STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, Volume XLIV, Number 3, September 1999

PROPERTIES OF A NEW CLASS OF ABSOLUTELY SUMMING OPERATORS

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Abstract. In [1] there was been introduced a new class of absolutely summing operators and there was been obtained some of its properties and, also, the relations with the known classes of absolutely summing operators.

In this article we go on with the study of the properties of this new operator class.

1. Preliminaries

We shall just refer briefly to those notions and results which are necessary for the proofs.

Let E, F be Banach spaces over the field Γ, where Γ is the set of the real or of the complex numbers. In the sequel we shall use the following notations:

1) $L(E,F) := \{T : E \to F : \ T \text{ is linear and bounded}\}$.

2) $E^* := L(E, \Gamma)$.

3) $U_E := \{x \in E : \|x\| \leq 1\}$.

4) For $a \in E^*$ and $x \in E$, let $\langle x, a\rangle := a(x)$.

5) Let $a \in E^*$ and $y \in F$. We denote by $a \otimes y$ the following operator

$$a \otimes y : E \to F, \ \ (a \otimes y)(x) = \langle x, a\rangle \cdot y, \ \text{ for all } x \in E.$$

6) We denote by l_∞ the set of all real number sequences, $\{x_n\}_n$, with the property

Key words and phrases. p-absolutely summing operators, p-q absolutely summing operators, symmetric norming function, Lorentz-Zygmund sequence ideals.

$$\|x\|_\infty := \sup_{n \text{ natural}} |x_n| < \infty.$$

7) We denote by c_0 the set of all real number sequences, $\{x_n\}_n$, with the property

$$\lim_{n\to\infty} |x_n| = 0.$$

8) We denote by l_p, $0 < p < \infty$, the set of all real number sequences, $\{x_n\}_n$, with the property

$$\|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} < \infty.$$

Definition 1. ([7])

For $x = \{x_n\}_n \in l_\infty$, let
$s_n(x) := \inf\{\sigma \geq 0 : card\{i : |x_i| \geq \sigma\} < n\}$.

Remark 1. ([7])

If the sequence $x = \{x_n\}_n \in l_\infty$ is ordered such that $|x_n| \geq |x_{n+1}|$, for any natural n, then

$$s_n(x) = |x_n|.$$

Proposition 2. ([7])

The numbers $s_n(x)$ have the following properties:

1. $\|x\|_\infty = s_1(x) \geq s_2(x) \geq ... \geq 0$, for all $x = \{x_n\}_n \in l_\infty$,

2. $s_{n+m-1}(x+y) \leq s_n(x) + s_m(y)$, for all $x = \{x_i\}_i \in l_\infty, y = \{y_i\}_i \in l_\infty$,

   and $n, m \in \{1, 2, ...\}$, where $x + y = \{x_i + y_i\}_i$,

3. $s_{n+m-1}(x \cdot y) \leq s_n(x) \cdot s_m(y)$, for all $x = \{x_i\}_i \in l_\infty, y = \{y_i\}_i \in l_\infty$,

   and $n, m \in \{1, 2, ...\}$, where $x \cdot y = \{x_i \cdot y_i\}_i$,

4. If $x = \{x_i\}_i \in l_\infty$ and $card\{i : x_i \neq 0\} < n$ then $s_n(x) = 0$.

Let us remark the similarity between the properties of the sequence $s_n(x)$, where $x = \{x_n\}_n \in l_\infty$, and the axioms from the definition of an **additive and multiplicative** s**-scale,** an s**-scale** being a rule, $s : T \to \{s_n(T)\}_n$, which assigns to every linear and bounded operator a scalar sequence with the following properties:

1. $\|T\| = s_1(T) \geq s_2(T) \geq ... \geq 0$, for all $T \in L(E, F)$,

2. $s_{n+m-1}(T + S) \leq s_n(T) + s_m(S)$, for all $T, S \in L(E, F)$

   and $n, m \in \{1, 2, ...\}$,

3. $s_{n+m-1}(T \circ S) \leq s_n(T) \cdot s_m(S)$, for all $T \in L(F, F_0), S \in L(E, F)$

   and $n, m \in \{1, 2, ...\}$,

4. $s_n(T) = 0$, $\dim T < n$,

5. $s_n(I_E) = 1$, if $\dim E \geq n$, where $I_E(x) = x$, for all $x \in E$.

We call $s_n(T)$ the n-th $s-$**number** of the operator T.

For properties, examples of $s-$**numbers** and relations between diferent s-numbers it can be seen [3], [4], [5], [6], [7].

We continue by giving some basic facts about the classical real interpolation method, called the K-method.

For those interested to find an introduction on interpolation theory we recommend, for example, [2], [9].

Definition 3. ([2], [8])

For a compatible couple (X_0, X_1), in the sense of the interpolation theory, of normed or quasi-normed spaces, and $t > 0$ consider the functional:

$$K(t, x) := \inf\{\|x_0 \mid X_0\| + t \cdot \|x_1 \mid X_1\| : x = x_0 + x_1, x_i \in X_i, i = 0, 1\}.$$

Let $0 < \theta < 1$ and $0 < q \leq \infty$. The **interpolation space** $(X_0, X_1)_{\theta,q}$ is defined as follows:

$$(X_0, X_1)_{\theta,q} := \left\{ x = x_0 + x_1, x_i \in X_i, i = 0, 1 : \left(\int_0^\infty \left[t^{-\theta} \cdot K(t, x)\right]^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\},$$

if $q < \infty$, and

$$(X_0, X_1)_{\theta,\infty} := \left\{ x = x_0 + x_1, x_i \in X_i, i = 0, 1 : \sup_{t>0} t^{-\theta} \cdot K(t, x) < \infty \right\}.$$

The operator classes $P_{p,q,\gamma}$, introduced in [1], are closed related to the Lorentz-Zygmund sequence spaces. For that we shall recall here a few things about these sequence spaces and the Lorentz-Zygmund operator ideals.

Definition 4. ([7])

Let $0 < p, q < \infty$ and $-\infty < \gamma < \infty$. The **Lorentz-Zygmund sequence spaces** are defined as follows

$$l_{p,q,\gamma} := \left\{ x = \{x_i\}_i \in c_0 : \sum_{i=1}^\infty \left[i^{\frac{1}{p} - \frac{1}{q}} \cdot (1 + \log i)^\gamma \cdot s_i(x) \right]^q < \infty \right\}.$$

These are quasi-normed spaces, with the quasi-norm

$$\|x\|_{p,q,\gamma} := \left(\sum_{i=1}^{\infty} \left[i^{\frac{1}{p}-\frac{1}{q}} \cdot (1+\log i)^{\gamma} \cdot s_i(x) \right]^q \right)^{\frac{1}{q}}.$$

Definition 5. ([7], [8])

Let E, F be Banach spaces, s an additive s-number and $0 < p < \infty$, $0 < q < \infty$, $-\infty < \gamma < \infty$. We introduce the following operator classes:

$$L^{(s)}_{p,q,\gamma}(E,F) := \left\{ T \in L(E,F) : \ \|T\|^{(s)}_{p,q,\gamma} := \left(\sum_{n=1}^{\infty} \left[n^{\frac{1}{p}} \cdot (1+logn)^{\gamma} \cdot s_n(T) \right]^q \cdot n^{-1} \right)^{\frac{1}{q}} < \infty \right\},$$

and for $q = \infty$

$$L^{(s)}_{p,\infty,\gamma}(E,F) := \left\{ T \in L(E,F) : \ \|T\|^{(s)}_{p,\infty,\gamma} := \sup_n n^{\frac{1}{p}} \cdot (1+logn)^{\gamma} \cdot s_n(T) < \infty \right\}.$$

We denote by $L^{(s)}_{p,q,\gamma} := \bigcup_{E,F \text{ Banach spaces}} L^{(s)}_{p,q,\gamma}(E,F)$.

Remark 2. ([7], [8])

Let s be an additive s-number and $0 < p < \infty$, $0 < q \leq \infty$, $-\infty < \gamma < \infty$, then $\left(L^{(s)}_{p,q,\gamma}, \|\cdot\|^{(s)}_{p,q,\gamma} \right)$ is a quasi-normed operator ideal.

We are giving now an interpolation result obtained by classical methods

Proposition 6. ([8])

Let E, F be Banach spaces, $0 < p_0 < p_1 < \infty$, $0 < q_0, q_1, q \leq \infty$, $0 < \gamma_0, \gamma_1 < \infty$ and $0 < \theta < 1$. Then

$$\left(L^{(s)}_{p_0,q_0,\gamma_0}(E,F), L^{(s)}_{p_1,q_1,\gamma_1}(E,F) \right)_{\theta,q} \subseteq L^{(s)}_{p,q,\gamma}(E,F), \text{ where } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

and $\gamma = (1-\theta) \cdot \gamma_0 + \theta \cdot \gamma_1$.

At the end of this section we shall remind the construction of the operator classes $P_{p,q,\gamma}$ and one of the properties proved in [1].

Definition 7. ([4])

Let E be a Banach space and I an index set. An E-valued family $\{x_i\}_{i \in I}$ is said to be **absolutely r-summable** if $\{\|x_i\|\} \in l_r(I)$. The set of these families is denoted by $[l_r(I), E]$.

For $\{x_i\}_{i\in I} \in [l_r(I), E]$ we define:

$$\|\{x_i\} \mid [l_r(I), E]\| := \left(\sum_i \|x_i\|^r\right)^{1/r}.$$

If there is no risk of confusion, then we use the shortened symbol $\|\{x_i\} \mid l_r\|$. Moreover,we write $[l_r, E]$ instead of $[l_r(N), E]$.

Proposition 8. ([4])

$[l_r(I), E]$ is a Banach space.

Definition 9. ([4])

Let E be a Banach space and I an index set. An E-valued family $\{x_i\}_{i\in I}$ is said to be **weakly** r**-summable if** $\{\langle x_i, a\rangle\} \in l_r(I)$ for all $a \in E^*$.

The set of these families is denoted by $[w_r(I), E]$.For $\{x_i\}_{i\in I} \in [w_r(I), E]$ we define:

$$\|(x_i) \mid [w_r(I), E]\| = sup\left\{\left(\sum_i |\langle x_i, a\rangle|^r\right)^{1/r} : a \in U_{E^*}\right\}.$$

If there is no risk of confusion, then we use the shortened symbol $\|\{x_i\} \mid w_r\|$. Moreover,we write $[w_r, E]$ instead of $[w_r(N), E]$.

Proposition 10. ([4])

$[w_r(I), E]$ is a Banach space.

Remark 3. ([1])

Using the Lorentz-Zygmund sequence spaces, $l_{p,q,\gamma}$, we can define, in a similar way, the spaces $[l_{p,q,\gamma}(I), E]$ and $[w_{p,q,\gamma}(I), E]$.

Definition 11. ([1])

Let E, F be Banach spaces and $0 < p_1, p_2 < \infty$, $1 \leq q_2 \leq q_1 < \infty$, $-\infty < \gamma_1, \gamma_2 < \infty$. An operator $T \in L(E,F)$ is called **absolutely** $(p_{12}, q_{12}, \gamma_{12})$ **−summing** if there exists a constant $c \geq 0$ such that

$$\left(\sum_{i=1}^{n}\left[i^{\frac{1}{p_1}-\frac{1}{q_1}}\cdot(1+\log i)^{\gamma_1}\cdot\|Tx_i\|\right]^{q_1}\right)^{\frac{1}{q_1}} \leq$$

$$\leq c\cdot\sup_{a\in U_{E^*}}\left(\sum_{i=1}^{n}\left[i^{\frac{1}{p_2}-\frac{1}{q_2}}\cdot(1+\log i)^{\gamma_2}\cdot|\langle x_i, a\rangle|\right]^{q_2}\right)^{\frac{1}{q_2}}$$, for every finite family of elements $x_1, \ldots x_n \in E$. The set of these operators is denoted by $P_{p_{12},q_{12},\gamma_{12}}(E,F)$.

For $T \in P_{p_{12},q_{12},\gamma_{12}}(E,F)$ we define $\pi_{p_{12},q_{12},\gamma_{12}}(T) := \inf c$, the infimum being taken over all constants $c \geq 0$ for which the above inequality holds.

Theorem 12. ([1])

$P_{p_{12},q_{12},\gamma_{12}}$ is an injective Banach operator ideal.

2. **Results**

We start by giving a result concerning the "lexicografic order" of the Lorentz-Zygmund sequence spaces.

Proposition 13.

We have the following inclusion:

$$l_{p,q_0,\gamma} \subseteq l_{p,q_1,\gamma}, \text{ where } 0 < p < \infty,\ 0 < q_0 < q_1 \leq \infty,\ \gamma > 0.$$

Proof. We whall need the following result, established by N. Ţiţa, in [8], for the operator ideal case.

Proposition 14.

Let $0 < p < \infty$, $0 < q \leq \infty$ and $0 < \gamma < \infty$ then

$$\{x_n\}_n \in l_{p,q,\gamma} \Leftrightarrow \left\{2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(x)\right\}_n \in l_{r,q}, \text{ where } \gamma = \frac{1}{r} - \frac{1}{q}.$$

Moreoever there are the constants c and $\overline{c}$, which depend on p, q, γ, such that:

$$c \cdot \left\| \left\{2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(x)\right\}_n \right\|_{r,q} \leq \|x\|_{p,q,\gamma} \leq \overline{c} \cdot \left\| \left\{2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(x)\right\}_n \right\|_{r,q}.$$

We start now our proof.

Let $\xi = \{\xi_n\}_n \in l_{p,q_0,\gamma} \Leftrightarrow \left\{2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(\xi)\right\}_n \in l_{r,q_0}$, where $\gamma = \frac{1}{r} - \frac{1}{q_0}$.

Let $q_1 > q_0$ and r_1 such that $\gamma = \frac{1}{r_1} - \frac{1}{q_1}$. It follows that

$\frac{1}{r} - \frac{1}{q_0} = \frac{1}{r_1} - \frac{1}{q_1} \Leftrightarrow \frac{1}{r_1} = \frac{1}{r} + \left(\frac{1}{q_1} - \frac{1}{q_2}\right) \Rightarrow \frac{1}{r_1} < \frac{1}{r} \Rightarrow r_1 > r.$

From the "lexicografic orderliness" of the Lorentz sequence spaces, [4], [7], we know that $l_{r,q_0} \subseteq l_{r_1,q_1}$.

So $\left\{2^{\frac{n-1}{p}} \cdot s_{2^{n-1}}(\xi)\right\}_n \in l_{r,q_1} \Leftrightarrow \xi = \{\xi_n\}_n \in l_{p,q_1,\gamma}$.

In conclusion $l_{p,q_0,\gamma} \subseteq l_{p,q_1,\gamma}$, for $0 < p < \infty$, $0 < q_0 < q_1 \leq \infty$, $\gamma > 0$. □

Proposition 15.

Let $0 < p_0 < p_1 < \infty$, $0 < q_0, q_1, q \leq \infty$, $0 < \gamma_0, \gamma_1 < \infty$ and $0 < \theta < 1$. Then $(l_{p_0,q_0,\gamma_0}, l_{p_1,q_1,\gamma_1})_{\theta,q} \subseteq l_{p,q,\gamma}$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\gamma = (1-\theta) \cdot \gamma_0 + \theta \cdot \gamma_1$.

Proof. If we take account of the similarity between the properties of the sequences $\{s_n(T)\}_n$, where s is an additive s-scale, $T \in L(E, F)$, and $\{s_n(x)\}_n$, where $x = \{x_n\}_n \in l_\infty$, the proof of the above inclusion will have the same course like the proof of the Proposition 6.

We shall consider $q < \infty$, the proof of the case $q = \infty$ being similar.

From the Proposition 13 it follows that we have the inclusion

$(l_{p_0,q_0,\gamma_0}, l_{p_1,q_1,\gamma_1})_{\theta,q} \subseteq (l_{p_0,\infty,\gamma_0}, l_{p_1,\infty,\gamma_1})_{\theta,q}$.

So it will be enough to prove the relation

$(l_{p_0,\infty,\gamma_0}, l_{p_1,\infty,\gamma_1})_{\theta,q} \subseteq l_{p,q,\gamma}$.

Let $x = \{x_n\}_n \in (l_{p_0,\infty,\gamma_0}, l_{p_1,\infty,\gamma_1})_{\theta,q}$. We shall consider the arbitrary decomposition

$x = x^0 + x^1$, where $x^i = \{x_n^i\}_n \in l_{p_i,\infty,\gamma_i}$, $i \in \{0,1\}$.

Let $i \in \{0,1\}$ then

$$x^i = \{x_n^i\}_n \in l_{p_i,\infty,\gamma_i} \Leftrightarrow$$
$$\Leftrightarrow \left\|x^i\right\|_{p_i,\infty,\gamma_i} = \sup_n \left[n^{\frac{1}{p_i}} \cdot (1+\log n)^{\gamma_i} \cdot s_n\left(x^i\right)\right] < \infty \Rightarrow$$
$$\Rightarrow s_n\left(x^i\right) \le n^{-\frac{1}{p_i}} \cdot (1+\log n)^{-\gamma_i} \cdot \left\|x^i\right\|_{p_i,\infty,\gamma_i}, \text{ for any natural } n.$$

We shall evaluate $\|x\|_{p,q,\gamma}$.

$$\left(\|x\|_{p,q,\gamma}\right)^q = \sum_{n=1}^{\infty}\left[n^{\frac{1}{p}} \cdot (1+\log n)^{\gamma} \cdot s_n(x)\right]^q \cdot \frac{1}{n} =$$
$$= \sum_{n=1}^{\infty}\left[(2\cdot n-1)^{\frac{1}{p}-\frac{1}{q}} \cdot (1+\log(2\cdot n-1))^{\gamma} \cdot s_{2\cdot n-1}(x)\right]^q +$$
$$+\sum_{n=1}^{\infty}\left[(2\cdot n)^{\frac{1}{p}-\frac{1}{q}} \cdot (1+\log(2\cdot n))^{\gamma} \cdot s_{2\cdot n}(x)\right]^q \le$$
$$\le \sum_{n=1}^{\infty}\left[(2\cdot n-1)^{\frac{1}{p}-\frac{1}{q}} \cdot (1+\log(2\cdot n-1))^{\gamma} \cdot s_{2\cdot n-1}(x)\right]^q +$$
$$+\sum_{n=1}^{\infty}\left[(2\cdot n)^{\frac{1}{p}-\frac{1}{q}} \cdot (1+\log(2\cdot n))^{\gamma} \cdot s_{2\cdot n-1}(x)\right] \le$$
$$\le c(p,q,\gamma) \cdot \sum_{n=1}^{\infty}\left[n^{\frac{1}{p}-\frac{1}{q}} \cdot (1+\log n)^{\gamma} \cdot s_{2\cdot n-1}(x)\right]^q \le$$
$$\le c(p,q,\gamma) \cdot \sum_{n=1}^{\infty}\left[n^{\frac{1}{p}-\frac{1}{q}} \cdot (1+\log n)^{\gamma} \cdot \left(s_n\left(x^0\right)+s_n\left(x^1\right)\right)\right]^q \le$$
$$\le c(p,q,\gamma) \cdot$$
$$\sum_{n=1}^{\infty}\left[n^{\frac{1}{p}-\frac{1}{q}}(1+\log n)^{\gamma-\gamma_0} n^{-\frac{1}{p_0}}\left(\left\|x^0\right\|_{p_0,\infty,\gamma_0} + n^{\frac{1}{p_0}-\frac{1}{p_1}}(1+\log n)^{\gamma_0-\gamma_1}\left\|x^1\right\|_{p_1,\infty,\gamma_1}\right)\right]^q.$$

The decomposition $x = x^0 + x^1$ being arbitrary and taking account of the K-functional's definition

$K\left(x, n^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+\log n)^{\gamma_0-\gamma_1}, l_{p_0,\infty,\gamma_0}, l_{p_1,\infty,\gamma_1}\right)$ we obtain that

$$\left\|x^0\right\|_{p_0,\infty,\gamma_0} + n^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+\log n)^{\gamma_0-\gamma_1} \cdot \left\|x^1\right\|_{p_1,\infty,\gamma_1} \le$$
$$\le K\left(x, n^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+\log n)^{\gamma_0-\gamma_1}, l_{p_0,\infty,\gamma_0}, l_{p_1,\infty,\gamma_1}\right).$$

So $\left(\|x\|_{p,q,\gamma}\right)^q \le$

$$\le c \cdot \sum_{n=1}^{\infty}\left[n^{\frac{1}{p}-\frac{1}{p_0}}(1+\log n)^{\gamma-\gamma_0} K\left(x, n^{\frac{1}{p_0}-\frac{1}{p_1}}(1+\log n)^{\gamma_0-\gamma_1}\right)\right]^q \cdot \frac{1}{n} \le$$
$$\le c_1 \cdot \int_1^{\infty}\left[t^{\frac{1}{p}-\frac{1}{p_0}}(1+\log t)^{\gamma-\gamma_0} K\left(x, t^{\frac{1}{p_0}-\frac{1}{p_1}}(1+\log t)^{\gamma_0-\gamma_1}\right)\right]^q \frac{dt}{t} =$$
$$= c_1 \cdot \int_1^{\infty}\left[t^{\frac{1-\theta}{p_0}+\frac{\theta}{p_1}-\frac{1}{p_0}}(1+\log t)^{(1-\theta)\gamma_0+\theta\gamma_1-\gamma_0} K\left(x, t^{\frac{1}{p_0}-\frac{1}{p_1}}(1+\log t)^{\gamma_0-\gamma_1}\right)\right]^q \frac{dt}{t} =$$

$$= c_1 \cdot \int_1^\infty \left[t^{-\theta\left(\frac{1}{p_0}-\frac{1}{p_1}\right)} (1+logt)^{-\theta(\gamma_0-\gamma_1)} K\left(x, t^{\frac{1}{p_0}-\frac{1}{p_1}} (1+logt)^{\gamma_0-\gamma_1}\right)\right]^q \frac{dt}{t} =$$

$$= c_1 \cdot \int_1^\infty \left[\left(t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1}\right)^{-\theta} \cdot K\left(x, t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1}\right)\right]^q \frac{dt}{t}.$$

Let now define $f : (1,\infty) \to (0,\infty)$, $f(t) = t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1}$.

$f'(t) \cdot t = \left(\frac{1}{p_0} - \frac{1}{p_1}\right) \cdot t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1} +$

$+ (\gamma_0 - \gamma_1) \cdot t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1} \cdot \frac{1}{1+logt} \cdot c_2 =$

$= t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1} \cdot \left(\frac{1}{p_0} - \frac{1}{p_1} + (\gamma_0-\gamma_1) \cdot \frac{1}{1+logt} \cdot c_2\right) \le c_3 \cdot f(t).$

Hence we obtain $\left(\|x\|_{p,q,\gamma}\right)^q \le$

$$\le c_1 \cdot \int_1^\infty \left[\left(t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1}\right)^{-\theta} \cdot K\left(x, t^{\frac{1}{p_0}-\frac{1}{p_1}} \cdot (1+logt)^{\gamma_0-\gamma_1}\right)\right]^q \frac{dt}{t} =$$

$$= c_1 \cdot \int_1^\infty \left[f(t)^{-\theta} \cdot K(x, f(t))\right]^q \cdot f'(t) \cdot \frac{1}{f'(t) \cdot t} dt \le c_3 \cdot \int_1^\infty \left[f(t)^{-\theta} \cdot K(x, f(t))\right]^q \cdot \frac{1}{f(t)} \cdot f'(t) \cdot dt =$$

$$= c_3 \cdot \int_0^\infty \left[s^{-\theta} \cdot K(x,s)\right]^q \cdot \frac{ds}{s} < \infty.$$

(We have made the following change of variable $f(t) = s$.)

In conclusion $x \in l_{p,q,\gamma}$. □

Proposition 16.

Let I be any infinite index set. An operator $T \in L(E,F)$ is absolutely $(p_{12}, q_{12}, \gamma_{12})$ –summing if and only if $T(I) : \{x_i\}_{i\in I} \to \{Tx_i\}_i$ defines a linear and bounded operator from $[w_{p_2,q_2,\gamma_2}(I), E]$ into $[l_{p_1,q_1,\gamma_1}(I), F]$. When this is so, then

$$\pi_{p_{12},q_{12},\gamma_{12}}(T) = \|T(I) : [w_{p_2,q_2,\gamma_2}(I), E] \to [l_{p_1,q_1,\gamma_1}(I), F]\|.$$

Proof. It is similar to the proof for the similar result for absolutely (r,s) –summing operators, see Proposition 1.2.2 from [3].

Suppose that $T \in P_{p_{12},q_{12},\gamma_{12}}(E,F)$ and $(x_i)_{i\in I} \in [w_{p_2,q_2,\gamma_2}(I), E]$. Then we have

$$\left(\sum_i \left[i^{\frac{1}{p_1}-\frac{1}{q_1}} \cdot (1+\log i)^{\gamma_1} \cdot \|Tx_i\|\right]^{q_1}\right)^{\frac{1}{q_1}} \le$$

$\le \pi_{p_{12},q_{12},\gamma_{12}}(T) \cdot \sup_{a\in U_{E^*}} \left(\sum_i \left[i^{\frac{1}{p_2}-\frac{1}{q_2}} \cdot (1+\log i)^{\gamma_2} \cdot |\langle x_i, a\rangle|\right]^{q_2}\right)^{\frac{1}{q_2}}$, for all F, $F \in \mathbf{F}(I)$. Passing to the limit I yields

$$\left\|(Tx_i)_{i\in I} \mid [l_{p_1,q_1,\gamma_1}(I), F]\right\| \le \pi_{p_{12},q_{12},\gamma_{12}}(T) \cdot \left\|(x_i)_{i\in I} \mid [w_{p_2,q_2,\gamma_2}(I), E]\right\|.$$

This proves that

$$\|T(I) : [w_{p_2,q_2,\gamma_2}(I), E] \to [l_{p_1,q_1,\gamma_1}(I), F]\| \le \pi_{p_{12},q_{12},\gamma_{12}}(T).$$

The reverse inequality is obvious. □

Theorem 17. *(interpolation theorem)*

Let E, F be Banach spaces. If $0 < p_1 < p_3 < \infty$, $0 < p_2 < \infty$, $0 < q_1, q_2, q_3, q_4 < \infty$, $0 < \gamma_1, \gamma_3 < \infty$ and $0 < \theta < 1$, then

$\left(P_{p_{12},q_{12},\gamma_{12}}(E,F), P_{p_{32},q_{32},\gamma_{32}}(E,F)\right)_{\theta,q_4} \subseteq P_{p_{42},q_{42},\gamma_{42}}(E,F)$, where $\frac{1}{p_4} = \frac{1-\theta}{p_1} + \frac{\theta}{p_3}$ and $\gamma_4 = (1-\theta)\cdot\gamma_1 + \theta\cdot\gamma_3$.

Proof. We use the idea from the proof of the **interpolation theorem** for the **absolutely** (p,q) **−summing operators.** This theorem can be found in [4], Proposition 1.2.6.

Let $\{x_i\}_i \in [w_{p_2,q_2,\gamma_2}, F]$. We define the operator

$X : T \in L(E,F) \to \{Tx_i\}_i$. From the Proposition 16 it follows that, for $T \in P_{p_{12},q_{12},\gamma_{12}}(E,F)$, $\{Tx_i\}_i \in [l_{p_1,q_1,\gamma_1}, F]$ and, for $T \in P_{p_{32},q_{32},\gamma_{32}}(E,F)$, $\{Tx_i\}_i \in [l_{p_3,q_3,\gamma_3}, F]$.

So for

$T \in \left(P_{p_{12},q_{12},\gamma_{12}}(E,F), P_{p_{32},q_{32},\gamma_{32}}(E,F)\right)_{\theta,q_4} \Rightarrow$

$\Rightarrow \{Tx_i\}_i \in \left([l_{p_1,q_1,\gamma_1}, F], [l_{p_3,q_3,\gamma_3}, F]\right)_{\theta,q_4} \subseteq [l_{p_4,q_4,\gamma_4}, F]$. We have applied the Proposition 15.

In conclusion

$X : T \in \left(P_{p_{12},q_{12},\gamma_{12}}(E,F), P_{p_{32},q_{32},\gamma_{32}}(E,F)\right)_{\theta,q_4} \to \{Tx_i\}_i \in [l_{p_4,q_4,\gamma_4}, F]$.

Hence the assertion follows from the Proposition 16. □

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