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INEQUALITIES FOR GENERALIZED CONVEX FUNCTIONS WITH APPLICATIONS, I.

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Abstract. The Theory of Inequalities has a majore role in Mathematical Analysis, and in almost all areas of Mathematics, too. In this theory, the convex functions and the generalized convexity plays a special role. The author has published a series of papers with applications of convexity inequalities in various fields of Mathematics. We quote applications in geometry (see e.g. [16], [22]), special functions ([19], [18], [23]); number theory (see many articles collected in the monograph [34]); the theory of means ([24], [25], [31], [33]), etc.

The aim of this series of papers (planned to have 4 parts) is to survey the most important ideas and results of the author in the theory of convex inequalities. In the course of this survey, many new results and applications will be obtained. In most cases only the new results will be presented with a proof; the other results will be stated only, with connections and/or applications to known theorems. All material is centered around three most important inequalities, namely: Jensen's inequality, Jensen-Hadamard's (or Hermite-Hadamard's) inequality and Jessen's inequality.

1. Jensen's inequality

One of the most important inequalities is Jensen's inequality either in its discrete or in its integral form. In what follows we will discuss various generalizations, extensions, special cases, or refinements.

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A. Let start with $f : [a,b] \to \mathbf{R}$, a convex function in the classical sense, and let $L : C[a,b] \to \mathbf{R}$ be a linear and positive functional defined on the space $C[a,b]$ of all continuous functions on $[a,b]$. Put $e_k(x) = x^k$, $x \in [a,b]$, $(k \in \mathbf{N})$.

Theorem 1.1. (see [9] and [22]) *If the above conditions are satisfied and the functional L has the property $L(e_0) = 1$, then the following double-inequality holds true:*

$$f(L(e_1)) \leq L(f) \leq L(e_1)\left[\frac{f(b)-f(a)}{b-a}\right] + \frac{bf(a)-af(b)}{b-a}. \tag{1}$$

Proof. See [22].

Corollary 1.1. Let $L(f) = \dfrac{1}{b-a}\displaystyle\int_a^b f(t)dt$. Then $L(e_0) = 1$ and the relation (1) gives us the classical Jensen-Hadamard inequality

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\left[\frac{f(a)+f(b)}{2}\right] \tag{2}$$

which will be considered later.

Corollary 1.2. Let $w_i \geq 0$ $(i = \overline{1,n})$ with $\displaystyle\sum_{i=1}^n w_i = 1$, and let $a_i \in [a,b]$, $(i = \overline{1,n})$. Let us define the functional $L(f) = \displaystyle\sum_{i=1}^n w_i f(a_i)$, which is linear and positive. From (1) we can deduce the double relation

$$f\left(\sum_{i=1}^n w_i a_i\right) \leq \sum_{i=1}^n w_i f(a_i) \leq \left(\sum_{i=1}^n w_i a_i\right)\left[\frac{f(b)-f(a)}{b-a}\right] + \frac{bf(a)-af(b)}{b-a}. \tag{3}$$

The left side of this relation is the famous discrete (pondered) inequality by Jensen ([5], [8], [10]).

Corollary 1.3. Let $p : [a,b] \to \mathbf{R}$ be a strictly positive, integrable function, and let $g : [a,b] \to \mathbf{R}$ continuous, strictly monotone on $[a,b]$. Define

$$L_g(f) = \frac{\displaystyle\int_a^b p(x)f[g(x)]dx}{\displaystyle\int_a^b p(x)dx}.$$

We can deduce from (1) the important integral inequality by Jensen:

$$f\left[\frac{\int_a^b p(x)g(x)dx}{\int_a^b p(x)dx}\right] \leq \frac{\int_a^b p(x)f[g(x)]dx}{\int_a^b p(x)dx}, \tag{4}$$

with various applications in different branches of Mathematics. We will see later, how can be applied (4) in the theory of means.

Remark. From the proof of the left side of (1) one can see that in place of convex functions one can consider **invex functions related to** $\eta : [a,b] \times [a,b] \to [a,b]$ (see [32]). This gives the following result:

Theorem 1.2. *If the function* $f : [a,b] \to \mathbf{R}$ *in invex related to a given function* η, *and the following condition is satisfied:*

$$L(\eta(e_1, L(e_1))) = 0, \tag{5}$$

then one has

$$f(L(e_1)) \leq L(f). \tag{6}$$

Corollary 1.4. Under the above conditions, as well as the conditions of Corollary 1.2, if in addition we assume that

$$\sum_{i=1}^n \eta\left(a_i, \sum_{i=1}^n w_i a_i\right) = 0,$$

then

$$f\left(\sum_{i=1}^n w_i a_i\right) \leq \sum_{i=1}^n w_i f(a_i). \tag{7}$$

We note that this relation holds true (with the analogous proof) for invex functions $f : S \to \mathbf{R}$ with $S \subset \mathbf{R}^n$ (see [32]).

B. Let $f : [a,b] \to \mathbf{R}$ and put $a = (a_1, \ldots, a_n) \in ([a,b])^n$. Let us consider the following expression

$$A_{k,n} = A_{k,n}(a) = \frac{1}{C_n^k} \sum_{1 \leq i < \cdots < i_k \leq n} f\left[\frac{a_{i_1} + \cdots + a_{i_k}}{k}\right] \tag{8}$$

(where $C_n^k = \binom{n}{k}$). Clearly

$$A_{n,n} = f\left(\frac{a_1 + \cdots + a_n}{a}\right); \quad A_{1,n} = \frac{f(a_1) + \cdots + f(a_n)}{n}.$$

This expression was considered for the first time by S. Gabler [7]. A more general (pondered) form is given by

$$A_{k,n}(a,w) = \frac{1}{C_{n-1}^{k-1} W_n} \sum_{1 \le i_1 < \cdots < i_k \le n} (w_{i_1} + \cdots + w_{i_k}) f\left(\frac{w_{i_1} a_{i_1} + \cdots + w_{i_k} a_{i_k}}{w_{i_1} + \cdots + w_{i_k}}\right) \tag{9}$$

with $W_n = \sum_{i=1}^{n} w_i$. The following refinement of the Jensen inequality holds true:

Theorem 1.3. ([29]) *One has*

$$f\left(\frac{\sum_{i=1}^{n} w_i a_i}{W_n}\right) = A_{n,n} \le \cdots \le A_{k+1,n} \le A_{k,n} \le \ldots A_{1,n} =$$

$$= \frac{\sum_{i=1}^{n} w_i f(a_i)}{\sum_{i=1}^{n} w_i}. \tag{10}$$

Corollary 1.5.

$$\frac{1}{n-1} \sum (w_1 + \cdots + \widehat{w}_i + \cdots + w_n) f\left(\frac{w_1 a_1 + \cdots + \widehat{w}_i \widehat{a}_i + \cdots + w_n a_n}{w_1 + \cdots + \widehat{w}_i + \cdots + w_n}\right) \le$$

$$\le \frac{1}{n} \frac{\sum \widehat{w}_i f(\widehat{a}_i)}{\sum \widehat{w}_i} \tag{11}$$

where $\widehat{w}_i$ denotes the fact that the term w_i is missing in the summation with $n-1$ terms (between n terms).

Proof. Apply (10) for $k = n-1$.

Another refinement of Jensen's inequality is contained in

Theorem 1.4. ([29]) *Let* $w_i \geq 0$, $\sum_{i=1}^{n} w_i = W_n > 0$, $a_i \in [a,b]$ $(i = \overline{1,n})$. *If* $f : [a,b] \to \mathbf{R}$ *is convex, then for all* $u, v \geq 0$ *with* $u + v > 0$ *one has the inequality*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right) \leq \left(\frac{1}{W_n}\right)^2 \sum_{i,j=1}^{n} w_i w_j f\left(\frac{ua_i + va_j}{u+v}\right) \leq$$

$$\leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i). \tag{12}$$

C. The above theorems still hold in arbitrary linear spaces, by considering the elements a_i $(i = \overline{1,n})$ to be contained in a convex subset.

Let now X be a real prehilbertian space with scalar product $(,)$ and norm $\|\cdot\|$. Let $S \subset X$ be a convex subset of X. The function $f : S \to \mathbf{R}$ will be called **uniformly-convex** on S if

$$\lambda f(x) + (1-\lambda)f(y) - f[\lambda x + (1-\lambda)y] \geq \lambda(1-\lambda)\|x-y\|^2 \tag{13}$$

for all $x, y \in S$, $\lambda \in [0,1]$.

Holds true the following characterization of uniformly-convex functions:

Proposition 1.1. ([27]) Let $f : S \to \mathbf{R}$ defined on the convex subset $S \subset X$. Then the following assertions are equivalent:

(i) f is uniformly-convex on S

(ii) $f - \|\cdot\|^2$ is convex on S.

Examples. 1) Let $A : \mathcal{D}(A) \subset X \to X$ let be a linear, symmetric operator on the subspace $\mathcal{D}(A)$ of X, which is coerciv, i.e. satisfying the relation

$$(Ax, x) \geq \gamma\|x\|^2, \; \forall\, x \in \mathcal{D}(A) \; (\gamma > 0).$$

Then the function $f_A : \mathcal{D}(A) \to \mathbf{R}$, $f_A(x) = \frac{1}{\gamma}(Ax, x)$ is uniformly-convex on $\mathcal{D}(A)$.

2) Let $f : (a,b) \subseteq \mathbf{R} \to \mathbf{R}$ be a twice differentiable functions satisfying $f''(x) \geq 0 > 0$, $x \in (a,b)$. Let $g(x) = \frac{2}{m}f(x)$, $x \in (a,b)$. Then g is uniformly-convex.

The following theorem gives also a refinement of Jensen's inequality, in case of uniformly-convex functions:

Theorem 1.6. ([27]) *Let $f : S \subset X \to \mathbf{R}$ be uniformly-convex functions on the convex set S; let $w_i \geq 0$, $W_n > 0$, (where $W_n = \sum_{i=1}^{n} w_i$) and let $a_i \in S$ $(i = \overline{1,n})$. Then*

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right) \geq$$

$$\geq \frac{1}{W_n}\sum_{i=1}^{n} w_i \|a_i\|^2 - \left\|\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right\|^2 \geq 0. \tag{14}$$

Corollary 1.6. Let $A : \mathcal{D}(A) \subset X \to X$ be an operator defined as in Example 1. Then for all $a_i \in \mathcal{D}(A)$, $w_i \geq 0$, $W_n > 0$ $(i = \overline{1,n})$, holds true the following inequality:

$$W_n \sum_{i=1}^{n} w_i (Aa_i, a_i) - \left(A\left(\sum_{i=1}^{n} a_i w_i\right), \sum_{i=1}^{n} a_i w_i\right) \geq$$

$$\geq \gamma \left(W_n \sum_{i=1}^{n} w_i \|a_i\|^2 - \left\|\sum_{i=1}^{n} w_i a_i\right\|\right)^2 \geq 0. \tag{15}$$

Corollary 1.7. Let $f : (a, b) \to \mathbf{R}$ be defined as in Example 2. Then for all $a_i \in (a, b)$, $w_i \geq 0$ with $W_n > 0$ $(i = \overline{1,n})$, we have

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right) \geq$$

$$\geq \frac{m}{2}\left[\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i^2 - \frac{1}{W_n^2}\left(\sum_{i=1}^{n} w_i x_i\right)^2\right] \geq 0.$$

D. The **convex functions of order** n were introduced in the science by Tiberiu Popoviciu [11]. The following result is related to the discrete inequality by Jensen:

Theorem 1.7. *Let $f : (a, b) \to \mathbf{R}$ be a concave and 3-convex function. Let (a_i), (b_i) $(i = \overline{1,n})$ two sequences in (a, b) having the properties*

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq b_n \leq \cdots \leq b_2 \leq b_1$$

$$a_{i+1} - a_i \geq b_i - b_{i+1} \quad (i = 1, 2, \ldots, n-1) \ (n \geq 2).$$

Then

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a_i\right) \leq$$

$$\leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(b_i) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i b_i\right). \tag{16}$$

Proof. We will use induction with respect to n. Let $n = 2$. For simplicity, let us assume that $W_2 = w_1 + w_2 = 1$. Let $a_0 = w_1 a_1 + w_2 a_2$, $b_0 = w_1 b_1 + w_2 b_2$. If $b_1 = b_2$, by concavity of f it results $w_1 f(a_1) + w_2 f(a_2) - f(w_1 a_1 + w_2 a_2) \leq 0$, which shows that (16) is true in this case. If $b_1 \neq b_2$, then $a_1 < a_0 < a_2 \leq b_2 < b_0 < b_1$, so f being 3-convex, we can write:

$$\frac{f(a_1)}{(a_1-a_2)(a_1-a_0)} + \frac{f(a_2)}{(a_2-a_1)(a_2-a_0)} + \frac{f(a_0)}{(a_0-a_1)(a_0-a_2)} \leq$$

$$\leq \frac{f(b_1)}{(b_1-b_2)(b_1-b_0)} + \frac{f(b_2)}{(b_2-b_1)(b_2-b_0)} + \frac{f(b_0)}{(b_0-b_1)(b_0-b_2)}. \tag{$*$}$$

By definition, $a_0 - a_1 = w_2(a_2 - a_1)$; $a_2 - a_0 = w_1(a_2 - a_1)$, so by multiplying both sides of $(*)$ with $w_1 w_2 (a_2 - a_1)^2$, one can deduce

$$w_1 f(a_1) + w_2 f(a_2) - f(a_0) \leq [w_1 f(b_1) + w_2 f(b_2) - f(b_0)]\frac{(a_1-a_2)^2}{(b_1-b_2)^2}.$$

By concavity of f it results $w_1 f(b_1) + w_2 f(b_2) - f(b_0) \leq 0$. From $a_2 - a_1 \geq b_1 - b_2 > 0$ we get $w_1 f(a_1) + w_2 f(a_2) - f(a_0) \leq w_1 f(b_1) + w_2 f(b_2) - f(b_0)$, proving (16) for $n = 2$.

Let us assume now that (16) holds true for all arguments from 2 to $n-1$. Then

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(a_i) - \frac{1}{W_n}\sum_{i=1}^{n} w_i f(b_i) =$$

$$= \frac{W_{n-1}}{W_n}\left\{\frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i f(a_i) - \frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i f(b_i)\right\} +$$

$$+\frac{w_n}{W_n}[f(a_n) - f(b_n)] \leq \frac{W_{n-1}}{W_n}\left\{f\left(\frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i a_i\right) -\right.$$

$$\left.-f\left(\frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i b_i\right)\right\} + \frac{w_n}{W_n}[f(a_n) - f(b_n)].$$

Let

$$c_1 = \frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i a_i, \quad c_2 = a_n,$$

$$d_1 = \sum_{i=1}^{n-1} w_i b_i / W_{n-1}, \quad d_2 = b_n.$$

Then the sequences $\{c_1, c_2\}$ and $\{d_1, d_2\}$ satisfy the conditions of the theorem. Applying the above proved case $n = 2$ with W_{n-1} and w_n in place of w_1, w_2, we obtain the desired inequality.

Corollary 1.8. Let $b > 0$ and $a_i \in (0, b]$ $(i = \overline{1,n})$. Let $f : (0, 2b] \to \mathbf{R}$ have a negative second derivative and a nonnegative third derivative. Then

$$\frac{1}{W_n} \sum_{i=1}^{n} w_i f(a_i) - f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i a_i\right) \leq$$

$$\leq \frac{1}{W_n} \sum_{i=1}^{n} w_i f(2b - a_i) - f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i (2b - a_i)\right). \tag{17}$$

Proof. Put $b_i = 2b - a_i$, where $a_1 \leq a_2 \leq \cdots \leq a_n$. Then the conditions of Theorem 1.7 are satisfied, and we get relation (17). This inequality has been obtain by N. Levinson (see [5]) as a generalization of the famous inequality of Ky Fan ([5], [4], [13]).

Let $a_i \in \left(0, \frac{1}{2}\right]$, $(i = \overline{1,n})$,

$$A_n(a) = \frac{1}{W_n} \sum_{i=1}^{n} w_i a_i, \quad G_n(a) = \prod_{i=1}^{n} a_i^{w_i / W_n},$$

where $a = (a_1, \ldots, a_n)$. Put $A'_n(a) = A_n(1 - a)$, $G'_n(a) = G_n(1 - a)$. Then

$$\frac{G_n}{G'_n} \leq \frac{A_n}{A'_n}. \tag{18}$$

Proof. Apply (17) with $b = \frac{1}{2}$ to $f(x) := \ln x$. Then $f''(x) = -\frac{1}{x^2} < 0$, $f'''(x) = \frac{2}{x^3} > 0$, and after certain elementary computations we obtain Ky Fan's inequality (18).

Let now, for simplicity, $w_i \equiv 1$, $(i = \overline{1,n})$. Then relation (17) can be written also as

$$\frac{1}{n} \sum_{i=1}^{n} f(a_i) - f(A_n) \leq \frac{1}{n} \sum_{i=1}^{n} f(1 - a_i) - f(A'_n) \tag{19}$$

where A_n is the (unweighted) arithmetic mean of (a), and $A\prime_n$ is the (unweighted) arithmetic mean of $(1-a)$.

Let us introduce also

$$H_n = H_n(a) = n/\sum_{i=1}^{n} \frac{1}{a_i}, \quad H\prime_n = H_n(a\prime) = H_n(1-a),$$

the corresponding harmonic means. Let $f(x) = -\frac{1}{x}$ in (19). Then we can deduce the following "additive variant" of the Ky Fan inequality:

$$\frac{1}{A_n} - \frac{1}{H_n} \leq \frac{1}{A\prime_n} - \frac{1}{H\prime_n}. \tag{20}$$

For other variants and refinements we quote the author's papers [13], [14]. See also [4].

E. Inequalities for nondifferentiable η-invex functions generally are fairly difficult to obtain. More precisely, either we must assume that the function f satisfies certain complicated functional equations (see [32]), or if we do not admit such relations, the informations contained in these inequalities are more restrictive.

Let us remind that the function $f : S \to \mathbf{R}$ is called η-invex on the η-invex domain S, if one has

$$f(u + \lambda\eta(x,u)) \leq \lambda f(x) + (1-\lambda) f(u) \text{ for all } x, u \in S,\ \lambda \in [0,1]. \tag{21}$$

Let $\lambda = \dfrac{p}{p+q}$ $(p, q > 0)$. From (21) it follows

$$f\left[\frac{(p+q)u + p\eta(x,u)}{p+q}\right] \leq \frac{pf(x) + qf(u)}{p+q}. \tag{22}$$

Let now $S \subset \mathbf{R}_+ = [0,\infty)$ and apply relation (22) to $p := x_1$, $q := x_2$, $x := x_1 + x_2$, yielding:

$$f\left[\frac{(x_1+x_2)u + x_1\eta(x_1+x_2,u)}{x_1+x_2}\right] \leq \frac{x_1 f(x_1+x_2) + x_2 f(u)}{x_1+x_2}.$$

By interchanging x_1 with x_2 we can write

$$f\left[\frac{(x_1+x_2)u + x_2\eta(x_1+x_2,u)}{x_1+x_2}\right] \leq \frac{x_2 f(x_1+x_2) + x_1 f(u)}{x_1+x_2}.$$

By addition we get

$$f(x_1+x_2)+f(u)\geq f[u+\alpha_1\eta(x_1+x_2,u)]+f[u+\alpha_2\eta(x_1+x_2,u)] \qquad (23)$$

where $\alpha_1+\alpha_2=1$, $\alpha_1>0$, $\alpha_2>0$.

Put $u:=0$ in (23) and assume that f satisfies

$$f(a\theta(b))\geq f(ab) \text{ cu } a,b>0 \qquad (24)$$

where $\theta(x)=\eta(x,0)$. By taking into account of

$$f\left[\frac{x_1}{x_1+x_2}\theta(x_1+x_2)\right]\geq f(x_1) \quad \text{şi} \quad f\left[\frac{x_2}{x_1+x_2}\theta(x_1+x_2)\right]\geq f(x_2),$$

one gets the inequality

$$f(x_1)+f(x_2)\leq f(x_1+x_2)+f(0), \quad \text{with} \quad x_1>0,\ x_2>0. \qquad (25)$$

By mathematical induction it easily follows now that

$$f(x_1)+\cdots+f(x_n)\leq f(x_1+\cdots+x_n)+(n-1)f(x_0), \quad x_i>0\ (i=\overline{1,n})\ (n\geq 1) \qquad (26)$$

So, we have proved the following result:

Theorem 1.8. Let $f:[0,\infty)\to\mathbf{R}$ be η-invex function and let $\theta(x)=\eta(x,0)$ with $x>0$. Let us assume that for $a,b>0$ one has the inequality $f(a\theta(b))\geq f(ab)$. Then, for all $x_i>0$ $(i=\overline{1,n})$, $(n\geq 1)$ we have the inequality (26).

Remark. For convex f and $\theta(x)=x$ we can reobtain from (26) the known inequality by M. Petrović ([10]).

In what follows we shall introduce the notion of **invex combination**. Let X be a linear space and let $S\subset X$ be an invex subset of X. We say that z is an invex combination of x_1 and x_2, in notation $z\in\operatorname{inv}(x_1,x_2)$ if there exists $\lambda\in[0,1]$ such that $z=x_2+\lambda\eta(x_1,x_2)$. Let $x_1,\ldots,x_n\in S$. Then $z\in\operatorname{inv}(x_1,x_2,\ldots,x_n)$ (invex combination of n elements) if there exist $y\in\operatorname{inv}(x_1,\ldots,x_{n-1})$ and there exists $\lambda\in[0,1]$ such that

$$z=y+\lambda\eta(x_n,y)\in\operatorname{inv}(y,x_n).$$

We can prove the following analogue of Jensen's inequality:

Theorem 1.9. Let $f : S \to \mathbf{R}$ be η-invex function. Then for all $n \geq 2$ and $x_1, x_2, \dots, x_n \in S$ and $z \in \operatorname{inv}(x_1, x_2, \dots, x_n)$ there exists $Z \in \operatorname{conv}(f(x_1), \dots, f(x_n))$ with the property

$$f(z) \leq Z \tag{27}$$

where conv *is the convex combination.*

Proof. We shall proceed by mathematical induction. For $n = 2$ we have $z \in \operatorname{inv}(x_1, x_2) \in S$, so $z = x_2 + \lambda\eta(x_1, x_2)$ and from (21) we can deduce $f(z) \leq \lambda f(x_1) + (1-\lambda)f(x_2) = Z \in \operatorname{conv} f(x_1), f(x_2))$. Let us assume that relation (27) holds for n elements, and let $z\prime \in \operatorname{inv}(x_1, x_2, \dots, x_{n+1})$, where $z \in \operatorname{inv}(x_1, x_2, \dots, x_n)$. Then $z\prime$ has a form $z\prime = z + \lambda\eta(x_{n+1}, z)$ so we can write $f(z\prime) \leq \lambda f(x_{n+1}) + (1-\lambda)f(z) = \lambda f(x_{n+1}) + (1-\lambda)[\overline{\lambda}_1 f(x_1) + \overline{\lambda}_2 f(x_2) + \cdots + \overline{\lambda}_n f(x_n)]$ where $\overline{\lambda}_1 + \cdots + \overline{\lambda}_n = 1$. Therefore, $f(\overline{z}) \leq \overline{\lambda}_1(1-\lambda)f(x_1) + \overline{\lambda}_2(1-\lambda)f(x_2) + \cdots + \overline{\lambda}_n(1-\lambda)f(x_n) + \lambda f(x_{n+1})$. Remarking that $\overline{\lambda}_1(1-\lambda) + \cdots + \overline{\lambda}_n(1-\lambda) + \lambda = 1$ we get $f(z\prime) \leq Z\prime \in \operatorname{conv}(f(x_1), \dots, f(x_{n+1}))$, finishing the proof of Theorem 1.9.

F. In this final subsection on Jensen's inequality we mention certain applications. First we reobtain the classical inequality of weighted means. This inequality plays a central role in information theory (Shannon's theory of entropy) [1], in the theory of codes (Kraft's inequality), in the theory of functional equations and rational group decision [2], etc. (See e.g. [3] for applications and economics, and [6] for geometric programming).

Theorem 1.10. (Theorem of means) *Let $a_j > 0$, $q_j > 0$ $(j = \overline{1,n})$ with $\sum_{j=1}^{n} q_j = 1$. Then we have*

$$\prod_{j=1}^{n} a_j^{q_j} \leq \sum_{j=1}^{n} q_j a_j \tag{28}$$

Proof. Select $b_j := \log a_j$ and the convex function $f(t) = e^t$ $(t \in (-\infty, \infty))$ and apply Jensen's discrete inequality.

By letting $n = 2$, $q_1 = \dfrac{1}{p}$, $q_2 = \dfrac{1}{q}$; $a_1 = x^p$, $a_2 = y^q$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we obtain:

Corollary 1.9. a) (Young's inequality)

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q, \text{ where } \frac{1}{p} + \frac{1}{q} = 1,\ p > 1 \tag{29}$$

b) (Hölder's inequality)

$$\sum_{j=1}^{n} x_j y_j \leq \left(\sum_{j=1}^{n} x_j^p\right)^{1/p} \left(\sum_{j=1}^{n} y_j^q\right)^{1/q} \quad (x_j, y_j > 0) \tag{30}$$

Proof. It is sufficient to consider

$$u := \left(\sum_{j=1}^{n} x_j^p\right)^{1/p}, \quad v = \left(\sum_{j=1}^{n} y_j^q\right)^{1/q}$$

and apply (29) for $x := x_j/u$, $y := y_j/v$. After summation we get (30).

The following little known refinement of (23) is due to the author [15]:

Theorem 1.11. *Let* $\lambda > 0$, $p > 0$ *and let*

$$J(a_i, q_i, p, \lambda) = \left\{ p \int_0^{\infty} \left[\frac{\prod_{j=1}^{n} (1 + \lambda a_j + \lambda x)^{q_j} - 1}{\lambda} \right]^{-p-1} dx \right\}^{1/p}$$

and

$$J(a_i, q_i, p) = \left\{ p \int_0^{\infty} \left[\prod_{j=1}^{n} (x + a_j)^{q_j} \right]^{-p-1} dx \right\}^{-1/p}.$$

Then we have the following inequalities:

$$\prod_{j=1}^{n} a_j^{q_j} \leq J(a_i, q_i, p) \leq J(a_i, q_i, p, \lambda) \leq \sum_{j=1}^{n} q_j a_j. \tag{31}$$

Proof. Since this result has been published in a journal with reduced circulation, we give here the proof of (31). First we prove that

$$\prod_{j=1}^{n} a_j^{q_j} \leq \frac{1}{\lambda} \left[\prod_{j=1}^{n} (1 + \lambda a_j)^{q_j} - 1 \right] \leq \sum_{j=1}^{n} a_j q_j. \tag{32}$$

Indeed, let $f(x) = \ln(1 + \lambda e^x)$, $x \in \mathbf{R}$, which is strictly convex since $f''(x) = \lambda e^x/(1 + \lambda e^x)^2 > 0$. By Jensen's inequality we have

$$\ln\left(1 + \lambda e^{\sum\limits_{j=1}^{n} a_j q_j}\right) \leq \sum_{j=1}^{n} q_j \ln(1 + \lambda e^{a_j}).$$

By the substitution $e^{a_j} \to a_j$ we obtain

$$1 + \lambda \prod_{j=1}^{n} a_j^{q_j} \leq \prod_{j=1}^{n} (1 + \lambda a_j)^{q_j}.$$

On the other hand, from the inequality of means we can write

$$\prod_{j=1}^{n} (1 + \lambda a_j)^{q_j} \leq 1 + \lambda \sum_{j=1}^{n} a_j q_j,$$

which combined with the above inequality gives (32). Apply now this inequality to $a_j + x$ in place of a_j and integrate the obtained relation. We can successively deduce

$$\prod_{j=1}^{n} (a_j + x)^{q_j} \leq \frac{1}{\lambda}\left[\prod_{j=1}^{n} (1 + \lambda a_j + \lambda x)^{q_j} - 1\right] \leq \sum_{j=1}^{n} q_j a_j + x$$

and since $p > 0$, we have

$$\int_0^\infty \left[\sum_{j=1}^{n} (x + a_j)^{q_j}\right]^{-p-1} dx \geq \int_0^\infty \left[\frac{\prod\limits_{j=1}^{n} (x + \lambda a_j + \lambda x)^{q_j} - 1}{\lambda}\right]^{-p-1} dx \geq$$

$$\geq \int_0^\infty \left[x + \sum_{j=1}^{n} q_j a_j\right]^{-p-1} dx = \frac{1}{p}\left(\sum_{j=1}^{n} q_j a_j\right)^{-p}. \tag{33}$$

By Hölder's integral inequality for n functions (which for 2 functions is in fact a consequence of (30), while for n functions follows by mathematical induction, see e.g. [8]) we can write

$$\int_0^\infty \left[\prod_{j=1}^{n} (x + a_j)^{q_j}\right]^{-p-1} dx = \int_0^\infty \prod_{j=1}^{n} [(x + a_j)^{-p-1}]^{q_j} dx \leq$$

$$\leq \prod_{j=1}^{n} \left[\int_0^\infty (x + a_j)^{-p-1} dx\right]^{q_j} = \prod_{j=1}^{n} \frac{1}{p} a_j^{-pq_j},$$

which combined with (33) gives us

$$\prod_{j=1}^{n} \frac{1}{p}(a_j)^{-pq_j} \geq \int_0^\infty \left[\sum_{j=1}^{n}(x+a_j)^{q_j}\right]^{-p-1} dx \geq$$

$$\geq \int_0^\infty \left[\frac{\prod_{j=1}^{n}(1+\lambda a_j+\lambda x)^{q_j}-1}{\lambda}\right]^{-p-1} dx \geq \frac{1}{p}\left(\sum_{j=1}^{n} q_j a_j\right)^{-p},$$

finishing the proof of theorem.

Corollary 1.10. $(a_1 a_2 \ldots a_n)^{1/n} \leq \left\{p\int_0^\infty [(x+a_1)\ldots(x+a_n)]^{-(p+1)/n} dx\right\}^{1/p} \leq$

$$\leq \left\{p\int_0^\infty \left[\frac{(1+\lambda a_1+\lambda x)^{1/n}\ldots(1+\lambda a_n+\lambda x)^{1/n}-1}{\lambda}\right]^{-p-1} dx\right\}^{-1/p} \leq$$

$$\leq \frac{1}{n}(a_1+\cdots+a_n). \tag{34}$$

(Put $q_1 = \cdots = q_n = \frac{1}{n}$ in (31)).

Application. Let $n = 3$ in (34). We shall apply this relation in the theory of **geometric inequalities**. Let ABC be a triangle of sides a, b, c; with r as the inscribed circle radius, R as the circumscribed circle radius. Then (see [16]) it is known that

$$R \geq \frac{a+b+c}{3\sqrt{3}} \quad \text{and} \quad 2r \leq \frac{(abc)^{1/3}}{\sqrt{3}}.$$

From the above inequality for $n = 3$ we can obtain the following refinements

$$2r\sqrt{3} \leq (abc)^{1/3} \leq J(a,b,c,p) \leq J(a,b,c,p,\lambda) \leq \frac{1}{3}(a+b+c) \leq R\sqrt{3},$$

implying in fact infinitely many refinements of the classical Euler inequality $2r \leq R$.

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