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SCHUNCK CLASSES OF π-SOLVABLE GROUPS

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Abstract. The paper deals with some properties of $\underline{X}$-maximal subgroups, $\underline{X}$-pro-
jectors and $\underline{X}$-covering subgroups in finite π-solvable groups related to a π-closed Schunck class $\underline{X}$, where π is an arbitrary set of primes. The main results are: 1) an existence and conjugacy theorem for $\underline{X}$-maximal subgroups; 2) the proof of a property of covering subgroups in the more general case of projectors and some important corollaries if π is the set of all primes.

1. Preliminaries

The aim of this paper is to study in the case of finite π-solvable groups some special subgroups introduced by W. Gaschütz in [6] and [7].
All groups considered in the paper are finite. We denote by π an arbitrary set of primes and by π' the complement to π in the set of all primes.
The notions in the paper are resumed in the following definitions.

Definition 1.1. a) ([7]) We call $\underline{X}$ a *class* of groups if the members of $\underline{X}$ are finite groups and $\underline{X}$ has the properties:
(1) $1 \in \underline{X}$;
(2) if $G \in \underline{X}$ and f is an isomorphism of G then $f(G) \in \underline{X}$.
b) ([8]) A class $\underline{X}$ of groups is a *homomorph* if $\underline{X}$ is closed under homomorphisms, i.e. if $G \in \underline{X}$ and N is a normal subgroup of G imply $G/N \in \underline{X}$.
c) A group G is *primitive* if there is a maximal subgroup W of G with $\mathrm{core}_G W = 1$, where

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$\mathrm{core}_G W = \cap \{ W^g \ / \ g \in G \}$.

d) ([8]) A homomorph $\underline{X}$ is a *Schunck class* if $\underline{X}$ is *primitively closed*, i.e. if any group G , all of whose primitive factor groups are in $\underline{X}$, is itself in $\underline{X}$.

Definition 1.2. Let $\underline{X}$ be a class of groups, G a group and H a subgroup of G.

a) ([7]) H is *$\underline{X}$-maximal* in G if:

(1) $H \in \underline{X}$;

(2) $H \leq K \leq G$, $K \in \underline{X} \Rightarrow H = K$.

b) ([7]) H is an *$\underline{X}$-projector* of G if for any normal subgroup N of G, HN/N is $\underline{X}$-maximal in G/N. c) ([6]) H is an *$\underline{X}$-covering subgroup* of G if:

(1) $H \in \underline{X}$;

(2) $H \leq K \leq G$, $K_0 < K$, $K/K_0 \in \underline{X} \Rightarrow K = HK_0$.

Definition 1.3. a) ([5]) A group is *π-solvable* if every chief factor is either a solvable π-group or a π'-group. If π is the set of all primes, we obtain the notion of *solvable group*.

b) A class $\underline{X}$ of groups is *π-closed* if:

$G/O_{\pi'}(G) \in \underline{X} \Rightarrow G \in \underline{X}$,

where $O_{\pi'}(G)$ denotes the largest normal π'-subgroup of G. We shall call ***π-homomorph*** (*π-Schunck class*) a π-closed homomorph (Schunck class).

We shall use in the paper the following result given by R. Baer in [1]:

Theorem 1.4. *A solvable minimal normal subgroup of a group is abelian.*

2. Basic properties of special subgroups

We remind here some basic properties of special subgroups defined in 1.2.

Theorem 2.1. *([6]; [8]) Let $\underline{X}$ be a homomorph, G a group and H a subgroup of G.*

a) If H is an $\underline{X}$-covering subgroup of G, then:

(1) for any $x \in G$, H^x is an $\underline{X}$-covering subgroup of G;

(2) for any normal subgroup N of G, HN/N is an $\underline{X}$-covering subgroup of G/N;

(3) for any subgroup K with $H \leq K \leq G$, it follows that H is an $\underline{X}$-covering subgroup of

K.

b) If N is a normal subgroup of G and $H \leq H^* \leq G$ *such that* $N \subseteq H^*$*, H is an* $\underline{X}$*-covering subgroup of* H^* *and* H^*/N *is an* $\underline{X}$*-covering subgroup of G/N, then H is an*
$\underline{X}$*-covering subgroup of G.*

Theorem 2.2. *([7]) Let* $\underline{X}$ *be a class of groups, G a group and H a subgroup of G.*
a) If H is an $\underline{X}$*-projector of G and* $x \in G$*, then* H^x *is an* $\underline{X}$*-projector of G.*
b) H is an $\underline{X}$*-projector of G if and only if:*
(1) H is $\underline{X}$*-maximal in G;*
(2) HM/M is an $\underline{X}$*-projector of G/M for all minimal normal subgroups M of G.*
c) If H is an $\underline{X}$*-projector of G and N is a normal subgroup of G, then HN/N is an* $\underline{X}$*-projector of G/N.*

Theorem 2.3. *Let* $\underline{X}$ *be a class of groups, G a group and H an* $\underline{X}$*-maximal subgroup of G. Then:*
a) for any $x \in G$, H^x *is an* $\underline{X}$*-maximal subgroup of G;*
b) for any subgroup K with $H \leq K \leq G$*, it follows that H is* $\underline{X}$*-maximal in K.*

Concerning to the connection between $\underline{X}$-maximal subgroups, $\underline{X}$-projectors and
$\underline{X}$-covering subgroups in finite groups we give:

Theorem 2.4. *([4]) Let* $\underline{X}$ *be a class of groups, G a group and H a subgroup of G.*
a) If H is an $\underline{X}$*-covering subgroup or an* $\underline{X}$*-projector of G, then H is* $\underline{X}$*-maximal in G.*
b) If further $\underline{X}$ *is a homomorph,then: H is an* $\underline{X}$*-covering subgroup of G if and only if H is an* $\underline{X}$*-projector in any subgroup K with* $H \leq K \leq G$*. Particularly, any* $\underline{X}$*-covering subgroup of G is an* $\underline{X}$*-projector of G.*

Remaark. The converse of the last assertion does not hold, as the following example shows: Let $\underline{A}$ be the homomorph of all finite abelian groups. Any subgroup of order

4 which is not normal in the symmetric group S_4 is an $\underline{A}$-projector, but is not an $\underline{A}$-covering subgroup in S_4.

3. Existence and conjugacy theorems

The fundamental problem on the special subgroups defined in 1.2. is to prove the existence and conjugacy theorems. We give below such theorems for finite π-solvable groups.

All groups in this section are finite π-solvable.

Theorem 3.1. *([2]) Let $\underline{X}$ be a π-homomorph.*

a) $\underline{X}$ is a Schunck class if and only if any π-solvable group has $\underline{X}$-covering subgroups.

b) Any two $\underline{X}$-covering subgroups of a π-solvable group G are conjugate in G.

Theorem 3.2. *([3]; [4]) Let $\underline{X}$ be a π-homomorph. Then: $\underline{X}$ is a Schunck class if and only if any π-solvable group has $\underline{X}$-projectors.*

Corollary 3.3. *Let $\underline{X}$ be a π-homomorph. The following conditions are equivalent:*

(1) $\underline{X}$ is a Schunck class;

(2) any π-solvable group has $\underline{X}$-covering subgroups;

(3) any π-solvable group has $\underline{X}$-projectors.

Theorem 3.4. *([3]) If $\underline{X}$ is a π-Schunck class, then any two $\underline{X}$-projectors of a π-solvable group G are conjugate in G.*

In the proof of 3.4. given in [3], we use a lemma, important in itself, because it can be considered as an existence and conjugacy theorem for $\underline{X}$-maximal subgroups in finite π-solvable groups.

Theorem 3.5. *([3]) Let $\underline{X}$ be a π-Schunck class, G a π-solvable group and A an abelian normal subgroup of G with $G/A \in \underline{X}$. Then:*

a) there is a subgroup S of G with $S \in \underline{X}$ and $AS = G$ (which imply that there is an $\underline{X}$-maximal subgroup S of G such that $AS = G$);

b) if S_1 and S_2 are $\underline{X}$-maximal subgroups of G with $AS_1 = G = AS_2$, then S_1 and S_2 are conjugate in G.

4. New results on projectors

In our intention to study some properties of special subgroups in finite π-solvable groups we raised the following question: Does an analogous property of 2.1.b) hold for projectors? The answer is affirmative in finite π-solvable groups, as the result below shows.

Theorem 4.1. *Let $\underline{X}$ be a π-Schunck class, G a π-solvable group such that for any minimal normal subgroup M of G which is a π'-group we have $G/M \in \underline{X}$ and let B be a normal abelian subgroup of G such that:*

(1) S is $\underline{X}$-maximal in BS;

(2) BS/B is an $\underline{X}$-projector of G/B.

Then S is an $\underline{X}$-projector of G.

Proof. We consider two cases:

1) $B = 1$. Then $BS/B \cong S$ and $G/B \cong G$. By (2), S is an $\underline{X}$-projector of G.

2) $B \neq 1$. To prove that S is an $\underline{X}$-projector of G we use 2.2.b).

(1) S is $\underline{X}$-maximal in G. Indeed, if we put $S^* = BS$, our assumptions (1) and (2) imply that S is $\underline{X}$-maximal in S^* and S^*/B is an $\underline{X}$-projector of G/B. Then $S \in \underline{X}$. Let $S \leq T \leq G$ and $T \in \underline{X}$. We show that $S = T$. From $BT/B \cong T/B \cap T$ and $\underline{X}$ being a homomorph we obtain $BT/B \in \underline{X}$. By 2.4.a), S^*/B is $\underline{X}$-maximal in G/B. This and $BS/B \leq BT/B$, where $BT/B \in \underline{X}$, imply $BS/B = BT/B$, hence $S^* = BS = BT$ and $T \leq S^*$. But $S \leq T \leq S^*$, $T \in \underline{X}$ and S $\underline{X}$-maximal in S^* imply $S = T$.

(2) For any minimal normal subgroup M of G, MS/M is an $\underline{X}$-projector of G/M. Indeed, M being a minimal normal subgroup of the π-solvable group G, two cases are possible:

a) *M is a solvable π-group.* Then, by 1.4., M is abelian. $\underline{X}$ being a π-Schunck class, 3.2. shows that the π-solvable group G/M has an $\underline{X}$-projector T^*/M. We shall prove that MS/M and T^*/M are conjugate in G/M, hence, by 2.2.a), MS/M is an $\underline{X}$-projector of G/M.

We are in the hypotheses of 3.5. because T^* is a π-solvable group and M is an abelian normal subgroup of T^* with $T^*/M \in \underline{X}$. By 3.5.a), there is an $\underline{X}$-maximal subgroup T of T^* such that $MT = T^*$. We shall prove that T is $\underline{X}$-maximal in G. Indeed, $T \in \underline{X}$. Further, let $T \leq T' \leq G$ with $T' \in \underline{X}$. We show that $T = T'$. Since $T^* = MT \leq MT'$ it follows that

$T^*/M \leq MT'/M \cong T'/M \cap T' \in \underline{X}$.

Using that T^*/M is an $\underline{X}$-projector of G/M, that means that T^*/M is $\underline{X}$-maximal in G/M, we obtain $T^*/M = MT'/M$, hence $MT = T^* = MT'$. So $T \leq T' \leq T^*$. But T is an

$\underline{X}$-maximal subgroup of T^* and $T' \in \underline{X}$. Then $T = T'$. So T is $\underline{X}$-maximal in G.

Let $A = BM$. Clearly A is a normal abelian subgroup of G. Further AS/A and AT/A are $\underline{X}$-projectors of the π-solvable group G/A. By 3.4., AS/A and AT/A are conjugate in G/A. It follows that $AS^g = AT$ for some $g \in G$. But S and T are $\underline{X}$-maximal in G. By 2.3.b), S^g and T are $\underline{X}$-maximal in $AT = AS^g$. Applying now 3.5.b) to the π-solvable group AT and its abelian normal subgroup A with $AT/A \in \underline{X}$, it follows that S^g and T are conjugate in AT. Hence $M\ S^g/M$ and $MT/M = T^*/M$ are conjugate in G/M. Then MS/M and T^*/M are conjugate in G/M and so MS/M is an $\underline{X}$-projector of G/M.

B) *M is a π'-group.* Then $M \leq O\pi'(G)$ and

$G/\ O\pi'(G) \cong (G/M)/(\ O\pi'(G)/M)$.

But M being a minimal normal subgroup of G which is a π'-group, we have $G/M \in \underline{X}$. So, $\underline{X}$ being a homomorph, we also have $G/\ O\pi'(G) \in \underline{X}$. It follows, by the π-closure of $\underline{X}$, that $G \in \underline{X}$. But S is $\underline{X}$-maximal in G. Then $S = G$ is its own $\underline{X}$-projector, which means also that $MS/M = G/M$ is its own $\underline{X}$-projector. □

From now on let π be the set of all primes, i.e. all groups we consider are finite solvable groups. Theorem 4.1. has in this particular case the following immediate corollaries (given also in [7]).

Corollary 4.2. *Let $\underline{X}$ be a Schunck class, G a solvable group, S a subgroup of G and* $G = G_0 \geq G_1 \geq \ldots \geq G_r = 1$

such that for any i, $G_i < G$ and G_i/G_{i+1} is abelian. Then S is an $\underline{X}$-projector of G if and only if for any i, G_iS/G_i is $\underline{X}$-maximal in G/G_i.

Proof. By induction on $|G|$. If S is an $\underline{X}$-projector of G, then, by 1.2.b), for any i, G_iS/G_i is $\underline{X}$-maximal in G/G_i. Conversely, let, for any i, G_iS/G_i be $\underline{X}$-maximal in G/G_i.

By the induction, $G_{r-1}S/G_{r-1}$ is an $\underline{X}$-projector of G/G_{r-1}. Then putting in 4.1. B $= G_{r-1}$, we obtain that S is an $\underline{X}$-projector of G. □

Corollary 4.3. *Let $\underline{X}$ be a Schunck class, G a solvable group, H a subgroup of G and S an $\underline{X}$-projector of G such that $S \subseteq H$. Then S is an $\underline{X}$-projector of H.*

Proof. G being solvable, there is a chain

$G = G_0 \geq G_1 \geq \ldots \geq G_r = 1$

such that for any i, $G_i < G$ and G_i/G_{i+1} is abelian. We denote for any i, $H_i = H \cap G_i$. Then

$H = H_0 \geq H_1 \geq \ldots \geq H_r = 1$

is a chain with $H_i < H$ and H_i/H_{i+1} abelian for any i. Applying 4.2. for the $\underline{X}$-projector S of G, we obtain that for any i, G_iS/G_i is $\underline{X}$-maximal in G/G_i. But, for any i, we also have:

$H_iS/H_i \cong S/S\cap H_i = S/S\cap(H\cap G_i) = S/(S\cap H) \cap G_i = S/S\cap G_i \cong G_iS/G_i$

and

$H/H_i = H/H\cap G_i \cong HG_i/G_i \leq G/G_i$.

It follows that for any i, H_iS/H_i is $\underline{X}$-maximal in H/H_i, hence, by 4.2., S is an $\underline{X}$-projector of H. □

From 2.4.b) and 4.3. follows:

Corollary 4.4. *Let $\underline{X}$ be a Schunck class, G a solvable group and S a subgroup of G. Then S is an $\underline{X}$-covering subgroup of G if and only if S is an $\underline{X}$-projector of G.*

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