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AN EXTENSION OF RUSCHEWEYH'S UNIVALENCE CONDITION

IRINEL RADOMIR

Abstract. We obtain a new sufficient univalence condition, generalizing the univalence criterion of S.Ruscheweyh.

1. Introduction

We denote by U_r the disk of z–plane, $U_r = \{ z \in C : |z| < r \}$, where $r \in (0,1]$, $U_1 = U$ and $I = [0,\infty)$.

Let A be the class of functions f which are analytic in U with $f(0) = 0$ and $f'(0) = 1$.

Theorem 1.1. ([4]). *Let $s = \alpha + i\beta$, $\alpha > 0$ and $f \in A$. Assume that for a certain $c \in C$ and all $z \in U$,*

$$\left| c|z|^2 + s - \alpha(1-|z|^2)\left[s\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-s)\frac{zf'(z)}{f(z)}\right]\right| \le M\ , \tag{1}$$

where

$$M = \begin{cases} \alpha|s| + |s+c|(\alpha - 1)\ , & 0 < \alpha < 1, \\ |s|\ , & \alpha \ge 1. \end{cases} \tag{2}$$

Then the function f is univalent in U.

We will need Loewner's parametric method to prove our results.

2. Preliminaries

Theorem 2.1. ([3]. *Let r be a real number, $r \in (0,1]$. Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$, $a_1(t) \neq 0$, be analytic in U_r, for all $t \in I$, locally absolutely continuous in I and*

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locally uniform with respect to U_r . *For almost all* $t \in I$ *suppose*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t} \qquad (\forall) z \in U_r ,$$

where $p(z,t)$ *is analytic in* U *and satisfies* $\mathrm{Re}\,p(z,t) > 0$, $z \in U$, $t \in I$.

If $|a_1(t)| \to \infty$ *for* $t \to \infty$ *and* $\{ L(z,t)/a_1(t) \}$ *forms a normal family in* U_r , *then for each* $t \in I$, $L(z,t)$ *has an analytic and univalent extension to the whole disk* U.

3. Main results

Theorem 3.1. *Let* $f \in A$ *and let* s, c *be complex numbers,* $s = \alpha + i\beta$, $\alpha > 0$, $c \neq 0$, $|s + c| \leq |s|$. *If there exists an analytic function in* U, $p(z) = 1 + c_1 z + \ldots$, *such that*

$$\left|\frac{c}{p(z)} + s\right| \leq |s| , \tag{3}$$

$$\left|\frac{c}{p(z)}|z|^{2/\alpha} + s - \alpha(1 - |z|^{2/\alpha}) \left[s\left(1 + \frac{zf''(z)}{f'(z)}\right) + \right.\right. \tag{4}$$

$$\left.\left. +(1 - s)\frac{zf'(z)}{f(z)} + s\frac{zp'(z)}{p(z)}\right]\right| \leq |s| ,$$

for all $z \in U$, *then the function* f *is univalent in* U.

Proof. The conditions (3) and (4) implies that $p(z) \neq 0$ and $f(z)f'(z)/z \neq 0$ in U. If $c \neq 0$ let

$$f(z,t) = f(e^{-st}z)\left[1 - \frac{\alpha}{c}(e^{2t} - 1)p(e^{-st}z)e^{-st}z\frac{f'(e^{-st}z)}{f(e^{-st}z)}\right]^s \tag{5}$$

The inequalities $|c+s| \leq |s|$ and $\mathrm{Re}\, s > 0$ imply $\alpha/c \notin [0,\infty)$. It follows that there exists $r \in (0,1]$ such that

$$1 - \frac{\alpha}{c}(e^{2t} - 1)p(e^{-st}z)e^{-st}zf'(e^{-st}z)/f(e^{-st}z) \neq 0$$

for all $z \in U_r$ and $t \geq 0$, and hence the function $f(z,t)$ is analytic in U_r for all $t \geq 0$. Furthermore

$$\left|\frac{\partial f(0,t)}{\partial z}\right| = \left|\left[\left(1 + \frac{\alpha}{c}\right)e^{-t} - \frac{\alpha}{c}e^t\right]\right|^s \neq 0$$

in I, and $\lim_{t\to\infty}\left|\frac{\partial f(0,t)}{\partial z}\right| = \infty$ (we have chosen a fixed branch for $\frac{\partial f(0,t)}{\partial z}$). It follows that $\{ f(z,t)/\frac{\partial f(0,t)}{\partial z} \}$ forms a normal family in U_{r_0}, $r_0 < r$.

A simple calculation yields

$$\frac{\partial f(z,t)}{\partial t}/z \cdot \frac{\partial f(z,t)}{\partial z} = s\frac{1+P(e^{-st}z,t)}{1-P(e^{-st}z,t)} ,$$

where

$$P(z,t) = \frac{c}{\alpha}e^{-2t}\frac{1}{p(z,t)} + 1 - (1-e^{-2t})H_s(e^{-st}z) \quad \%; and \tag{6}$$

$$H_s(z) = s\left(1+\frac{zf''(z)}{f'(z)}\right) + (1-s)\frac{zf'(z)}{f(z)} + s\frac{zp'(z)}{p(z)} .$$

If $h(z,t) = \frac{\partial f(z,t)}{\partial t}/(z\frac{\partial f(z,t)}{\partial z})$ then the inequality $Re\ h(z,t) > 0$ for all $z \in U$ and $t \in I$ is equivalent to

$$|\ \alpha P(e^{-st}z,t) + i\beta\ | \le |s|, \quad z \in U,\ \ t \in I. \tag{7}$$

Replacing the function $P(z,t)$ defined from (6) in (7) we obtain

$$\left| e^{-2t}\left(\frac{c}{p(e^{-st}z)} + s\right) + (1-e^{-2t})[\ -\alpha H_s(e^{-st}z) + s] \right| \le |z| . \tag{8}$$

In order to prove the inequality (8) we consider the function

$$Q(z,t) = e^{-2t}\left(\frac{c}{p(e^{-st}z)} + s\right) + (1-e^{-2t})[\ -\alpha H_s(e^{-st}z) + s]$$

which for all $t \in I \setminus \{0\}$ is analytic in $\bar{U}$ and hence

$$\max_{|z|\le 1} |\ Q(z,t)\ | = |\ Q(e^{i\theta},t)\ |, \quad \theta \in R. \tag{9}$$

If $\xi = e^{-st}e^{i\theta}$, then $|\xi| = e^{-\alpha t}$, $e^{-t} = |\xi|^{1/\alpha}$ and by (8), (9) and (4) it results

$$|\ Q(z,t)\ | < |\ Q(e^{i\theta},t)\ | = \left| |\xi|^{2/\alpha}\left(\frac{c}{p(\xi)} + s\right) + \right.$$

$$\left. + \left(1 - |\xi|^{\frac{2}{\alpha}}\right)[\ -\alpha H_s(\xi) + s\ ] \right| \le |s| ,$$

for all $z \in U$ and $t \in I \setminus \{0\}$.

If $t = 0$, then $Q(z,0) = c/p(z) + s$ and by (3) it results that $|Q(z,0)| \le |s|$ for all $z \in U$ and hence the inequality (8) holds true for all $z \in U$ and $t \in I$.

Theorem 3.2. *Let $f \in A$ and let s, c be complex numbers, $s = \alpha + i\beta$, $\alpha \geq 1$, $c \neq 0$, $|s + c| \leq |s|$. If there exists an analytic function in U, $p(z) = 1 + c_1(z) + \ldots$, such that*

$$\left| \frac{c}{p(z)} + s \right| \leq |s| \tag{10}$$

$$\left| \frac{c}{p(z)}|z|^2 + s - \alpha(1 - |z|^2) \left[s\left(1 + \frac{zf''(z)}{f'(z)}\right) + \right.\right. \tag{11}$$

$$\left.\left. +(1 - s)\frac{zf'(z)}{f(z)} + s\frac{zp'(z)}{p(z)} \right]\right| \leq |s|,$$

for all $z \in U$, then the function f is univalent in U.

Proof. The function

$$w(z, \lambda) = \lambda\left(\frac{c}{p(z)} + s\right) + (1 - \lambda)[\, -\alpha H_s(z) + s\,]$$

is analytic in U for all $\lambda \in [0, 1]$. From (10) and (11) it results that

$$|\, w(z,\ |z|^2)\, | \leq |s| \qquad (\forall) z \in U\ ; \tag{12}$$

$$|\, w(z,\ 1)\, | \leq |s| \qquad (\forall) z \in U. \tag{13}$$

If λ increases from $\lambda_1 = |z|^2$ to $\lambda_2 = |z|^{2/\alpha}$, then the point $w(z, \lambda)$ moves on the segment whose endpoints are $A = w(z,\ |z|^2)$ and $B = w(z,\ 1)$, and hence from (12) and (13) it results that

$$|\, w(z,\ |z|^{2/\alpha})\, | \leq |s| \tag{14}$$

for all $z \in U$. Because

$$w(z,\ |z|^{2/\alpha}) = \frac{c}{p(z)}|z|^{2/\alpha} + s - \alpha(1 - |z|^{2/\alpha}) \left[\left(1 + \frac{zf''(z)}{f'(z)}\right) + \right. \tag{15}$$

$$\left. +(1 - s)\frac{zf'(z)}{f(z)} + s\frac{zp'(z)}{p(z)} \right]$$

from (14) and (15) it results that (4) holds true for all $z \in U$ and from Theorem 3.1 it results that the function f is univalent in U. *Remark.* For $\alpha \geq 1$ and $p(z) \equiv 1$, from Theorem 3.2 we obtain Theorem 1.1 .

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DEPARTMENT OF MATHEMATICS, "TRANSILVANIA" UNIVERSITY, BRAŞOV, ROMANIA