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PERTURBATION ANALYSIS OF MONOTONE GENERALIZED EQUATIONS

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Abstract. Our goal is to establish new methods and results in reflexive Banach spaces to the theory of local stability of the solutions of some non-compact generalized equations, including parametric variational inequalities. The continuity of the projections of a fixed point onto a family of nonempty, closed, convex sets will be also studied using these methods. The results from this paper generalize results proved in finit dimensional spaces and Hilbert spaces.

0. Introduction

Stability topics for parametric variational inequalities were studied in many papers in finite or in infinite dimensional Hilbert spaces [1, 3, 4, 8, 11]. The proofs in those papers are closely connected with the Hilbert spaces' properties (for example, the nonexpansivity of the metric projection onto a closed, convex set).

Our method is independent from the above mentioned properties and also from compactness assumptions (for compact perturbation of monotone operators see [7].

Papers [3, 11] use the strong-monotonicity condition in finite dimensional spaces. We will replace this condition by a weaker one, φ-uniform-monotonicity (used also in [1]). In Banach spaces this is a weaker and more usefull condition, (see Proposition 1.1, Examples 5.1 and 5.3) that the strong-monotonicity.

We will discuss also some aspects with respect to a consistency condition. Consistency conditions are frequently used in the theory of implicit function theorems [1, 2, 4]. Our condition is a generalization of those used in [1, 4]. We will show that the

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consistency condition is satisfied under reasonable conditions, as pseudo-continuity or lower-semicontinuity (see Corollary 2.1, Examples 5.1 and 5.2). The continuity of the projection of a fixed point onto a family of nonempty, closed, convex sets implies the consistency of the normal cone operator. A result similar to the Hölder continuity of the projections of a fixed point onto a pseudo-Lipschitz continuous family of closed convex sets [11] holds for uniformly-convex Banach spaces. We will use a generalization of the metric projection operator introduced by E. Zarantonello [12].

We will denote by Ω, Λ, W topological spaces and by X a reflexive Banach space. Throughout this paper we will work with the fixed points $x_0 \in X$, $\omega_0 \in \Omega$, $\lambda_0 \in \Lambda$, $w_0 \in W$ and with their neighborhoods $X_0 = B(x_0, r)$ (the closed ball centered at x_0 and radius r) of x_0, Ω_0 of ω_0, Λ_0 of λ_0, and W_0 of w_0. We need a single-valued mapping $f : X_0 \times \Omega_0 \to X^*$, a set-valued mapping $F : X_0 \times W_0 \rightsquigarrow X^*$ and an other set-valued mapping $C : \Lambda_0 \rightsquigarrow X$ with nonempty, closed, convex values.

Let us consider the following parametric variational inequality

$$\begin{cases} \text{find} \quad x \in C(\lambda) \quad \text{such that} \\ \langle f(x,\omega) , y - x \rangle \geq 0 , \quad \text{for all} \quad y \in C(\lambda) \end{cases} \qquad (VI(\omega,\lambda))$$

and the equivalent generalized equation

$$0 \in f(x,\omega) + N_{C(\lambda)}(x) , \qquad (GE(\omega,\lambda))$$

where

$$N_{C(\lambda)}(x) = \{x^* \in X^* : \langle x^* , y - x \rangle \leq 0 , \quad \text{for all} \quad y \in C(\lambda)\}$$

is the normal cone to the set $C(\lambda)$ at the point x.

The normal cone mappings $N_{C(\lambda)} : X \rightsquigarrow X^*$ are maximal-monotone, because the sets $C(\lambda)$ are nonempty, closed and convex.

In a reflexive Banach space we can introduce equivalent norms for which the space is strictly-convex with strictly-convex dual or locally uniform-convex with locally uniform-convex dual. The continuity and monotonicity properties from this paper remain the same when we use these equivalent norms, so we can use them when we need better properties for the duality mapping.

1. Preliminary results

In this section we present basic definitions and results.

Definition 1.1. [1] *The mappings $F(\cdot, w)$ are said φ-uniformly-monotone for all $w \in W_0$, if there exists an increasing function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$, with $\varphi(r) > 0$ when $r > 0$, such that for all $w \in W_0$, $x_1, x_2 \in Dom\, F(\cdot, w)$, $x_1^* \in F(x_1, w)$, $x_2^* \in F(x_2, w)$ hold*

$$\langle x_1^* - x_2^*\, ,\, x_1 - x_2 \rangle \ \geq\ \varphi\left(\|x_1 - x_2\|\right) \|x_1 - x_2\| \, .$$

If the function φ is defined as $\varphi(r) = ar$, with $a > 0$, then the mappings $F(\cdot, w)$ are said strongly-monotone with constant a.

The following proposition shows that φ-uniform-monotonicity is a natural one in uniformly-convex spaces.

Proposition 1.1. [10] *A Banach space X is uniformly-convex if and only if for each $R > 0$ there exists an increasing function $\varphi_R : \mathbf{R}_+ \to \mathbf{R}_+$, with $\varphi_R(r) > 0$ when $r > 0$, such that the normalized duality mapping $J : X \rightsquigarrow X^*$, defined by*

$$J(x) \ = \ \left\{ \, x^* \in X^* \ : \ \langle x^*\, ,\, x \rangle \ = \ \|x\|^2 \, ,\ \|x\| = \|x^*\| \, \right\} \ ,$$

is φ_R-uniformly-monotone in $B(0, R)$.

Definition 1.2. *Let $A, B \subset X$. The Hausdorff distance between A, B is defined as*

$$H(A, B) \ = \ \max\{e(A, B)\, ,\, e(B, A)\} \ ,$$

where

$$e(A, B) \ = \ \sup_{a \in A} \inf_{b \in B} \|a - b\| \, .$$

Definition 1.3. *Let (Λ, d) be a metric space.*

a) The set-valued mapping C is pseudo-continuous at $(\lambda_0, x_0) \in Graph\, C$ if there exist neighborhoods $V \subset \Lambda_0$ of λ_0, $U \subset X_0$ of x_0 and there exists a function $\beta : \mathbf{R}_+ \to \mathbf{R}_+$ continuous at 0, with $\beta(0) = 0$, such that

$$C(\lambda_0) \cap U \ \subset \ C(\lambda) \ + \ \beta\left(d(\lambda, \lambda_0)\right) \ B(0, 1) \tag{1.1}$$

and

$$C(\lambda) \cap U \subset C(\lambda_0) + \beta(d(\lambda, \lambda_0))\, B(0,1) \tag{1.2}$$

for all $\lambda \in V$.

b) The set-valued mapping C *is pseudo-continuous around* $(\lambda_0, x_0) \in Graph\, C$ *if there exist neighborhoods* $V \subset \Lambda_0$ *of* λ_0, $U \subset X_0$ *of* x_0 *and there exists a function* $\beta : \mathbf{R}_+ \to \mathbf{R}_+$ *continuous at* 0, *with* $\beta(0) = 0$, *such that*

$$C(\lambda_1) \cap U \subset C(\lambda_2) + \beta(d(\lambda_1, \lambda_2))\, B(0,1) \tag{1.3}$$

for all $\lambda_1, \lambda_2 \in V$.

c) If the function β *is defined as* $\beta(r) = Lr$, *with* $L \geq 0$, *then we say that* C *is pseudo-Lipschitz continuous at* (λ_0, x_0) *(resp. around* (λ_0, x_0)*).*

d) The set-valued mapping C *is pseudo-continuous on the set* $\Lambda_1 \subset \Lambda_0$, *if it is pseudo-continuous at each point* $(\lambda, x) \in Graph\, C$, $\lambda \in \Lambda_1$.

Remark 1.1. If the set-valued mapping $C(\cdot) \cap X_0$ is continuous with respect to the Hausdorff distance at λ_0 (resp. in a neighborhood of λ_0), then the set-valued mapping C is pseudo-continuous at (λ_0, x_0) (resp. around (λ_0, x_0)).

In [11] it is proved the following theorem:

Theorem 1.1. *In the case of* $\Omega \subset \mathbf{R}^m$, $\Lambda \subset \mathbf{R}^p$, $X = \mathbf{R}^n$, *let us suppose:*

(i) x_0 *is a solution of* $VI(\omega_0, \lambda_0)$;

(ii) there exists $l > 0$ *such that*

$$\|f(x_1, \omega_1) - f(x_2, \omega_2)\| \leq l\,(\|x_1 - x_2\| + \|\omega_1 - \omega_2\|) \ ,$$

for all $x_1, x_2 \in X_0$, $\omega_1, \omega_2 \in \Omega_0$;

(iii) the mappings $f(\cdot, \omega)$ *are strongly-monotone with a constant* $a > 0$, *for all* $\omega \in \Omega_0$;

(iv) the set-valued mapping C *is pseudo-Lipschitz continuous around* $(\lambda_0, x_0) \in Graph\, C$.

Then there exist constants l_{ω_0}, $l_{\lambda_0} \geq 0$ *and there exist neighborhoods* $\Omega' \subset \Omega_0$ *of* ω_0, $\Lambda' \subset \Lambda_0$ *of* λ_0 *such that:*

a) for every $(\omega, \lambda) \in \Omega' \times \Lambda'$ there exists a unique solution $x(\omega, \lambda)$ of $VI(\omega, \lambda)$;

b) for all (ω_1, λ_1), $(\omega_2, \lambda_2) \in \Omega' \times \Lambda'$ we have

$$\|x(\omega_1, \lambda_1) - x(\omega_2, \lambda_2)\| \leq l_{\omega_0}\|\omega_1 - \omega_2\| + l_{\lambda_0}\|\lambda_1 - \lambda_2\|^{\frac{1}{2}} .$$

The Hölder continuity with respect to λ it is the consequence of the following result :

Proposition 1.2. [11] *Let $\Omega \subset \mathbf{R}^p$ and $X = \mathbf{R}^n$. Let us assume that the set-valued mapping C is pseudo-Lipschitz continuous around (λ_0, x_0).*
Then there exist neghborhoods $\Omega' \subset \Omega_0$ of ω_0 and $X' \subset X_0$ of x_0 and there exists a constant $l' > 0$ such that

$$\left\|P_{C(\lambda_1)\cap X_0}(x) - P_{C(\lambda_2)\cap X_0}(x)\right\| \leq l'\|\lambda_1 - \lambda_2\|^{\frac{1}{2}} ,$$

for all λ_1, $\lambda_2 \in \Lambda'$ and $x \in X'$.

We denoted by $P_{C(\lambda)\cap X_0}(x)$ the metric projection of the point x onto the set $C(\lambda) \cap X_0$, i.e. the unique point in $C(\lambda) \cap X_0$ with minimal distance to x.
The continuity of $C(\cdot) \cap X_0$, with respect to the Hausdorff distance, at λ_0 is assumed in [3] and the continuity of $P_{C(\lambda)\cap X_0}$ at λ_0 is proved.

2. An implicit function theorem for monotone mappings

In this section we will show that Theorem 4.3 of [1] remains true when we suppose X a reflexive Banach space (Theorem 2.1). We will use this theorem to study the stability of the solutions of $VI(\omega, \lambda)$, using only the consistency of the normal cone operator which is a weaker property then the continuity of the projections of a fixed point onto a family of nonempty, closed, convex sets.

Lemma 2.1. [5] *Let $T : X \rightsquigarrow X^*$ be a maximal-monotone set-valued mapping. For all integers $k \geq 1$ we define the following single-valued mappings:*

$$P_k = (J + kT)^{-1} : X^* \to X .$$

If a sequence (x_k), with $x_{k+1} = P_k(Jx_k)$ is bounded, then there exists $\overline{x} \in X$ such that $0 \in T(\overline{x})$ and (x_k) has a subsequence weakly converging to $\overline{x}$.

Remark 2.1. From $x_{k+1} = P_k(Jx_k)$ we have

$$\frac{1}{k}(Jx_k - Jx_{k+1}) \in T(x_{k+1}) \, ,$$

so $x_{k+1} \in D(T)$.

If $D(T)$ is bounded, then (x_k) is also bounded and T has a zero in $D(T)$.

If $T_1 = T + N_{B(0,\varepsilon)}$ is maximal-monotone for an $\varepsilon > 0$, then there exists $x \in B(0,\varepsilon)$ such that $0 \in T(x) + N_{B(0,\varepsilon)}$. If $\|x\| < \varepsilon$, then $0 \in T(x)$.

Lemma 2.2. *Let $T : X \rightsquigarrow X^*$ be a maximal-monotone map. We suppose that there exist $0 < \delta < \varepsilon$ such that $D(T) \cap int B(0,\varepsilon) \neq \emptyset$ and $\langle y, x \rangle > 0$, for all $x \in X$, with $\delta \leq \|x\| \leq \varepsilon$ and for all $y \in T(x)$.*

Then there exists $\overline{x} \in B(0,\delta)$, such that $0 \in T(\overline{x})$.

Proof. Because of $D(T) \cap int B(0,\varepsilon) \neq \emptyset$, $T_1 = T + N_{B(0,\varepsilon)}$ is a maximal-monotone mapping with $D(T_1) = B(0,\varepsilon)$.

If $x \in B(0,\varepsilon)$ and $y_1 \in T_1(x)$, then there exist $y \in T(x)$, $n \in N_{B(0,\varepsilon)}(x)$ with $n = 0$ or $n = \lambda J(x)$, $\lambda > 0$, such that $y_1 = y + n$.

Then $\langle y_1, x \rangle = \langle y + n, x \rangle \geq \langle y, x \rangle$ and hence the assumptions of Lemma 2.2 are also satisfied by T_1.

Let us denote

$$P_k(x) = (J + kT_1)^{-1}(Jx) \, , \; x_1 = 0 \, , \; x_{k+1} = P_k(x_k) \, .$$

We will prove that $\|x_k\| \leq \delta$, for all $k \geq 1$.

Let us suppose the contrary and let k_0 be the first integer such that $\|x_{k_0}\| \leq \delta$ and $\|x_{k_0+1}\| > \delta$. Then

$$Jx_{k_0} \in Jx_{k_0+1} + k_0 T_1(x_{k_0+1}) \, ,$$

so $x_{k_0+1} \in D(T_1) = B(0,\varepsilon)$ and there exists $u_{k_0+1} \in T_1(x_{k_0+1})$, such that

$$Jx_{k_0} = Jx_{k_0+1} + k_0 u_{k_0+1} \, .$$

Then

$$\|Jx_{k_0}\| \|x_{k_0+1}\| \geq \langle x_{k_0+1}, Jx_{k_0} \rangle = \langle x_{k_0+1}, Jx_{k_0+1} \rangle + k_0 \langle x_{k_0+1}, u_{k_0+1} \rangle >$$

$$> \|x_{k_0+1}\|^2 .$$

Hence $\|x_{k_0}\| = \|Jx_{k_0}\| > \|x_{k_0+1}\| > \delta$, which is a contradiction.

So, $(x_k) \subset B(0,\delta)$ and using Lemma 2.1 together with the weakly-compactness of $B(0,\delta)$, we can found $\overline{x} \in B(0,\delta)$, such that $0 \in T_1(\overline{x})$.

But $N_{B(0,\varepsilon)}(\overline{x}) = \{0\}$, so $0 \in T(\overline{x})$.

Remark 2.2. Let us fix an $x_0 \in X$.

If we use Lemma 2.2 for $F(x) = T(x + x_0)$ we get:

Let $0 < \delta < \varepsilon$ be such that $D(T) \cap int B(x_0,\varepsilon) \neq \emptyset$ and $\langle y, x\rangle > 0$, for all $x \in X$ with $\delta \leq \|x\| \leq \varepsilon$ and for all $y \in T(x + x_0)$.

Then there exists $\overline{x} \in B(x_0,\delta)$, such that $0 \in T(\overline{x})$.

Definition 2.1. *Let $F : X \times W \rightsquigarrow X^*$ be a set-valued map and let $y_0 \in F(x_0, w_0)$. We say that F is consistent with respect to w at (x_0, w_0, y_0) in a neighborhood W_0 of w_0, if there exists a function $\beta : W_0 \to \mathbf{R}_+$, continuous at w_0, with $\beta(w_0) = 0$, such that for all $w \in W_0$ there exists $(x_w, y_w) \in GraphT(\cdot, w)$, satisfying $\|x_w - x_0\| \leq \beta(w)$ and $\|y_w - y_0\| \leq \beta(w)$.*

Remark 2.3. For example, F is consistent in w at (x_0, w_0, y_0), if $F(\cdot, w_0)$ has a continuous selection trough $(x_0, y_0) \in \text{Graph}F(\cdot, w_0)$.

In [4] it is used a stronger assumption ($x_w = x_0$ for all $w \in W_0$), but in the study of the parametric variational inequalities this form cannot be used. We will show also that the normal cone operator is consistent if the projection operator is continuous with respect to the parameter λ.

The following theorem is the generalization of the Theorem 4.3 of [1], in the case of reflexive Banach spaces. We will suppose that X is renormed strictly-convex with strictly-convex dual.

Theorem 2.1. *Let us assume that:*

i) $0 \in F(x_0, w_0)$;

ii) F is consistent with respect to w at $(x_0, w_0, 0)$ in W_0;

iii) the set-valued mappings $F(\cdot, w)$ are maximal-monotone and φ-uniformly-monotone for all $w \in W_0$.

Then there exists a neighborhood W_1 of w_0 and a unique mapping $x : W_1 \to X_0$, continuous at w_0, such that $x(w_0) = x_0$ and $0 \in F(x(w), w)$, for all $w \in W_1$.

Proof. Let us fix $0 < \varepsilon < \varepsilon_1$ such that $B(x_0, \varepsilon_1) \subset X_0$. Let $W' \subset W_0$ be a neighborhood of w_0, such that $\beta(w) < \varepsilon_1$, for all $w \in W'$.

Let $0 < \delta \leq \varepsilon$ and $w \in W'$ be choosen arbitrarily.

Then $D(F(\cdot, w)) \cap intB(x_0, \varepsilon_1) \neq \emptyset$, because from assumption *ii)* , there exists $(x_w, y_w) \in \text{Graph}F(\cdot, w)$, such that $||x_w - x_0|| \leq \beta(w) < \varepsilon_1$ and $||y_w|| \leq \beta(w)$.

Let us choose $x \in X$, with $\delta \leq ||x|| \leq \varepsilon$ and $y \in F(x + x_w, w)$. Then

$$\varphi(\,||x||\,)||x|| \leq \langle y - y_w, x\rangle = \langle y, x\rangle - \langle y_w, x\rangle$$

and hence

$$\langle y, x\rangle \geq \varphi(\delta)\delta - \varepsilon\beta(w) .$$

Let us denote $M_w = \{\delta > 0 \ : \ \delta\varphi(\delta) > \varepsilon\beta(w)\}$.

We can see that $M_w \neq \emptyset$ for all w in a neighborhood $W_1 \subset W'$ of w_0 and $\inf M_w \to 0$, when $w \to w_0$. So, we can choose a selection $\delta(w) \in M_w$, such that $\delta(w) \to 0$, when $w \to w_0$.

Using Remark 2.2 we can find, for all $w \in W_1$, a solution $x(w) \in B(x_w, \delta(w))$ of [3] and this solution is unique because of the φ-uniform-monotonicity of $F(\cdot, w)$. We have also

$$||x(w) - x_0|| \leq ||x(w) - x_w|| + ||x_w - x_0|| \leq \delta(w) + \beta(w) \to 0 ,$$

when $w \to w_0$.

Remark 2.4. In the case when F is a single valued mapping the assumptions of Theorem 2.1 can be written as:

i) $0 = F(x_0, w_0)$;

ii) the mapping F is continuous at (x_0, w_0);

iii) the mappings $F(\cdot, w)$ are hemicontinuous and φ-uniformly-monotone on X_0, for *all* $w \in W_0$.

In the following two corollaries we will study the continuity of the solutions in a neighborhood of the fixed parameter ω_0.

Corollary 2.1. *Let* (W, d) *be a metric space.*
If we replace assumption ii) of Theorem 2.1 by:
ii') the set-valued mappings $F(x, \cdot)$ *are pseudo-continuous on* W_0*, for all* $x \in X_0$*, then the mapping* x *is continuous in a neighborhood of* w_0.

Proof. We will show that the pseudo-continuity at $(w_0, 0) \in \operatorname{Graph} F(x_0, \cdot)$ implies the consistency condition ii). Indeed, there exist neighborhoods U of 0_{X^*}, V of w_0 and a function $\beta_0 : \mathbf{R}_+ \to \mathbf{R}_+$continuous at 0, with $\beta_0(0) = 0$, such that

$$F(x_0, w_0) \cap U \subset F(x_0, w) + \beta_0(d(w, w_0))B(0, 1)$$

and

$$F(x_0, w) \cap U \subset F(x_0, w_0) + \beta_0(d(w, w_0))B(0, 1)$$

for all $w \in V$. Hence, for $0 \in F(x_0, w_0) \cap U$ and for all $w \in V$, there exists $z_w \in F(x_0, w)$ such that $\|z_w\| \leq \beta_0(d(w, w_0))$.
Now we use Theorem 2.1 to obtain a neighborhood W_1 of w_0 and a unique mapping $x : W_1 \to X_0$ continuous at w_0, with $x(w_0) = x_0$ and $0 \in F(x(w), w)$ for all $w \in W_1$. The continuity of the mapping x at w_0 implies that there exists an open neighborhood $W_1' \subset W_1$ of w_0, such that $x(w) \in \operatorname{int} X_0$ for all $w \in W_1'$.
If we choose $\overline{w} \in W_1'$ arbitrarily, a constant $\overline{r} > 0$, such that $B(x(\overline{w}), \overline{r}) \subset X_0$ and we use the pseudo-continuity of $F(x(\overline{w}), \cdot)$ at $(\overline{w}, 0)$, which implies the consistency at $(x(\overline{w}), \overline{w}, 0)$, then we can use Theorem 2.1 to find a neighborhood $\overline{W} \subset W_1$ of $\overline{w}$ and a unique mapping $\overline{x} : \overline{W} \to B(x(\overline{w}), \overline{r})$ continuous at $\overline{w}$, such that $\overline{x}(\overline{w}) = x(\overline{w})$, and $0 \in F(\overline{x}(w), w)$ for all $w \in \overline{W}$. The uniqueness of the mappings x and $\overline{x}$ implies that they coincide on $\overline{W}$, so we have proved the continuity of the mapping x at $\overline{w}$. Because $\overline{w}$ has been choosen arbitrarily, the continuity holds on W_1'.

Corollary 2.2. *Let (W,d) be a metric space. Let us suppose that in Theorem 2.1 we replace assumption ii) by:*

ii_L) there exists a constant $L > 0$ and for each $x \in X_0$ there exists a neighborhood U_x of 0_{X^*} such that

$$F(x,w_1) \cap U_x \subset F(x,w_2) + Ld(w_1,w_2)B(0,1)$$

for all $w_1, w_2 \in W_0$,

and assumption iii) by:

iii_L) the set-valued mappings $F(\cdot,w)$ are maximal-monotone and strongly-monotone with a constant $a > 0$, for all $w \in W_0$.

Then the mapping x is Lipschitz-continuous, with the constant $\frac{L}{a}$, in a neighborhood of w_0.

Proof. Using Corollary 2.1 we obtain a neighborhood W_1' of w_0, such that the mapping x is continuous on W_1'.

Let us choose $w_1, w_2 \in W_1'$ arbitrarily. Assumption ii_L) implies that for $x(w_1) \in X_0$ there exists a neighborhood U of 0_{X^*} such that

$$F(x(w_1),z) \cap U \subset F(x(w_1),t) + Ld(z,t)B(0,1)$$

for all $z,t \in W_0$.

Hence for $0 \in F(x(w_1),w_1) \cap U$ there exists $z_2 \in F(x(w_1),w_2)$ such that $||z_2|| \leq Ld(w_1,w_2)$. Then

$$a||x(w_1) - x(w_2)||^2 \leq \langle z_2 - 0\,,\, x(w_1) - x(w_2)\rangle \leq$$

$$\leq ||z_2||||x(w_1) - x(w_2)|| \,.$$

So,

$$||x(w_1) - x(w_2)|| \leq \frac{L}{a}d(w_1,w_2)\,.$$

In the followings we will apply Theorem 2.1 in the study of $VI(\omega,\lambda)$. We suppose the consistency of the normal cone operator instead of the continuity, with respect to parameters, of the projections. The adventage of this approache is that the assumptions *a-d*) of Theorem 2.2 are independent from the geometrical properties of the reflexive Banach space X. We will show also that, in locally-uniform convex Banach spaces with locally-uniform convex dual, the consistency is a weaker property that the continuity of the projection. Assumption *iii)* of Theorem 1.1, due to Proposition 1.2, implies the continuity of the projections which it is supposed also in [1] and [3].

Theorem 2.2. *Let us suppose that:*

a) $0 \in f(x_0, \omega_0) + N_{C(\lambda_0)}(x_0)$*;*

b) f *is continuous on* $X_0 \times \Omega_0$*;*

c) the mapping $N(x, \lambda) = N_{C(\lambda) \cap X_0}(x)$ *is consistent with respect to* λ *at* $(x_0, \lambda_0, -f(x_0, \omega_0))$ *in* Λ_0*;*

d) the mappings $f(\cdot, \omega)$ *are* φ*-uniformly-monotone on* X_0*, for all* $\omega \in \Omega_0$*.*

Then there exist neighborhoods Ω_1 *and* Λ_1 *of* ω_0 *and* λ_0 *and a unique mapping* $x : \Omega_1 \times \Lambda_1 \to X_0$*, continuous at* (ω_0, λ_0)*, such that* $x(\omega_0, \lambda_0) = x_0$ *and* $0 \in f(x(\omega, \lambda), \omega) + N_{C(\lambda)}(x(\omega, \lambda))$*.*

Proof. Let us denote $W = \Omega \times \Lambda$ and $F(x, w) = F(x, \omega, \lambda) = f(x, \omega) + N_{C(\lambda) \cap X_0}(x)$. The mappings $F(\cdot, w)$ are maximal-monotone. These mappings are also φ-uniformly-monotone on X_0 as a sum of a φ-uniformly-monotone and a monotone mapping.
Assumption *c)* implies the existence of a function $\beta_1 : \Lambda_0 \to \mathbf{R}_+$, continuous at λ_0, with $\beta_1(\lambda_0) = 0$, such that for all $\lambda \in \Lambda_0$ there exists $(x_\lambda, n_\lambda) \in \mathrm{Graph} N(\cdot, \lambda)$ such that $||x_\lambda - x_0|| \leq \beta_1(\lambda)$ and $||n_\lambda + f(x_0, \omega_0)|| \leq \beta_1(\lambda)$.
Let us choose $(\omega, \lambda) \in \Omega_0 \times \Lambda_0$.
We denote $x_{\omega,\lambda} = x_\lambda$ and $y_{\omega,\lambda} = n_\lambda + f(x_\lambda, \omega)$.
Then $y_{\omega,\lambda} \in F(x_{\omega,\lambda}, \omega, \lambda)$ and

$$||y_{\omega,\lambda}|| \leq ||n_\lambda + f(x_0, \omega_0)|| + ||f(x_\lambda, \omega) - f(x_0, \omega_0)|| \leq$$

$$\leq \beta_1(\lambda) + ||f(x_\lambda, \omega) - f(x_0, \omega_0)|| = \beta(\omega, \lambda) \; .$$

Using the continuity of f, we get $\beta(\omega, \lambda) \to 0$, when $(\omega, \lambda) \to (\omega_0, \lambda_0)$, hence the assumptions of Theorem 2.1 are satisfied and the existence and continuity at (ω_0, λ_0) of the solutions of $0 \in F(x, w)$ are proved.
When (ω, λ) is close enough to (ω_0, λ_0), then $x(\omega, \lambda) \in int X_0$ and hence $N_{C(\lambda)}(x(\omega, \lambda)) =$

$N_{C(\lambda)\cap X_0}(x(\omega,\lambda))$, so the proof is complete.

Remark 2.5. We observe that if in Theorem 2.2 we suppose the same type of pseudo-continuities for the set-valued mapping N, as in the previous corollaries for F, then we obtain same continuities for the mapping x.

Definition 2.2. [12] *Let $C \subset X$ be a nonempty, closed, convex set. The projection onto C is the mapping $P_C : X^* \to X$ defined by*

$$P_C(x^*) = (J + N_C)^{-1}(x^*) .$$

In the case when X is a Hilbert space, this is the metric projection onto C.
Let X be a locally-uniform convex, reflexive Banach space, with X^* locally-uniform convex. In this case the normalized duality mapping is continuous from the strong topology of X to the strong topology of X^*.
Let us define the mapping $P : X^* \times \Lambda \to X$ by

$$P(x^*,\lambda) = P_{C(\lambda)\cap X_0}(x^*) .$$

The following result shows that the continuity of the projection with respect to a parameter implies the consistency of the normal cone operator.

Proposition 2.1. *If for an $n_0 \in N(x_0,\lambda_0)$, $P(Jx_0+n_0,\cdot)$ is continuous at λ_0, then N is consistent with respect to λ at (x_0,λ_0,n_0) in a neighborhood of λ_0.*

Proof. Let us take $x_\lambda = P(Jx_0+n_0,\lambda)$. Because of $x_0 = P(Jx_0+n_0,\lambda_0)$ and the continuity of P hold $\|x_\lambda - x_0\| \to 0$, when $\lambda \to \lambda_0$.
We have also

$$Jx_0 + n_0 \in Jx_\lambda + N(x_\lambda,\lambda)$$

and hence

$$Jx_0 + n_0 - Jx_\lambda \in N(x_\lambda,\lambda) .$$

We can take

$$y_\lambda = Jx_0 + n_0 - Jx_\lambda ,$$

$$\beta(\lambda) = \max\{\,\|x_\lambda - x_0\|\,,\,\|Jx_\lambda - Jx_0\|\,\}$$

and the consistency of N is proved.

3. Continuity of the projection with respect to a parameter

In this section, assuming the pseudo-continuity of the set-valued mapping C, we will show that the continuity, with respect to a parameter, of the projection operator holds in a uniformly-convex Banach space. We cannot obtain the same type of Hölder-continuity as in Proposition 1.2 because, as will be shown, that holds only in Hilbert-spaces.

Proposition 3.1. *Let (Λ, d) be a metric space.*
Let $A : X_0 \to X^*$ be a continuous, φ-uniformly-monotone mapping. Let us suppose that the set-valued mapping C is pseudo-continuous at $(\lambda_0, x_0) \in Graph\, C$ and $0 \in A(x_0) + N_{C(\lambda_0)}(x_0)$.
Then there exist a neighborhood $V \subset \Lambda_0$ of λ_0, a function $\beta_1 : \mathbf{R}_+ \to \mathbf{R}_+$ continuous at 0, with $\beta_1(0) = 0$, and a constant $s > 0$ such that the generalized equation

$$0 \in A(x) + N_{C(\lambda)}(x) \tag{3.1}$$

has a unique solution $x(\lambda) \in B(x_0, s)$ for all $\lambda \in V$ and also hold

$$\varphi\left(\|x(\lambda) - x_0\|\right)\|x(\lambda) - x_0\| \leq \beta_1\left(d(\lambda, \lambda_0)\right)\ . \tag{3.2}$$

Proof. We choose a constant $0 < s < r$ such that the pseudo-continuity of C can be written as:
- there exist a function $\beta : \mathbf{R}_+ \to \mathbf{R}_+$ continuous at 0, with $\beta(0) = 0$, and a neighborhood $V' \subset \Lambda_0$ of λ_0 such that

$$C(\lambda_0) \cap B(x_0, s) \subset C(\lambda) + \beta\left(d(\lambda, \lambda_0)\right) B(0,1)$$

and

$$C(\lambda) \cap B(x_0, s) \subset C(\lambda_0) + \beta\left(d(\lambda, \lambda_0)\right) B(0,1)$$

for all $\lambda \in V'$.

Using the continuity of β at 0, we can choose $\varepsilon > 0$ such that $B(\lambda_0, \varepsilon) \subset V'$ and $\beta(d(\lambda, \lambda_0)) \leq s$, for all $\lambda \in B(\lambda_0, \varepsilon)$.

Let us define $V_\varepsilon = B(\lambda_0, \varepsilon)$.

Let $\lambda \in V_\varepsilon$ be choosen arbitrarly. Then the inclusion

$$x_0 \in C(\lambda_0) \cap B(x_0, s) \subset C(\lambda) + \beta(d(\lambda, \lambda_0))\, B(0,1)$$

implies the existence of an $u_\lambda \in C(\lambda)$ such that

$$||x_0 - u_\lambda|| \leq \beta(d(\lambda, \lambda_0)) \leq s .$$

This means that $C(\lambda) \cap B(x_0, s)$ is nonempty for all $\lambda \in V$. Corollary 32.35 of [13] shows that the generalized equation

$$0 \in A(x) + N_{C(\lambda)\cap B(x_0,s)}(x)$$

has a unique solution $x(\lambda) \in C(\lambda) \cap B(x_0, s)$. So

$$\langle A(x(\lambda))\, ,\, u - x(\lambda)\rangle \geq 0$$

for all $u \in C(\lambda) \cap B(x_0, s)$.

The pseudo-continuity of the set-valued mapping C implies that for $x(\lambda)$ there exists an element $u_0 \in C(\lambda_0)$ such that $||x(\lambda) - u_0|| \leq \beta(d(\lambda, \lambda_0))$.

Using the φ-uniform-monotonicity of A we obtain

$$\varphi(||x(\lambda) - x_0||)\, ||x(\lambda) - x_0|| \leq \langle A(x(\lambda)) - A(x_0)\, ,\, x(\lambda) - x_0\rangle \leq$$

$$\leq \langle A(x(\lambda)) - A(x_0)\, ,\, x(\lambda) - x_0\rangle + \langle A(x_0)\, ,\, u_0 - x_0\rangle +$$

$$+ \langle A(x(\lambda))\, ,\, u_\lambda - x(\lambda)\rangle =$$

$$= -\langle A(x(\lambda))\, ,\, u_\lambda - x_0\rangle + \langle A(x_0)\, ,\, u_0 - x(\lambda)\rangle \leq$$

$$\leq ||A(x(\lambda))||\, ||u_\lambda - x_0|| + ||A(x_0)||\, ||u_0 - x(\lambda)|| \leq$$

$$\leq 2M\beta(d(\lambda, \lambda_0)) ,$$

where $M = \sup \{||A(x)|| : x \in B(x_0, s)\}$ is finite, because a continuous, monotone mapping is bounded on the interior of its domain. The inequality

$$\varphi(||x(\lambda) - x_0||)\,||x(\lambda) - x_0|| \leq 2M\beta(d(\lambda, \lambda_0))$$

means that $x(\lambda) \to x_0$, when $\lambda \to \lambda_0$.

We can choose a neighborhood $V \subset V_\varepsilon$ of λ_0, such that $||x(\lambda) - x_0|| < s$, for all $\lambda \in V$. This means that for $\lambda \in V$

$$N_{C(\lambda)}(x(\lambda)) = N_{C(\lambda)\cap B(x_0,s)}(x(\lambda))$$

and hence $x(\lambda)$ is a solution of the problem (3.1).

The inequality (3.2) is satisfied with $\beta_1(r) = 2M\beta(r)$.

Corollary 3.1. *If in Proposition 3.1 we suppose that the set-valued mapping C is pseudo-continuous around (λ_0, x_0), then*

$$\varphi(||x(\lambda_1) - x(\lambda_2)||)\,||x(\lambda_1) - x(\lambda_2)|| \leq \beta_1(d(\lambda_1, \lambda_2)) ,$$

for all $\lambda_1, \lambda_2 \in V$.

Proof. Let us choose the constant $0 < s < r$ and the neighborhood $V' \subset \Lambda_0$ such that

$$C(\lambda_1) \cap B(x_0, s) \subset C(\lambda_2) + \beta(d(\lambda_1, \lambda_2))B(0, 1)$$

for all $\lambda_1, \lambda_2 \in V'$.

As in the proof of the Proposition 3.1 we obtain the neighborhood V of λ_0 and the solution $x(\lambda)$ of (3.1), for all $\lambda \in V$.

Let us choose $\lambda_1, \lambda_2 \in V$. For $x(\lambda_1) \in C(\lambda_1) \cap B(x_0, s)$ there exists $u_2 \in C(\lambda_2)$, such that

$$||x(\lambda_1) - u_2|| \leq \beta(d(\lambda_1, \lambda_2)) .$$

For $x(\lambda_2) \in C(\lambda_2) \cap B(x_0, s)$ there exists $u_1 \in C(\lambda_1)$ such that

$$||x(\lambda_2) - u_1|| \leq \beta(d(\lambda_1, \lambda_2)) .$$

Then

$$\varphi(||x(\lambda_1) - x(\lambda_2)||)\,||x(\lambda_1) - x(\lambda_2)|| \leq$$

$$\leq \langle A(x(\lambda_1)) - A(x(\lambda_2)), x(\lambda_1) - x(\lambda_2)\rangle \leq$$

$$\leq \langle A(x(\lambda_1)) - A(x(\lambda_2)), x(\lambda_1) - x(\lambda_2)\rangle + \langle A(x(\lambda_1)), u_1 - x(\lambda_1)\rangle +$$

$$+ \langle A(x(\lambda_2)), u_2 - x(\lambda_2)\rangle =$$

$$= \langle A(x(\lambda_1)), u_1 - x(\lambda_2)\rangle - \langle A(x(\lambda_2)), x(\lambda_1) - u_2\rangle \leq$$

$$\leq \|A(x(\lambda_1))\| \, \|u_1 - x(\lambda_2)\| + \|A(x(\lambda_2))\| \, \|x(\lambda_1) - u_2\| \leq$$

$$\leq 2M\beta(d(\lambda_1, \lambda_2)) = \beta_1(d(\lambda_1, \lambda_2)) .$$

Corollary 3.2. *Let (Λ, d) be a metric space and let X be a uniformly-convex Banach space. If the set-valued mapping C is pseudo-continuous at (λ_0, x_0) (resp. around (λ_0, x_0)), then $P(\cdot, x^*)$ is continuous at λ_0 (resp. in a neighborhood of λ_0) for all $x^* \in X^*$.*

Proof. Let us choose $r, R > 0$ such that $x_0 \in \operatorname{int} B(0, R)$, $B(x_0, r) \subset B(0, R)$. Let us fix an element $x^* \in X^*$. We use Proposition 3.1 (resp. Corollary 3.1) in the case of the mapping $A : B(x_0, r) \to X^*$, defined by $A(x) = J(x) - x^*$, which is φ_R-uniformly-monotone due to Proposition 1.1. In this way we obtain a neighborhood V of λ_0 and a unique mapping $x : V \to X_0$ continuous at λ_0 (resp. in a neighborhood of λ_0), such that for all $\lambda \in V$ we have

$$0 \in J(x(\lambda)) - x^* + N_{C(\lambda)}(x(\lambda)) ,$$

which means that $x(\lambda) = P(x^*, \lambda)$.

Remark 3.1. If we suppose that C is pseudo-continuous around (λ_0, x_0), then $P(\cdot, x^*)$ is continuous in a neighborhood of λ_0.

We observe that in a uniformly-convex Banach space, which is not a Hilbert space, we cannot prove the Hölder-continuity of Proposition 1.2, even when C is pseudo-Lipschitz continuous. The Hölder-continuity holds only in a Hilbert space because only in this case is the normalized duality mapping strongly-monotone.

4. Parametric variational inequalities

In this section we generalize Theorem 1.1, on the continuity of the solutions of $VI(\omega,\lambda)$, in the case of reflexive Banach spaces. The continuity of the projection operator or the consistency of the normal cone operator will be not supposed. A consequence of Theorem 4.1 is that all the results from [1], [3], [8], [11] remain true in reflexive Banach spaces.

We suppose that X is renormed strictly-convex with strictly-convex dual and let (Λ, d) be a metric space.

Theorem 4.1. *Let us suppose that:*

a) $0 \in f(x_0,\omega_0) + N_{C(\lambda_0)}(x_0)$*;*

b) f *is continuous on* $X_0 \times \Omega_0$*;*

c) the mappings $f(\cdot,\omega)$ *are* φ*-uniformly-monotone on* X_0*, for all* $\omega \in \Omega_0$*;*

d) the set-valued mapping C *is pseudo-continuous at* (λ_0, x_0)*.*

Then there exist neighborhoods Ω' *of* ω_0*,* Λ' *of* λ_0 *and a unique mapping* $x : \Omega' \times \Lambda' \to X_0$ *continuous at* (ω_0,λ_0)*, such that* $x(\omega_0,\lambda_0) = x_0$ *and*

$$0 \in f(x(\omega,\lambda), \omega) + N_{C(\lambda)}(x(\omega,\lambda))$$

for all $\omega \in \Omega'$, $\lambda \in \Lambda'$.

Proof. We choose the positive constants ε, r small enough to

$$\beta(d(\lambda,\lambda_0)) \leq r ,$$

$$C(\lambda_0) \cap B(x_0,r) \subset C(\lambda) + \beta(d(\lambda,\lambda_0))B(0,1)$$

and

$$C(\lambda) \cap B(x_0,r) \subset C(\lambda_0) + \beta(d(\lambda,\lambda_0))B(0,1) ,$$

for all $\lambda \in B(\lambda_0,\varepsilon)$.

As in the previous proofs, for all $\lambda \in B(\lambda_0,\varepsilon)$, the set $C(\lambda) \cap B(x_0,r)$ is nonempty. Hence, for all $(\omega,\lambda) \in \Omega_0 \times B(\lambda_0,\varepsilon)$ there exists a unique element $x(\omega,\lambda)$ such that

$$0 \in f(x(\omega,\lambda), \omega) + N_{C(\lambda)}(x(\omega,\lambda)) .$$

Using Proposition 3.1 we deduce that for all $\omega \in \Omega_0$ the mappings $x(\omega, \cdot)$ are continuous at λ_0 and Theorem 2.2 implies that $x(\cdot, \lambda_0)$ is continuous at ω_0.
The above continuities shows that the mapping x is continuous at (ω_0, λ_0).

Remark 4.1. As in the previous section, if we suppose that the set-valued mapping C is pseudo-continuous around (λ_0, x_0), then the solution mapping x is continuous in a neighborhood of (ω_0, λ_0).

5. Applications

The reason of the following examples is to show that Theorem 2.1 is usefull in the study of the continuity of the solutions of parametric integral equations and evolution differential inclusions. We will also show that the consistency condition appear under well-known assumptions and it is important because some of the mappings are not defined everywhere, they have only dense domains.

Example 5.1. Let $(a, b) \subset \mathbf{R}$ be an open interval, let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, let $\lambda_0 \in \mathbf{R}$, let Λ_0 be a neighborhood of λ_0 and let $u_0 \in L^p(a, b)$.
We suppose that the mappings $F : (a, b) \times \mathbf{R} \times \Lambda_0 \to \mathbf{R}$ and $K : (a, b) \times (a, b) \to \mathbf{R}$ satisfy the following conditions:
(F_1) *the mappings* $F(\cdot, r, \lambda)$ *are measurable for all* $r \in \mathbf{R}$ *and* $\lambda \in \Lambda_0$;
(F_2) *the mappings* $F(x, \cdot, \lambda)$ *are continuous a.e.* $x \in (a, b)$ *and* $\lambda \in \Lambda_0$;
(F_3) *for each* $\lambda \in \Lambda_0$ *there exist* $g_\lambda \in L^q(a, b)$ *and* $c_\lambda > 0$ *such that*

$$|F(x, r, \lambda)| \leq g_\lambda(x) + c_\lambda |r|^{p-1}$$

for all $r \in \mathbf{R}$ *and* $x \in (a, b)$;
(F_4) *there exists a constant* $d > 0$ *such that*

$$(F(x, r_1, \lambda) - F(x, r_2, \lambda))(r_1 - r_2) \geq d|r_1 - r_2|^p ,$$

for all $x \in (a, b)$, $r_1, r_2 \in \mathbf{R}$, $\lambda \in \Lambda_0$;
(F_5) *the mappings* $F(\cdot, u_0(\cdot), \lambda)$ *converge uniformly to* $F(\cdot, u_0(\cdot), \lambda_0)$ *on (a,b), when*

$\lambda \to \lambda_0$;

(K_1) *there exist constants* $c_1, c_2 > 0$, $s, t > 1$ *such that*

$$\left(\int_a^b |K(x,y)|^s \, dx\right)^{\frac{1}{s}} \leq c_1 \,, \quad a.e. \quad y \in (a,b) \,,$$

$$\left(\int_a^b |K(x,y)|^t \, dy\right)^{\frac{1}{t}} \leq c_2 \,, \quad a.e. \quad x \in (a,b) \,,$$

$$p \geq s \,, \quad \left(1 - \frac{s}{p}\right) p \leq t \,;$$

(K_2) *for all* $u \in L^p(a,b)$, $u \neq 0$ *we have*

$$\int_a^b \int_a^b K(x,y) u(x) u(y) \, dx \, dy > 0 \,.$$

Remark 5.1. Assumptions (F_3) and (F_4) do not exclude. We can take, for example, $F(x,r,\lambda) = x\lambda + |r|^{p-2}r$.

Assumptions (F_1), (F_2), (F_3) imply that ([9]) for all $\lambda \in \Lambda_0$ the mappings $H(\cdot,\lambda) : L^p(a,b) \to L^q(a,b)$, defined by $H(u,\lambda)(x) = F(x,u(x),\lambda)$, are well-defined and continuous.

Assumption (F_4) implies the φ-uniform-monotonicity of the mappings $H(\cdot,\lambda)$, with $\varphi(r) = dr^{p-1}$. This means that the strong-monotonicity is satisfied locally when $1 < p \leq 2$ and is not satisfied when $2 < p$.

Assumption (K_1) implies that ([9]) the mapping $G : L^q(a,b) \to L^p(a,b)$, defined by

$$G(v)(x) = \int_a^b K(x,y) v(y) \, dx \,,$$

is well-defined and continuous (not nescesarrily compact).

Assumption (K_2) implies the strict-monotonicity of G.

Let us consider the following parametric Hammerstein integral equation:

$$u(x) + \int_a^b K(x,y) F(y, u(y), \lambda) \, dy = \omega(x) \,. \tag{5.1}$$

Proposition 5.1. *Let us consider that assumptions* $(F_1) - (F_5)$, $(K_1) - (K_2)$ *are satisfied and there exist* $u_0, \omega_0 \in L^p(a,b)$ *such that*

$$u_0(x) + \int_a^b K(x,y)F(y,u_0(y),\lambda_0)\,dy = \omega_0(x)\ .$$

Then there exists a unique mapping $u : L^p(a,b) \times \Lambda_0 \to L^p(a,b)$ *such that* $u(\omega,\lambda)$ *is a solution of (5.1) for all* $(\omega,\lambda) \in L^p(a,b) \times \Lambda_0$, $u(\omega_0,\lambda_0) = u_0$ *and the mapping* u *is continuous at* (ω_0,λ_0).

Proof. The existence of the solutions $u(\omega,\lambda)$ is proved in [9]. We will prove now the continuity.

Equation (5.1) can be written as

$$u + G \circ H(u,\lambda) = \omega$$

or equivalently

$$0 \in H(u,\lambda) - G^{-1}(\omega - u)\ .$$

We define the mapping $T : L^p(a,b) \times L^p(a,b) \to L^q(a,b)$ by

$$T(\omega,u) = -G^{-1}(\omega - u)\ .$$

The mappings $T(\omega,\cdot)$ are linear, continuous, maximal-monotone and strictly-monotone and hence G^{-1} is linear, continuous, maximal-monotone on $Dom\, G^{-1}$. We observe also that in this case $Dom\, G^{-1}$ is dense in $L^p(a,b)$ ([12]).

Let us fix $\omega \in L^p(a,b)$. Then we can choose $u_\omega \in L^p(a,b)$ such that $||u_\omega - u_0|| \le ||\omega - \omega_0||$ and $\omega - u_\omega \in Dom\, G^{-1}$. Hence $G^{-1}(\omega - u_\omega) \to G^{-1}(\omega_0 - u_0)$, when $\omega \to \omega_0$, so we proved the consistency of T with respect to ω at $(u_0, \omega_0, T(\omega_0,u_0))$.

Assumption (F_5) implies that the mapping $H(u_0,\cdot)$ is continuous at λ_0 and using the continuity of the mappings $H(\cdot,\lambda)$ we conclude that H is continuous at (u_0,λ_0). This continuity together with the consistency of T implies that $H + T$ is consistent with respect to (ω,λ) at $(u_0,\omega_0,\lambda_0,0)$.

Now we can use Theorem 2.1 for the mapping $H + T$ to get the desired continuity.

Example 5.2. Let H be a Hilbert space. Let us consider the following problem:

$$\begin{cases} u' + A(u,\lambda) \ni f \\ u(0) = 0 \end{cases} \tag{5.2}$$

in the case when $T > 0$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^q(0,T;H)$, $\lambda_0 \in \mathbf{R}$, Λ_0 is a neighborhood of λ_0, $A : L^p(0,T;H) \times \Lambda_0 \to L^q(0,T;H)$.

Definition 5.1. *Let X and Z be topological spaces. A set-valued mapping $F : X \to Z$ is called lower-semicontinuous at $(x_0, z_0) \in Graf F$, if for all neighborhood Z_0 of z_0 there exists a neighborhood X_0 of x_0 such that $F(x) \cap Z_0 \neq \emptyset$, for all $x \in X_0$.*

Lemma 5.1. [13] *The linear mapping $L : L^p(0,T;H) \to L^q(0,T;H)$, defined by $L(u) = u'$ and $Dom\, L = \{u \in W^{1,p}(0,T;H) \ : \ u(0) = 0\}$, is maximal-monotone.*

Proposition 5.2. *Let us suppose that:*

a) there exist $u_0 \in Dom\, L$ and $v_0 \in A(u_0, \lambda_0)$ such that $u_0' + v_0 - f = 0$;

b) the set-valued mapping A is lower-semicontinuous at $((u_0, \lambda_0)\ ,\ v_0)$;

c) the set-valued mappings $A(\cdot, \lambda)$ are maximal-monotone and φ-uniformly-monotone for all $\lambda \in \Lambda_0$.

Then there exist a neighborhood Λ' of λ_0 and a unique mapping $u : \Lambda' \to L^p(0,T;H)$ such that $u(\lambda_0) = u_0$, $u(\lambda)$ is the unique solution for each $\lambda \in \Lambda'$ for (5.2) and u is continuous at λ_0.

Proof. Let us denote $X = L^p(0,T;H)$. Assumption $b)$ implies that for all $\varepsilon > 0$ there exists $\eta > 0$ such that, for all $(u, \lambda) \in X \times \Lambda_0$, with $\|u - u_0\| < \eta$, $\|\lambda - \lambda_0\| < \eta$, hold $A(u,\lambda) \cap B(v_0, \varepsilon) \neq \emptyset$.

Hence for all $\varepsilon > 0$ there exists $v_{u,\lambda} \in A(u,\lambda)$ such that $\|v_{u,\lambda} - v_0\| \leq \varepsilon$.

Let us consider the sequence $(\varepsilon_n)_{n \in \mathbf{N}}$, $\varepsilon_n = \frac{1}{n}$, and a corresponding sequence $(\eta_n)_{n \in \mathbf{N}}$ converging to 0 such that $B(\lambda_0, \eta_1) \subset \Lambda_0$.

Let us choose arbitrarily $\lambda \in B(\lambda_0, \eta_1)$. Then for all $u \in B(u_0, \eta_1)$ we have $A(u,\lambda) \cap B(u_0, 1) \neq \emptyset$, so $B(u_0, \eta_1) \subset \operatorname{Dom} A(\cdot, \lambda)$. In this way we can see that $L + A(\cdot, \lambda)$ is maximal-monotone and as a sum between a monotone and another φ-uniformly-monotone mapping, is φ-uniformly-monotone.

Let η_{n_λ} be the smallest number in the sequence $(\eta_n)_{n\in\mathbf{N}}$ for which $\lambda \in B(\lambda_0, \eta_{n_\lambda})$. Then there exists $v_\lambda \in A(u_0, \lambda)$ such that $||v_\lambda - v_0|| \leq \frac{1}{n_\lambda}$. Hence

$$||L(u_0) + v_\lambda - f|| = ||L(u_0) + v_\lambda - L(u_0) - v_0|| = ||v_\lambda - v_0|| \leq \frac{1}{n_\lambda} .$$

We define the function $\beta : B(\lambda_0, \eta_1) \to \mathbf{R}_+$ by

$$\beta(\lambda) = \max \left\{ \eta_{n_\lambda}, \frac{1}{n_\lambda} \right\} .$$

Using this function β we conclude that the mapping $L + A - f$ is consistent with respect to λ at $(u_0, \lambda_0, 0)$ in $B(\lambda_0, \eta_1)$. The conclusion of this proposition follows now from Theorem 2.1.

Example 5.3. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain, $p, q \in \mathbf{R}_+$ such that $2 \leq p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and let $\lambda \in \mathbf{R}_+$.

We denote $X = W_0^{1,p}(\Omega)$ and

$$a(u, v, \lambda) = \int_\Omega \left(\sum_{p=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + \lambda uv \right) dx ,$$

$$F_f(v) = \int_\Omega f(x) v(x) \, dx .$$

Let us consider the following problem:

- for $f \in L^q(\Omega)$ and $\lambda \in \mathbf{R}_+$, find $u \in X$ such that

$$a(u, v, \lambda) = F_f(v) , \quad \text{for all } v \in X . \tag{5.3}$$

Let us define the mapping $A : \mathbf{R}_+ \times X \to X^*$ by

$$A(\lambda, u)(v) = a(u, v, \lambda) = \int_\Omega \left(\sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + \lambda uv \right) dx . \tag{5.4}$$

Proposition 5.3. [13] *For all $\lambda \in \mathbf{R}_+$ and $u \in X$, the mapping $A(\lambda, u)$ is well-defined and the mappings $A(\lambda, \cdot) : X \to X^*$ are continuous, φ-uniformly-monotone, with $\varphi(r) = c_1 r^{p-1}$.*

Proposition 5.4. *For all $\lambda \in \mathbf{R}_+$ and $f \in L^q(\Omega)$, the problem (2.3) has a unique solution $u(\lambda, f) \in W_0^{1,p}(\Omega)$ and these solutions are continuous in λ and f.*

Proof. Using the surjectivity and the φ-uniform-monotonicity of the mappings $A(\cdot,\lambda)$, we deduce that for all $f \in L^q(\Omega)$ and $\lambda \in \mathbf{R}_+$ there exists a unique element $u(\lambda, f) \in X$ such that $A(u,\lambda) = F_f$.

For all $\lambda_1, \lambda_2 \in \mathbf{R}_+$ and $u, v \in X$ hold

$$(A(u,\lambda_1) - A(u,\lambda_2)\,,\,v) = \int_\Omega (\lambda_1 - \lambda_2)uv\,dx \le$$

$$\le |\lambda_1 - \lambda_2|\,||u||_{L^2}\,||v||_{L^2} \le |\lambda_1 - \lambda_2|c||u||\,||v||\,.$$

Hence

$$||A(u,\lambda_1) - A(u,\lambda_2)|| \le c|\lambda_1 - \lambda_2|\,||u||\,,$$

which means that the mappings $A(u,\cdot)$ are continuous on $\mathbf{R}_+$.

Let us fix $\lambda_0 \in \mathbf{R}_+$ and $f_0 \in L^q(\Omega)$.

Theorem 2.1 implies the existence of a neighborhood $\Lambda_0 \times U_0$ of (λ_0, f_0) and of a unique mapping $u_0 : \Lambda_0 \times U_0 \to X$ continuuous at (λ_0, f_0), such that $u_0(\lambda, f)$ is the unique solution of the problem (2.3) for all $(\lambda, f) \in \Lambda_0 \times U_0$. The uniqueness of the solutions implies that the mappings u_0 and u coincide on $\Lambda_0 \times U_0$. Hence the continuity of u at (λ_0, f_0) is proved.

(λ_0, f_0) being choosed arbitrarily, the continuty holds for all $(\lambda, f) \in \Lambda_0 \times U_0$.

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