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# ON DARBOUX LINES

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**Abstract.** In this paper, we first prove that the only surface (other then a sphere) on which the two families of Darboux lines form a Tschebycheff net and the third family of Darboux lines is transversal to one of the two families of Darboux lines is a cylinder of revolution. We next show that the surface (other than a sphere or a developable surface) on which the Darboux lines correspond to those on its parallel surface is a surface of constant mean curvature. Moreover, if the two families of Darboux lines on a surface of constant mean curvature form a Tschebyscheff net, then a surface becomes a cylinder of revolution or a plane. Furthermore, we prove that surfaces (other than a sphere or a plane) whose Darboux lines are preserved under inversion are Dupin's cyclides or, in particular, a pipe surface of revolution. Finally, we show that Molure surface on which the two families of Darboux lines which are different from the lines of curvature form two semi-Tschebyscheff nets together with a family of lines of curvature are either a surface of revolution or a pipe surface.

## 1. Introduction

Let $S$ be a surface of class $C^4$ in Euclidean 3-space and let $C$ be a line on $S$. $C$ is said to be a Darboux line if the relation

$$\mathcal{D} = \frac{d\tau_g}{ds} + (\rho - \overline{\rho}_n)\rho_g = 0 \tag{1.1}$$

holds all along $C$, where $\mathcal{D}$ is the Darboux's direction function and $\rho_n, \rho_g, \tau_g$ and $s$ are, respectively, the normal curvature, the geodesic curvature, the geodesic torsion and the arc-length of $C$, $\overline{\rho}_n$ being the normal curvature of the orthogonal trajectories of $C$.

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Let a Darboux line make angles $\gamma^*$ and $\gamma^{**}$ $(\gamma^* + \gamma^{**} = \delta)$ with the parametric lines $v = \text{const.}$ and $u = \text{const.}$ respectively. Then, by using the respective generalised Euler, Ossian-Bonnet and Liouville formulae

$$\rho_m = \{r^* \cos\gamma^* \sin\gamma^{**} + r^{**} \sin\gamma^* \cos\gamma^{**} + (t^* - t^{**}) \sin\gamma^* \sin\gamma^{**}\}/\sin\delta$$

$$\tau_g = \{t^* \cos\gamma^* \sin\gamma^{**} + t^{**} \sin\gamma^* \cos\gamma^{**} + (r^{**} - r^*) \sin\gamma^* \sin\gamma^{**}\}/\sin\delta$$

$$\rho_g = \{(g^* + \gamma_1^*) \sin\gamma^{**} + (g^{**} - \gamma_2^{**}) \sin\gamma^*\}/\sin\delta$$

the equation (1.1) can be expressed in the form

$$\begin{aligned}
\mathcal{D} = &\left(\frac{t_1^* \sin\gamma^{**} + t_2^*\gamma^*}{\sin\delta}\right)\left(\frac{\cos\gamma^* \sin\gamma^{**}}{\sin\delta}\right) + \\
&+\left(\frac{t_1^{**} \sin\gamma^{**} + t_2^{**} \sin\gamma^*}{\sin\delta}\right)\left(\frac{\sin\gamma^* \cos\gamma^{**}}{\sin\delta}\right) + \\
&+\left[\frac{(r^{**} - r^*)_1 \sin\gamma^{**} + (r^{**} - r^*)_2 \sin\gamma^*}{\sin\delta}\right]\left(\frac{\sin\gamma^* \sin\gamma^{**}}{\sin\delta}\right) + \\
&+\frac{1}{\sin^3\delta}\{(\gamma_1^* \sin\gamma^{**} + \gamma_2^* \sin\gamma^*)[(r^{**} - r^*) \sin^2\gamma^{**} - \\
&-(t^* - t^{**}) \cos\gamma^{**} \sin\gamma^{**}] + (\gamma_1^{**} \sin\gamma^{**} + \gamma_2^{**} \sin\gamma^*)[(r^{**} - r^*) \sin^2\gamma^* + \\
&+(t^* - t^{**}) \cos\gamma^* \sin\gamma^*]\} + \frac{1}{\sin^2\delta}\{(r^* - r^{**}) \sin(\delta - 2\gamma^*) + \\
&+(t^* - t^{**}) \cos(\delta - 2\gamma^*)\}\{(g^* + \gamma_1^*) \sin\gamma^{**} + (g^{**} - \gamma_2^{**}) \sin\gamma^*\} = 0
\end{aligned} \tag{1.2}$$

where $r^*, r^{**}; g^*, g^{**}; t^*, t^{**}$ are, respectively, the normal curvatures, geodesic curvatures and geodesic torsions of the parametric lines and 1, 2 are the indices of the first order invariant derivatives.

The derivative of the differentiable function $f(u, v)$ in the direction of the curve $C$ is

$$\frac{df}{ds} = [(f)_1 \sin\psi + (f)_2 \sin\phi]/\sin\delta,$$

where $\psi, \phi$ are the angles between the tangent of the curve $C$ and the parametric lines $u$ =const., $v$ =const., respectively, and $s$ is the arc-length of $C$; $(f)_1$ and $(f)_2$ being the invariant derivatives of $f$ in the direction of the parametric lines $v$ =const., $u$ =const.

If the lines of curvature of $S$ are taken as parametric lines, then (1.2) becomes

$$(1.2)' \quad -\overline{r}_1 \sin^3\varphi + (\overline{r}_2 - 2r_2)\sin^2\varphi\cos\varphi + (2\overline{r}_1 - r_1)\sin\varphi\cos^2\varphi + r_2\cos^3\varphi = 0$$

where $\varphi$ is the angle between a Darboux line and the line $v$ =const., and $r, \overline{r}$ are the principal curvatures of $S$.

A vector field $\vec{u}$ making an angle $\varphi$ with the unit tangent vector field $\vec{t}$ of the curve $C$ on $S$ will undergo a parallel displacement along $C$ in the sense of Levi-Civita if

$$\frac{d\varphi}{ds} = -\rho_g, \tag{1.3}$$

where $\rho_g$ and $s$ are, respectively, the geodesic curvature and the arc-length of $C$ [1]. We assume that the angle $\varphi$ is measured in the positive sense around the normal of $S$ from $\vec{t}$ towards $\vec{u}$.

A Tschebyscheff net is by definition a set of parameter curves $p$ =const., $q$ =const., on a surface in terms of which the linear element takes the form

$$ds^2 = dp^2 + 2\cos\varphi dpdq + dq^2$$

or, equivalently, a net is a Tschebyschedd net if and only if the tangent vector field of either family of the net undergoes a parallel displacement in the sense of Levi-Civita along each curve of the other family [2].

A curve $C$ is called a transversal of a vector field, if the vector field undergoes a parallel displacement along $C$.

The couple $(\mathcal{D}_1, \mathcal{D}_2)$ of two families of curves on $S$ is called a 2-net. A 2-net is said to be a semi-Tschebyscheff net if one of the two families of this net is the transversal of the tangent vector field of the other family.

## 2. Tschebyscheff nets formed by Darboux lines

In this section surfaces on which the two families of Darboux lines form a Tschebyscheff net and the third family of Darboux lines is transversal to one of the two families of the Tschebyscheff net will be determined.

Let the three families of Darboux lines on the surface $S$ be denoted by $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$. Withput loss of generality, we can take the two families $\mathcal{D}_1, \mathcal{D}2$ as the families of parametric lines $v$ =const., $u$ =const., respectively. In this case, from (1.2) we obtain

$$t_1^* + [(r^* - r^{**}) + (t^* - t^{**})\text{cotg}\delta]g^* = 0 \tag{2.1}$$

$$t_2^{**} + [-(r^* - r^{**}) + (t^* - t^{**})\text{cotg}\delta]g^{**} = 0. \tag{2.2}$$

The conditions for the 2-net $\Delta \equiv (\mathcal{D}_1, \mathcal{D}_2)$ to be a Tschebyscheff net are, according to (1.3)

$$g^{**} - \delta_2 = 0, \quad g^* + \delta_1 = 0. \tag{2.3}$$

On the other hand, the condition for the third family $\mathcal{D}_3$ to be transversal of the tangent vector field to one of the two families $\mathcal{D}_1$ and $\mathcal{D}_2$, to $\mathcal{D}_1$ say, is, according to (1.3)

$$\frac{d\gamma^*}{ds} - (\rho_g)_{\mathcal{D}_3} = 0 \tag{2.4}$$

where $(\rho_g)_{\mathcal{D}_3}$ is the geodesic curvature of the curves belonging to $\mathcal{D}_3$, $s$ being the arc-length of $\mathcal{D}_3$.

The conditions (2.3) and (2.4) give

$$g^* = 0, \quad \delta = \delta(v). \tag{2.5}$$

Then, by using the Gauss equation [3]

$$K = r^*\{r^{**} + t^{**} - t^*)\text{cotg}\delta - t^{*2} =$$

$$= [g_2^* - g_1^* + qg^* - \overline{q}g^{**} + q\delta_1 + \delta_{12}]/\sin\delta$$

where

$$q = \frac{(g^{**} - \delta_2)\cos\delta - (g^* + \delta_1)}{\sin\delta}, \quad \overline{q} = \frac{(g^{**} - \delta_2) - (g^* + \delta_1)\cos\delta}{\sin\delta}$$

we have

$$K = r^*[r^{**} + (t^{**} - t^*)\text{cotg}\delta] - t^{*2} = 0 \tag{2.6}$$

so that $S$ is a developable surface.

Under the conditions (2.5), from (2.1) it follows that

$$t^* = h(v) \tag{2.7}$$

where $h(v)$ is an arbitrary differentiable function of its argument.

Differentiating (2.2) with respect to $u$ and making use of the relation

$$(r^{**} - r^*)\mathrm{cotg}\delta = t^* + t^{**}, \tag{2.8}$$

we obtain

$$t^{**}_{uv} = (2\delta_v \mathrm{cotg} 2\delta) t^{**}_u \tag{2.9}$$

where we have assumed that

$$\delta \neq \frac{\pi}{2}. \tag{2.8$'$}$$

For

$$t^{**}_u \neq 0, \tag{2.10}$$

integration of (2.9) gives

$$t^{**}_u = f(u) \sin 2\delta(v) \tag{2.10$'$}$$

where $f(u)$ is an arbitrary differentiable function of its argument.

Under the conditions (2.3), (2.4), (2.7), (2.10)', the Mainardi-Codazzi equations [3]

$$r^*_2 = [(t^* - t^{**})(g^{**} - \delta_2)/\sin\delta + \{r^* - r^{**} - (t^* - t^{**})\mathrm{cotg}\delta\}(g^* + \delta_1) + \\ + t^*_2 \cos\delta - t^{**}_1]/\sin\delta \tag{2.11}$$

$$r^{**}_1 = [(t^{**} - t^*)(g^* + \delta_1)/\sin\delta + \{r^* - r^{**} - (t^{**} - t^*)\mathrm{cotg}\delta\})g^{**} - \delta_2) + \\ + t^*_2 - t^{**}_1 \cos\delta]/\sin\delta \tag{2.12}$$

take the respective forms

$$r^*_v = h'(v)\mathrm{cotg}\delta(v) - \frac{f(u)\sin 2\delta(v)}{\sin\delta(v)} \tag{2.13}$$

$$r_u^{**} = \frac{h'(v)}{\sin\delta(v)} - f(u)\sin 2\delta(v)\mathrm{cotg}\delta(v). \tag{2.14}$$

Differentiating (2.8) with respect to $u$ and $v$ and using (2.13), (2.14), (2.10)' and (2.7) we have

$$r_u^{*} = \frac{h'(v)}{\sin\delta(v)} - 2f(u), \quad r_v^{**} = \frac{2h'(v)}{\sin 2\delta(v)} - 2f(u)\cos\delta(v) + 2t^{**}g^{**}. \tag{2.15}$$

With the help of (2.13) and (2.15), the integrability condition $r_{uv}^{*} = r_{vu}^{*}$ for $r^*$ gives

$$\frac{1}{\cos\delta(v)}\frac{d}{dv}\left[\frac{h'(v)}{\sin\delta(v)}\right] = -2f'(u) = A = const. \tag{2.16}$$

Differentiating (2.6) with respect to $u$ and using (2.7), (2.8), (2.10)' and the first equation of (2.15) we get

$$h^2(v)\left[\frac{h'(v)}{\sin\delta(v)} - 2f(u)\right] + r^{*2}\frac{h'(v)}{\sin\delta(v)} = 0 \tag{2.17}$$

or

$$r^{*2} = -h^2(v)\left[1 - \frac{2f(u)}{h'(v)/\sin\delta(v)}\right] \tag{2.17$'$}$$

with

$$r^* \neq 0, \quad h'(v) \neq 0. \tag{2.17$''$}$$

Differentiating (2.17)' with respect to $u$ and using (2.15) and (2.16) we obtain

$$r^* = \frac{-(A/2)h^2(v)}{l(v)[l(v) - 2f(u)]}, \quad l(v) = h'(v)/\sin\delta(v) \tag{2.18}$$

where

$$l(v) - 2f(u) \neq 0. \tag{2.18$'$}$$

Substituting of $r^*$ in (2.17)' gives

$$\left[\frac{(A/2)^2h^2(v)}{l(v)}\right]^{1/3} - l(v) = -2f(u) = B = const. \tag{2.19}$$

From (2.16) and (2.19) it follows that $r^* = 0$ which contradicts the condition $r^* \neq 0$ in (2.17)".

Now we consider the cases where the conditions (2.18)', (2.17)", (2.10), (2.8)' are not satisfied, namely the cases

I. $h'(v)/\sin\delta(v) - 2f(u) = 0$, II. $h'(v) = 0$, III. $r^* = 0$, IV. $t_u^{**} = 0$, V. $\delta(v) = \pi/2$.

It is easy to see that the cases I and III cannot hold.

In case II, from (2.17) it follows that

$$h^2(v)f(u) = 0. \qquad (2.20)$$

If, in (2.20) $f(u) = 0$, then (2.10)' gives $t_u^{**} = 0$ which contradicts (2.10).

If, in (2.20) $h(v) = 0$, then from (2.16) and (2.7) we get $A = 0$, $f(u) = C_1 = const.$ and $t^* = 0$. Substituting of $f(u), h(v)$ and $t^*$ in (2.15), (2.14), (2.13), (2.10)', (2.2) yields

$$r_u^* = -2C_1, \quad r_v^* = -2C_1\cos\delta(v) \qquad (2.21)$$

$$r_u^{**} = -2C_1\cos^2\delta(v), \quad r_v^{**} = -2C_1\cos\delta(v) + 2t^{**}g^{**} \qquad (2.22)$$

$$t_u^{**} = C_1\sin 2\delta(v), \quad t_v^{**} = 2t^{**}g^{**}\mathrm{cotg}2\delta(v) \qquad (2.23)$$

With the help of the condition (2.3), (2.23) gives

$$t^{**} = (C_1u + C_2)\sin 2\delta(v), \quad (C_1, C_2) = const. \qquad (2.24)$$

Then, from (2.24) and (2.22), we obtain

$$r^{**} = -2(C_1u + C_2)\cos^2\delta(v) - 2C_1\int\cos\delta(v)dv + C_4 \quad (C_4 = const.) \qquad (2.25)$$

Substituting the values of $r^{**}, t^*, t^{**}$ in (2.6) and remembering that $r^* \neq 0$, we get

$$2C_1\int\cos\delta(v)dv = C_4$$

from which it follows that $C_1 = C_4 = 0$. In this case, from the first equation of (2.23) we find that $t^{**} = 0$ which contradicts (2.10). Therefore, the case II can not hold.

In case IV, we have

$$t^{**} = k(v) \qquad (2.26)$$

where $k(v)$ is an arbitrary differentiable function of its argument. Then, Mainardi-Codazzi equations become

$$r_v^* = h'(v)\text{cotg}\delta(v), \quad r_u^{**} = \frac{h'(v)}{\sin\delta(v)}. \tag{2.27}$$

Differentiating (2.8) with respect to $u$ and $v$ and using (2.27), (2.28), we get

$$r_u^* = \frac{h'(v)}{\sin\delta(v)}, \quad r_v^{**} = \frac{2h'(v)}{\sin 2\delta(v)} + 2k(v)g^{**}. \tag{2.28}$$

By means of (2.28) and (2.29) it follows that

$$r^* = A_1 \int \cos\delta(v)dv + A_1 u + A_2 \quad \left(\frac{h'(v)}{\sin\delta(v)} = A_1 = const., \ A_2 = const.\right),$$

$$r^{**} = A_1 u + A_3 + \int \left[\frac{A_1}{\cos\delta(v)} + 2k(v)g^{**}(v)\right] dv \quad (A_3 = const.). \tag{2.29}$$

If (2.30), (2.27) and (2.7) are taken into consideration, the Gauss equation (2.6) reduces to

$$A_1^2 u^2 + A_1[\alpha(v) + \beta(v)]u + \alpha(v)\beta(v) - h^2(v) = 0$$

where we have put

$$\alpha(v) = A_2 + A_1 \int \cos\delta(v),$$

$$\beta(v) = A_2 + \int \left[\frac{A_1}{\cos\delta(v)} + 2k(v)g^{**}(v)\right] dv + (k(v) - h(v))\text{cotg}\delta(v),$$

from which it follows that

$$A_1 = 0, \quad \alpha(v)\beta(v) - h^2(v) = 0. \tag{2.30}$$

Combining (2.31) with (2.30) we get

$$h(v) = const., \quad \int 2k(v)g^{**}(v)dv + (k(v) - h(v))\text{cotg}\delta(v) = \frac{B^2}{A_2} - A_3 \tag{2.31}$$

where we have assumed that

$$A_2 \neq 0. \tag{2.32}$$

Differentiating (2.32) with respect to $v$, we obtain

$$-k(v)g^{**}\cos 2\delta(v) + \frac{1}{2}k'(v)\sin 2\delta(v) + Bg^{**} = 0, \quad (g^{**} = \delta_v)$$

which, by the condition $g^{**} - \delta_v = 0$, gives

$$k(v) = B\cos 2\delta(v) + D\sin 2\delta(v), \quad (D = const.) \tag{2.33}$$

On the other hand, (2.34) and (2.32) give

$$k(v) = B\cos 2\delta(v) + \left(\frac{B^2}{A_2} - A_3\right)\sin 2\delta(v). \tag{2.34)'$$

With the use of (2.34)', (2.32) and (2.31), the equations (2.30), (2.27) and (2.7) become respectively

$$t^* = B, \quad t^{**} = B\cos 2\delta(v) + \left(\frac{B^2}{A_2} - A_3\right)\sin 2\delta(v)$$

$$r^* = A_2, \quad r^{**} = A_3 + B\sin 2\delta(v) - \left(\frac{B^2}{A_2} - A_3\right)\cos 2\delta(v).$$

Then it can be easily seen that one of the principal curvatures is zero while the other one is a constant which means that $S$ is a cylinder of revolution.

We next consider the case where the condition (2.33) is not satisfied. Namely, the case of $A_2 = 0$. Then, from (2.31) we get $B = 0$ by which the equations (2.30), (2.27) and (2.7) reduce to

$$t^* = 0, \quad t^{**} = k(v), \quad r^* = 0, \quad r^{**} = 2\int k(v)g^{**}dv + A_3. \tag{2.34}$$

If these values of $t^*, t^{**}, r^*, r^{**}$ are substituted in (2.8) we obtain

$$\left(2\int k(v)g^{**}dv + A_3\right)\mathrm{cotg}\delta(v) = k(v). \tag{2.35}$$

Differentiating (2.36) with respect to $v$ we get

$$\frac{k'(v)}{k(v)} = 2\delta'(v)\mathrm{cotg}2\delta(v)$$

where $k(v) \neq 0$. (In case of $k(v) = 0$, $S$ is a plane), from which it follows that

$$k(v) = C_1\sin 2\delta(v) \quad (C_1 = const.)$$

Then, from (2.36) we find that $A_3 = 0$, by which (2.35) becomes

$$t^* = 0, \quad t^{**} = C_1\sin 2\delta(v), \quad r^* = 0, \quad r^{**} = 2C_1\sin^2\delta(v)$$

which means that, $S$ is a cylinder of revolution.

In case V, from (2.1), (2.2), (2.3), (2.5) we obtain

$$g^* = g^{**} = 0, \quad t^* = -t^{**} = A \tag{2.36}$$

by which Mainardi-Codazzi equations reduce to

$$r^* = l(u), \quad r^{**} = m(v), \tag{2.37}$$

where $l(u)$ and $m(v)$ are arbitrary functions of their arguments.

By the use of (2.37) and (2.38), the Gauss equation yields

$$m(v) = \frac{A^2}{l(u)} = B = const. \tag{2.38}$$

where we have assumed that

$$l(u) \neq 0. \tag{2.39}$$

By (2.39), the equations (2.38) become

$$r^* = l(u), \quad r^{**} = B$$

stating that $S$ is a cylinder of revolution or a plane.

Finally, we consider the case where the condition (2.41) is not satisfied. Namely, the case of $l(u) = 0$. In this case, we have $A = 0$. Then the equations (2.38) and (2.37) become

$$t^* = t^{**} = 0, \quad r^* = 0, \quad r^{**} = m(v).$$

With these values of $t^*, t^{**}, r^*$ and $r^{**}$, (1.2) reduces to

$$m'(v) \sin^2 \gamma^* \sin \gamma^{**} = 0$$

from which it follows that

$$m(v) = const.$$

Therefore, $S$ is a cylinder of revolution or a plane.

Summing up what we have found above we obtain the

**Theorem 2.1.** *The only surface (other than a plane) on which the two families of Darboux lines form a Tschebycheff net and the third family of Darboux lines is* *transversal to one of the two families of the Tschebycheff net is a cylinder of revolution.*

## 3. Darboux lines on parallel and inverse surfaces

**3.1. Surfaces on which the Darboux lines correspond to those on their parallel surfaces.** Let $S$ be a real surface in Euclidean 3-space with vector equation $\vec{x} = \vec{x}(u,v)$. A surface $\overline{S}$ parallel to $S$ is defined by $\vec{\overline{x}} = \vec{x} + C\vec{n}$ ($C = const.$) where $\vec{n}$ is the unit normal vector of $S$. If the lines of curvature of $S$ are taken as the parametric lines, the coefficients of the first fundamental form of $\overline{S}$ denoted by $\overline{E}, \overline{F}, \overline{G}$ are given by

$$\overline{E} = \left(\frac{C-\alpha}{\alpha}\right)^2 E, \quad \overline{F} = 0, \quad \overline{G} = \left(\frac{C-\beta}{\beta}\right)^2 G \tag{3.1}$$

where $E$ and $G$ are the coefficients of the first fundamental form of $S$, $\alpha = 1/r$ and $\beta = 1/\overline{r}$ being its the radii of principal curvatures. The principal radii of curvatures, denoted by $R$ and $\overline{R}$, of $\overline{S}$ are given by

$$R = \frac{\varepsilon}{(\alpha - C)}, \quad \overline{R} = \frac{\varepsilon}{(\beta - C)} \quad (\varepsilon = \pm 1) \tag{3.2}$$

If the lines of curvatures of $S$ are taken as the parametric lines, the equation (1.2) becomes

$$-\overline{r}_1 \tan^3\varphi + (\overline{r}_2 - 2r_2)\tan^2\varphi + (2\overline{r}_1 - r_1)\tan\varphi + r_2 = 0. \tag{3.3}$$

Using the fact that $\tan\varphi = (G/E)^{1/2}\dfrac{dv}{du}$, (3.3) takes the form

$$-\overline{r}_u G^2(dv)^3 + (\overline{r}_v - 2r_v)EG(dv)^2du + (2\overline{r}_u - r_u)EGdv(du)^2 + r_vE^2(du)^3 = 0. \tag{3.4}$$

Using this equation and remembering that the lines of curvature on $S$ and $\overline{S}$ correspond, the differential equation of the Darboux lines of $\overline{S}$ is obtained in the form

$$-\overline{R}_u\overline{G}^2(dv)^3 + (\overline{R}_v - 2R_v)\overline{E}\,\overline{G}(dv)^2du + (2\overline{R}_u - R_u)\overline{E}\,\overline{G}dv(du)^2 + R_v\overline{E}^2(du)^3 = 0. \tag{3.5}$$

The Darboux lines of $S$ and $\overline{S}$ will correspond to each other, if and only if the respective coefficients of the equations (3.4) and (3.5) are proportional. Then, with the help of (3.1) and (3.2), from (3.4) and (3.5) it follows that

$$\frac{\beta^2}{(\beta-C)^2}=\frac{-\beta_v\alpha^2+2\alpha_v\beta^2}{-\beta_v(\alpha-C)^2+2\alpha_v(\beta-C)^2}=\frac{-2\beta_u\alpha^2+\alpha_u\beta_2}{-2\beta_u(\alpha-C)^2+\alpha_u(\beta-C)^2}=\frac{\alpha^2}{(\alpha-C)^2} \tag{3.6}$$

where we have assumed that

$$\alpha_v\neq 0,\quad \beta_u\neq 0. \tag{3.6)'}$$

From (3.6) we get

$$\beta_v(\beta-\alpha)(1-CW)=0,\quad (\beta-\alpha)(1-CW)(\alpha_u\beta_v-4\alpha_v\beta_u)=0 \tag{3.7}$$

$$(\beta-\alpha)(1-CW)=0,\quad \alpha_u(\beta-\alpha)(1-CW)=0 \tag{3.8}$$

where $W$ is the mean curvature of $S$ and $K\neq 0$.

In the case of $K=0$, the Darboux lines of $S$ and $\overline{S}$ do correspond.

If $S$ is neither a sphere nor a developable surface, from (3.7) and (3.8), we get $1-CW=0$ which means that $S$ is a surface of constant mean curvature.

Now we consider the cases where the conditions in (3.6)' are not satisfied, namely

$$I.\quad \beta_u=0,\qquad II.\quad \alpha_v=0.$$

In case I, by (3.1) and (3.2), equations (3.4) and (3.5) take the respective forms

$$Edu\left\{\left[-\frac{\beta_v}{\beta^2}+2\frac{\alpha_v}{\alpha^2}\right]G(dv)^2+\frac{\alpha_u}{\alpha^2}Gdvdu-\frac{\alpha_v}{\alpha^2}E(du)^2\right\}=0 \tag{3.9}$$

$$\overline{E}du\left\{\left[-\frac{\beta_v}{(\beta-C)^2}+\frac{2\alpha_v}{(\alpha-C)^2}\right]\overline{G}(dv)^2+\frac{\alpha_u}{(\alpha-C)^2}\overline{G}dvdu-\frac{\alpha_v}{(\alpha-C)^2}\overline{E}(du)^2\right\}=0 \tag{3.10}$$

showing that one of the families of Darboux lines on $S$ and $\overline{S}$ coincides with the family of lines of curvature $u = const.$ The other two families of Darboux lines on $S$ and $\overline{S}$ are given by

$$\left[-\frac{\beta_v}{\beta^2} + 2\frac{\alpha_v}{\alpha^2}\right] G(dv)^2 + \frac{\alpha_u}{\alpha^2} G dv du - \frac{\alpha_v}{\alpha^2} E(du)^2 = 0,$$

$$\left[-\frac{\beta_v}{(\beta - C)^2} + \frac{2\alpha_v}{(\alpha - C)^2}\right] \overline{G}(dv)^2 + \frac{\alpha_u}{(\alpha - C)^2} \overline{G} dv du - \frac{\alpha_v}{(\alpha - C)^2} \overline{E}(du)^2 = 0.$$

These lines will be in correspondence provided that

$$\beta_v(\beta - \alpha)(1 - CW) = 0 \tag{3.11}$$

$$(\beta - \alpha)(1 - CW) = 0 \tag{3.12}$$

$$\alpha_u \neq 0, \quad \alpha_v \neq 0 \tag{3.13}$$

By (3.14) we get $1 - CW = 0$ from which it follows that $\alpha_u = 0$ contradicting with (3.15).

Next we consider the cases where the conditions in (3.15) are not satisfied. Namely,

$$I'. \quad \alpha_i = 0, \qquad II''. \quad \alpha_v = 0.$$

In case $I'$, using (3.1) and (3.2), from (3.4) and (3.5) we find that

$$\left[-\frac{\beta_v}{\beta^2} + 2\frac{\alpha_v}{\alpha^2}\right] G(dv)^2 - \frac{\alpha_v}{\alpha^2} E(du)^2 = 0, \tag{3.14}$$

$$\left[-\frac{\beta_v}{(\beta - C)^2} + \frac{2\alpha_v}{(\alpha - C)^2}\right] \overline{G}(dv)^2 - \frac{\alpha_v}{(\alpha - C)^2} \overline{E}(du)^2 = 0. \tag{3.15}$$

Since the coefficients of (3.16) and (3.17) must be proportional, we obtain

$$(\beta - \alpha)(1 - CW) = 0. \tag{3.16}$$

(3.18) gives $1 - CW = 0$ for $\alpha \neq \beta$ showing that $S$ is a surface of revolution of constant mean curvature other then a sphere.

In case II", we have $\beta_u = 0, \ \alpha_v = 0$ implying that $S$ is a cylinder of revolution or a plane.

In case II, we apply the same reasoning as above and obtain the same results included in case I.

We therefore obtain the

**Theorem 3.1.** *Let $\overline{S}$ be a surface parallel to $S$ and suppose that $S$ is neither a sphere nor a developable surface. If the Darboux lines of $S$ and $\overline{S}$ correspond to each other, then $S$ is a surface of constant mean curvature.*

3.1.1. *Surfaces of constant mean curvature on which the two families of Darboux lines form a Tschebycheff net.* It is well known that [4] the Darboux lines of a surface of constant mean curvature other than a sphere, a cylinder of revolution or a plane, cut each other under an angle of 120°.

Suppose that the lines of curvature on $S$ are taken as the parametric lines. Let the lines of the two families $\mathcal{D}_1$ and $\mathcal{D}_2$ of Darboux lines make, respectively, the angles $\varphi$ and $\phi$ with the parametric line $v$ =const. Then $\varphi - \phi = 120°$.

The conditions for the 2-net $\Delta \equiv (\mathcal{D}_1, \mathcal{D}_2)$ to be a Tschebycheff net are, according to (1.3),

$$\frac{d(\varphi - \phi)}{ds_2} = B - (\rho_g)_2, \quad \frac{d(\phi - \varphi)}{ds_1} = -(\rho_g)_1 \tag{3.17}$$

where $s_1, s_2; (\rho_g)_1, (\rho_g)_2$ are the arc-lengths and the geodesic curvatures of $\mathcal{D}_1$ and $\mathcal{D}_2$ given by

$$(\rho_g)_1 = \varphi_1 \cos\varphi + \varphi_2 \sin\varphi + g\cos\varphi + \overline{g}\sin\varphi = 0, \tag{3.18}$$

$$(\rho_g)_2 = \phi_1 \cos\phi + \phi_2 \sin\phi + g\cos\phi + \overline{g}\sin\phi = 0. \tag{3.19}$$

Using the fact that $\varphi - \phi = 120°$, by (3.20) and (3.21), (3.19) becomes

$$[\varphi_1 + g]\cos\varphi + [\varphi_2 + \overline{g}]\sin\varphi = 0,$$

$$[\varphi_1 + g]\cos(\varphi + 120°) + [\varphi_2 + \overline{g}]\sin(\varphi + 120°) = 0$$

from which it follows that

$$\varphi_1 + g = 0, \quad \varphi_2 + \overline{g} = 0. \tag{3.20}$$

With the help of (3.22), the integrability condition $\varphi_{12} - \varphi_{21} = g\varphi_1 + \overline{g}\varphi_2$ for $\varphi$ gives

$$g_2 - \overline{g}_1 - g^2 - \overline{g}^2 = K = 0$$

which means that $S$ is a developable surface. On the other hand, since $2W = \dfrac{1}{\alpha} + \dfrac{1}{\beta} = r + \overline{r} =$const., $S$ is a cylinder of revolution or a plane.

Thus, we obtain the

**Theorem 3.2.** *A surface of constant mean curvature on which the two families of Darboux lines form a Tschebycheff net is a cylinder of revolution or a plane.*

3.2. **Surfaces on which the Darboux lines are preserved by inversion.** Let the surface $S$ be given by the vector equation $\vec{x} = \vec{x}(u, v)$ and let $S^*$ be its inverse. Then $S^*$ is defined by the equation $\vec{x}^* = \frac{c^2}{x^2}\vec{x}$, $c$ being the radius of inversion. Denoting by $E, F, G$ and $E^*, F^*, G^*$ the coefficients of the first fundamental forms of $S$ and $S^*$ respectively, we have

$$E^* = \frac{c^4}{x^4}E, \quad F^* = \frac{c^4}{x^4}F, \quad G^* = \frac{c^4}{x^4}G. \tag{3.21}$$

If the lines of curvatures on $S$ are taken as parametric lines, the principal curvatures, denoted by $r^*$ and $\overline{r}^*$ of $S^*$ are given by

$$r^* = -\frac{1}{c^2}(rx^2 + 2p), \quad \overline{r}^* = -\frac{1}{c^2}(\overline{r}x^2 + 2p) \tag{3.22}$$

where $p$ is the perpendicular distance, measured in the sense of the unit normal vector of $S$, from the centre of inversion to the tangent plane of $S$ at the point considered. The quantities $r, \overline{r}, p, x$ are related by

$$2p_u = -(x^2)_u r, \quad 2p_v = -(x^2)_v \overline{r}. \tag{3.24$'$}$$

Using the fact that $\tan\varphi = (G/E)^{1/2}\dfrac{dv}{du}$, (1.2) takes the form

$$-\overline{r}_u G^2(dv)^3 + (\overline{r}_v - 2r_v)EG(dv)^2du + (2\overline{r}_u - r_u)EGdv(du)^2 + r_v E^2(du)^3 = 0. \tag{3.23}$$

Since the lines of curvature on $S$ and $S^*$ correspond, the differential equation of the Darboux lines of $S^*$ is obtained in the form

$$-\overline{r}_u^* G^{*2}(dv)^3 + (\overline{r}_v^* - 2r_v^*)E^* G^*(dv)^2 du + \tag{3.24}$$

$$+(2\overline{r}_u^* - r_u^*)E^* G^* dv(du)^2 + r_v^* E^{*2}(du)^3 = 0.$$

The Darboux lines of $S$ and $S^*$ will correspond, if and only if the respective coefficients of the equations (3.25) and (3.26) are proportional. Consequently, by using (3.23), (3.24), (3.24)', from (3.25) and (3.26) we get

$$2\overline{r}_u(x^2)_v - (\overline{r}_v - 2r_v)(x^2)_u = 0, \tag{3.25}$$

$$r_u(x^2)_u = 0, \quad \overline{r}_v(x^2)_v = 0, \tag{3.26}$$

$$\overline{r}_u(x^2)_v + r_v(x^2)_u = 0, \tag{3.27}$$

$$(\overline{r}_v - 2r_v)(x^2)_u - (x^2)_v(2\overline{r}_u - r_u) = 0, \tag{3.28}$$

$$2r_v(x^2)_u + (2\overline{r}_u - r_u)(x^2)_v = 0 \tag{3.29}$$

from which it follows that

$$r_u = \overline{r}_v = 0, \tag{3.30}$$

$$\overline{r}_u(x^2)_v + r_v(x^2)_u = 0 \tag{3.31}$$

where we have assumed that

$$(x^2)_u \neq 0, \quad (x^2)_v \neq 0. \tag{3.32}$$

It is well known that the conditions (3.32) characterize the Dupin's Cyclides [5].

The general solution of (3.33) is

$$x^2 = \phi(r - \overline{r})$$

which means that the curves $r - \overline{r}$ =const. are spherical.

If, in (3.34), $(x^2)_u = 0$, $(x^2)_v = 0$ it is clear that $S$ is a sphere.

Now suppose that one of the quantities $(x^2)_u$ and $(x^2)_v$, say $(x^2)_u$ is zero. Then, we find that

$$\overline{r} = const., \quad r_u = 0, \quad \overline{g} = 0$$

showing that $S$ is a pipe surface of revolution.

We therefore obtain the

**Theorem 3.3.** *If the Darboux lines of the surfaces $S$ (other then a sphere) and $S^*$ correspond to each other, then $S$ belongs to one of the following two classes:*

*1. $S$ is a Dupin's cyclide and the curves $r - \overline{r}$ =const., are spherical,*

*2. $S$ is a pipe surface of revolution.*

3.3. **Molure surfaces admitting two semi-Tschebycheff nets formed by Darboux lines.** In this section, we consider Molure surfaces which include pipe surfaces as a special case and are characterized by the condition $\overline{r}_1 = 0$ ($\overline{r} = f(v)$, $f(v)$ being a differentiable function of its argument).

It is easy to see from (3.4) that one of the families of Darboux lines on a Molure surface coincides with the family of lines of curvature $u$ =const. Then the other two families of Darboux lines are given, according to (3.3), by

$$\tan^2\varphi - \frac{r_1}{\overline{r}_2 - 2r_2}\tan\varphi + \frac{r_2}{\overline{r}_2 - 2r_2} = 0 \tag{3.33}$$

where we have assumed that

$$\overline{r}_2 - 2r_2 \neq 0. \tag{3.35)'}$$

Let the two families of Darboux lines make, respectively the angles $\phi$ and $\psi$ with the parametric line $v$ =const.

The conditions for the family $\mathcal{D}_1$ : $u$ =const. to be transversal of the tangent vector fields of the other two families are, according to (1.3),

$$\left(\frac{\pi}{2} - \phi_1\right)_2 = \overline{g}, \quad \left(\frac{\pi}{2} - \psi_2\right)_2 = \overline{g}. \tag{3.34}$$

Since the geodesic curvature $\overline{g}$ of the lines of curvature $u$ =const. is zero, by a suitable choice of the parameter $v$, we can make $G = 1$. Then, (3.36) becomes

$$\phi_2 = \psi_2 = 0. \tag{3.36)'}$$

Taking the invariant derivative of (3.35) in the direction of $u$ =const., and using (3.36)' and replacing $\varphi$ by $\phi$ and $\psi$ we respectively get

$$-\left(\frac{r_1}{f'(v)-2r_2}\right)_2 \tan\phi + \left(\frac{r_2}{f'(v)-2r_2}\right)_2 = 0, \quad -\left(\frac{r_1}{f'(v)-2r_2}\right)_2 \tan\psi + \left(\frac{r_2}{f'(v)-2r_2}\right)_2 :$$

from which it follows that

$$\frac{r_1}{f'(v)-2r_2} = a(u), \quad \frac{r_2}{f'(v)-2r_2} = b(u) \tag{3.35}$$

where $a(u), b(u)$ are arbitrary differentiable functions of their arguments.

From (3.37), it follows that

$$r_1 = \frac{f'(v)a(u)}{1+2b(u)}, \quad r_2 = \frac{f'(v)b(u)}{1+2b(u)} \tag{3.36}$$

where we have assumed that

$$(3.38)' \qquad 1+2b(u) \neq 0.$$

Integration of the second equation in (3.38) we obtain

$$r = \frac{f(v)b(u)}{1+2b(u)} + C(u) \tag{3.37}$$

$C(u)$ being an arbitrary differentiable function of its argument.

Substituting $r$ in (3.38) and assuming that

$$f'(v) \neq 0, \quad a(u) \neq 0 \tag{3.38}$$

we get

$$(3.40)' \qquad \sqrt{E} = \left(\frac{f(v)}{f'(v)}\right) U(u) + \left(\frac{1}{f'(v)}\right) \overline{U}(u)$$

where

$$(3.40)'' \qquad U(u) = \frac{b'(u)}{a(u)[1+2b(u)]}, \quad \overline{U}(u) = \frac{C'(u)[1+2b(u)]}{a(u)}.$$

With the help of (3.39) and (3.38) the Mainardi-Codazzi equation $r_2 = (r - \overline{r})g$ gives

$$[f(v)R(u) - C(u)]\left[\left(\frac{f(v)}{f'(v)}\right) U(u) + \left(\frac{1}{f'(v)}\right)' \overline{U}(u)\right] = \tag{3.39}$$

$$= [1 - R(u)][f(v)U(u) - \overline{U}(u)]$$

where

$$(3.41)' \qquad R(u) = \frac{1+b(u)}{1+2b(u)}.$$

Differentiating (3.41) with respect to $v$, we obtain

$$[X(v)R(u) - Y(v)C(u)]U(u) + [Z(v)R(u) - T(v)C(u)]\overline{U}(u) = [1 - R(u)]U(u) \tag{3.40}$$

where we have put

$$(3.42)' \qquad X(v) = \frac{\left[f(v)\left(\frac{f(v)}{f'(v)}\right)'\right]'}{f'(v)}, \quad Y(v) = \frac{\left(\frac{f(v)}{f'(v)}\right)''}{f'(v)},$$

$$Z(v) = \frac{\left[f(v)\left(\frac{1}{f'(v)}\right)'\right]'}{f'(v)}, \quad T(v) = \frac{\left(\frac{1}{f'(v)}\right)''}{f'(v)}.$$

Differentiation of (3.42) with respect to $v$ and division through by

$$T'(v) \neq 0 \tag{3.41}$$

gives the equation

$$(3.43)' \qquad [\overline{X}(v)R(u) - \overline{Y}(v)C(u)]U(u) + [\overline{z}(v)R(u) - C(u)]\overline{U}(u) = 0$$

where

$$(3.43)'' \qquad \overline{X}(v) = \frac{X'(v)}{T'(v)}, \quad \overline{Y}(v) = \frac{Y'(v)}{T'(v)}, \quad \overline{Z}(v) = \frac{Z'(v)}{T'(v)} \quad (T'(v) \neq 0).$$

Differentiating (3.43)' with respect to $v$ and dividing the resulting equation by

$$\overline{Z}' \neq 0 \tag{3.42}$$

we get

$$(3.44)' \qquad \frac{\overline{X}'(v)}{\overline{Z}'(v)}U(u)R(u) - \frac{\overline{Y}'(v)}{\overline{Z}'(v)}U(u)C(u) + \overline{U}(u)R(u) = 0.$$

Finally, differentiation of (3.44)' with respect to $v$ we have

$$U(u)\left[\left(\frac{\overline{X}'(v)}{\overline{Z}'(v)}\right)' R(u) - \left(\frac{\overline{Y}'(v)}{\overline{Z}'(v)}\right)' C(u)\right] = 0. \tag{3.43}$$

If, in (3.45) $U(u) = 0$, by (3.44)', (3.43)' and (3.40), it follows that $E = 0$ which is not possible.

For $U(u) \neq 0$, (3.45) becomes

$$(3.45)' \qquad \frac{\left(\frac{\overline{X}'(v)}{\overline{Z}'(v)}\right)'}{\left(\frac{\overline{Y}'(v)}{\overline{Z}'(v)}\right)'} = \frac{C(u)}{R(u)} = d = const.$$

where we have assumed that

$$(3.45)'' \qquad \left(\frac{\overline{Y}'(v)}{\overline{Z}'(v)}\right)' \neq 0, \quad R(u) \neq 0.$$

By (3.43)", (3.45)' gives

$$X(v) = dY(v) + eZ(v) + kT(v) + l, \quad C(u) = dR(u)$$

where $e, k, l$ are arbitrary constants. Substitution of $X(v)$ and $C(u)$ in (3.44)' and (3.43)', by (3.42), we obtain

$$eU + \overline{U} = 0, \quad k + ed = 0, \quad (1+l)\frac{1+b(u)}{1+2b(u)} = 1$$

from which it follows that $b(u)$ =const. Then, from (3.40)" we get $U(u) = 0$ which can not be the case.

Now, we consider the case where the conditions (3.45) are not satisfied. Here, we distinguish two cases:

$$1a. \quad \left(\frac{\overline{Y}'(v)}{\overline{Z}'(v)}\right)' = 0, \quad R(u) \neq 0; \qquad 1b. \quad R(u) = 0, \quad \left(\frac{\overline{Y}'(v)}{\overline{Z}'(v)}\right)' \neq 0$$

In case 1a, by (3.45) and (3.43)' we obtain

$$Y(v) = C_1 Z(v) + C_2 T(v) + C_3, \tag{3.44}$$

$$X(v) = C_4 Z(v) + C_5 T(v) + C_6 \tag{3.45}$$

where $C_1, C_2, C_3, C_4, C_5, C_6$ are arbitrary constants. Substitution of $X(v)$, $Y(v)$ in (3.44)', (3.43)' and (3.42) we have

$$[C)4R(u) - C_1C(u)]U(u) + \overline{U}(u)R(u) = 0, \tag{3.46}$$

$$[C_5R(u) - C_2C(u)]U(u) - \overline{U}(u)C(u) = 0, \tag{3.47}$$

$$(C_6 + 1)R(u) - C_3C(u) = 1. \tag{3.48}$$

Using (3.42)', from (3.46) and (3.47) we get

$$f(v)\left(\frac{f(v)}{f'(v)}\right)' = C_4f(v)\left(\frac{1}{f'(v)}\right)' + C_5\left(\frac{1}{f'(v)}\right)' + C_6f(v) + C_7, \tag{3.49}$$

$$\left(\frac{f(v)}{f'(v)}\right)' = C_1f(v)\left(\frac{1}{f'(v)}\right)' + C_2\left(\frac{1}{f'(v)}\right)' + C_3f(v) + C_8 \tag{3.50}$$

$C_7$ and $C_8$ being arbitrary constants. Combining (3.51) and (3.52), for

(3.52)′ $$(C_1 - 1)f(v) + C_2 \neq 0,$$

we obtain

$$C_3f^3(v) - [(1 - C_8) - (C_1 - 1)(C_6 - 1) + C_3C_4]f^2(v) -$$
$$-[C_3C_5 - C_7(C_1 - 1) - C_2(C_6 - 1) - C_4(1 - C_8)]f(v) +$$
$$+[C_5(1 - C_8) - C_2(1 - C_6) + C_2C_7] = 0.$$

Since $f(v) \neq$const., this equation gives

$$C_3 = 0, \quad C_7(C_1 - 1) + C_2(C_6 - 1) + C_4(1 - C_8) = 0,$$
$$(1 - C_8) - (C_1 - 1)(C_6 - 1) = 0,$$
$$C_5(1 - C_8) - C_2(1 - C_6) + C_2C_7 = 0.$$

With the help of these relations, from (3.50), (3.41)' and (3.40)" it follows that $U(u) = 0$ which is not possible.

Suppose now that the condition (3.52)' is not satisfied, i.e. $(C_1 - 1)f(v) + C_2 = 0$. We then have $C_1 = 1$, $C_2 = 0$ so that the equation (3.52) reduces to $(1 - C_8) - C_3f(v) = 0$. But this gives $C_8 = 1$, $C_3 = 0$. Under these conditions $U(u)$ becomes zero which cannot be the case.

In case 1b, by (3.41)' and (3.40) we find that $U(u) = 0$ which is impossible. Therefore the cases 1a and 1b can not hold.

If the condition (3.43) or (3.44) is not satisfied considerations similar to that given above show that these two cases cannot hold.

We next consider the case where (3.40) is not satisfied. Then, either 1°. $f'(v) = 0,\ a(u) = 0$ or 2°. $f'(v) \neq 0,\ a(u) = 0$. In case 1°, from (3.38) it follows that $r$ =const. from which we obtain $\overline{r}_2 - r_2 = 0$ which contradicts (3.35)'. In case 2°, from (3.38) and (3.39) we get

$$f(v) = \frac{C'(u)(1+2b(u))^2}{b'(u)} = \overline{D}_1 = const., \quad b'(u) \neq 0 \tag{3.51}$$

Then, by (3.38), we obtain $\overline{r}_2 = r_2 = 0$ which contradicts (3.35)'.

If, in (3.53) $b'(u) = 0$, then from (3.38) and (3.39) we have $C(u)$ =const., so that

$$r = r(v), \quad \overline{r} = \overline{r}(v)$$

which shows that $S$ is a surface of revolution.

Finally, suppose that the condition $1 + 2b(u) \neq 0$ in (3.38)' is not satisfied. In this case, from (3.37), it follows that $f'(v) = 0$ which means that $S$ is a pipe surface [5].

We therefore obtain the

**Theorem 3.4.** *If the two families of Darboux lines different from the lines of curvature on a Molure surface form two semi-Tschebycheff nets together a family of lines of curvature, then such a surface is either a surface of revoluion or a pipe surface.*

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