.$$

We choose the element $f_0 \in X$:

$$f_0(x,y) = \begin{cases} 0, & \text{a.e. on } \mathbb{R}^2_+ \\ \varphi(x,y), & (x,y) \in \mathbb{R}^2 \setminus \mathbb{R}^2_+. \end{cases}$$

Then

$$T(f_0) = \begin{cases} F(x, y, \varphi_0(\Delta, \tau), \varphi_1(\alpha), \varphi_2(\kappa), 0), & \text{a.e. on } \mathbb{R}^2_+ \\ \varphi(x,y),\ (x,y) \in \mathbb{R}^2 \setminus \mathbb{R}^2_+. \end{cases} \Rightarrow T(f_0) \in X$$

and consequently

$$\|T(f)\|_{L^p(K)} \le \|T(f) - T(f_0)\|_{L^p(K)} + \|T(f_0)\|_{L^p(K)} \le$$

$$\le (\max_{(x,y)\in K} e^{\lambda(|x|+|y|)})^{\frac{1}{p}} \rho_K(T(f), T(f_0)) + \|T(f_0)\|_{L^p(K)} \le$$

$$\le c(K, \lambda, p)\rho_{j(K)}(f, f_0) + \|T(f_0)\|_{L^p(K)},\ \forall\ f \in X.$$

But $\rho_{j(K)}(f, f_0) \leq \|f\|_{L^p(j(K)\cap\mathbb{R}^2_+)} \Rightarrow T(f) \in X$.

Besides the estimates $\rho_{j^l(K)}(f_0, T(f_0)) \leq \rho_{\widehat{K}}(f_0, T(f_0))$, $\rho_{j^l(K)}(f,g) \leq \rho_{\widehat{K}}(f,g)$ for any integer $l \geq 0$, $\forall\ f,g \in X$ (cf. (A5)) show that conditions 3 of theorem 1 and 4 of theorem 2 are fulfilled. Using once again (A5), we check that condition 2 of theorem 1 is also fulfilled, which completes the proof of theorem 4.

Acknowledgement. The author thanks Prof. V. Angelov for his very useful suggestions.

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