STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLIII, Number 2, June 1999

A NOTE ON THE TRIVIALITY OF THE BOHR-COMPACTIFICATION OF LIE GROUPS

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Abstract. We determine a class of connected Lie groups for which the triviality of the Bohr-compactification is equivalent to the triviality of the Bohr-compactification of the simply connected covering group. We derive from these results some information on the structure of the Bohr-compactification of some class of connected topological groups.

1. Introduction

A topological group G has a trivial Bohr-compactification if $(f, \{1\})$ is the Bohr-compactification of G, where $f: G \to \{1\}$ is the trivial homomorphism. One shows quickly that if a topological group G has a simply connected covering group $\widetilde{G}$ and if $\widetilde{G}$ has a trivial Bohr-compactification, then G itself must possess a trivial Bohr-compactification. The converse of this statement is not always true, i.e., that *the triviality of the Bohr-compactification of a topological group G does not imply the triviality of the Bohr-compactification of its simply connected covering group $\widetilde{G}$* (if this covering group exists). In the present paper we look for conditions when this converse is true. The main results are contained in Section 2: We find a class of connected Lie groups for which the triviality of the Bohr-compactification is equivalent to the triviality of the Bohr-compactification of the simply connected covering group (see Theorem 2.10). As we shall see in Section 3, Theorem 2.10 implies statements about the structure of the Bohr-compactification of connected simple Lie groups and of connected semisimple Lie groups. For the sake of completeness we include as a final result of this section the structure theorem for the Bohr-compactification of solvable

1991 *Mathematics Subject Classification.* 43A60,22E15.
Key words and phrases. topological groups, Lie gropus, Bohr compactification.

connected topological groups. We mention that Neeb investigates in Proposition X.1 of [5] the structure of the Bohr-compactification of Lie groups, too. But his methods differ essentially from ours. The last section of the paper contains an example for a connected topological group satisfying the property that it has a trivial Bohr-compactification while its simply connected covering group has a non-trivial Bohr-compactification.

We denote by (i_G, G^b) the Bohr-compactification of the topological group G. For the sake of simplicity we shall say that G^b is the Bohr-compactification of G. With this notation, the triviality of the Bohr-compactification of G is equivalent to the fact that $G^b = \{1\}$.

We recall the well-known fact that the Bohr-compactification of a topological group G is also the universal topological group compactification of G.

2. Passing to the universal covering group

It is easy to see that if the simply connected covering group $\widetilde{G}$ of a topological group G has a trivial Bohr-compactification then G has also a trivial Bohr-compactification. This fact follows from the following lemma.

Lemma 2.1. *Let $f\colon H \to K$ be a dense and continuous homomorphism between topological groups. If $H^b = \{1\}$, then $K^b = \{1\}$.*

Proof. The universality of (i_H, H^b) implies the existence of a continuous homomorphism $\phi\colon H^b \to K^b$ such that the diagram

$$\begin{tikzcd} H \arrow[r, "i_H"] \arrow[d, "i_K \circ f"'] & H^b \arrow[d, "\phi"] \\ K^b \arrow[r, equal] & K^b \end{tikzcd}$$

commutes. Since $H^b = \{1\}$, we deduce that

$$i_K(f(h)) = 1, \text{ for all } h \in H.$$

The density of f and the continuity of i_K now imply that $K^b = \{1\}$. $\square$

Corollary 2.2. *Let G be a connected topological group and $\widetilde{G}$ its simply connected covering group. If $(\widetilde{G})^b = \{1\}$, then $G^b = \{1\}$.*

Remark. The converse of Corollary 2.2 is not always true. We postpone the presentation of an example (see Proposition 4.1).

In the remainder of this section we look for conditions on a Lie group G which ensure that the converse of Corollary 2.2 is also true. The following well-known lemma on topological groups (whose proof we omit) will be very useful for our purposes.

Lemma 2.3. *Let G, H, and K be topological groups, $q\colon G \to H$ a quotient homomorphism of topological groups, and $f\colon G \to K$ a continuous homomorphism. If $\ker q \subseteq \ker f$, then there exists a unique continuous homomorphism $\bar{f}\colon H \to K$ such that the diagram*

$$\begin{tikzcd} G \arrow[r, "q"] \arrow[d, "f"'] & H \arrow[d, "\bar{f}"] \\ K \arrow[r, equal] & K \end{tikzcd}$$

is commutative.

We derive from Lemma 2.3 the following isomorphism results for topological groups:

Corollary 2.4. *Let G be a topological group, N a normal subgroup of G, and H an arbitrary subgroup of G. The map $\phi\colon H/(H\cap N) \to HN/N$ defined by $\phi(h(H\cap N)) = hN$, for all $h(H \cap N) \in H/(H \cap N)$, is a continuous algebraic isomorphism.*

Corollary 2.5. *Let G be a topological group, N a normal and closed subgroup of G, and H a compact subgroup of G. Then the map ϕ defined in* Corollary 2.4 *is a homeomorphism.*

The next theorem is basic for what follows.

Theorem 2.6. *Let $Z \xrightarrow{f} H \xrightarrow{p} G$ be a sequence of continuous homomorphisms of topological groups satisfying the following properties:*

(i) $f(Z) = \ker p$.

(ii) *Z is abelian.*

(iii) *p is a quotient map.*

(iv) $G^b = \{1\}$.

Then H^b is abelian.

Proof. The universality of (i_Z, Z^b) implies the existence of a continuous homomorphism $f'\colon Z^b \to H^b$ such that the following diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{i_Z} & Z^b \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\
H & \xrightarrow[i_H]{} & H^b
\end{array}
\tag{2.1}
$$

commutes.

We first prove that $f'(Z^b)$ is a closed normal subgroup of H^b. It is obvious that $f'(Z^b)$ is a closed subgroup of H^b.

The fact that $f(Z) = \ker p$ implies that the subgroup $f(Z)$ is normal in H. Thus for an arbitrary $h \in H$ we have

$$hf(Z)h^{-1} \subseteq f(Z).$$

Applying i_H to both sides of the above inclusion and taking into account (2.1), one obtains that

$$i_H(h)f'(i_Z(Z))(i_H(h))^{-1} \subseteq f'(i_Z(Z)) \subseteq f'(Z^b). \tag{2.2}$$

Since the inner automorphisms of H^b are continuous and since H^b is Hausdorff and compact, the following equality holds

$$\overline{i_H(h)f'(i_Z(Z))(i_H(h))^{-1}} = i_H(h)\overline{f'(i_Z(Z))}(i_H(h))^{-1}. \tag{2.3}$$

Using the continuity of f', the density of i_Z, and the fact that H^b is Hausdorff and compact, one gets the following equalities

$$\overline{f'(i_Z(Z))} = f'(\overline{i_Z(Z)}) = f'(Z^b). \tag{2.4}$$

Relations (2.2), (2.3), and (2.4) imply that

$$i_H(h)f'(Z^b)(i_H(h))^{-1} \subseteq f'(Z^b). \tag{2.5}$$

Taking into account that $\overline{i_H(H)} = H^b$, one concludes from (2.5) that $f'(Z^b)$ is normal in H^b.

Let $K := H^b/f'(Z^b)$ be endowed with the quotient topology. Since H^b is a compact topological group and since $f'(Z^b)$ is a closed normal subgroup of it, K is a compact Hausdorff topological group. Denote by $q\colon H^b \to K$ the canonical quotient map.

We now show that there is a continuous homomorphism $\phi\colon G \to K$ such that the diagram

$$\begin{tikzcd} H \arrow[r, "p"] \arrow[d, "q\circ i_H"'] & G \arrow[d, "\phi"] \\ K \arrow[r, equal] & K \end{tikzcd} \tag{2.6}$$

commutes. For this we observe that

$$\ker p \subseteq \ker(q \circ i_H). \tag{2.7}$$

Indeed, $\ker p = f(Z)$ and we know by (2.1) that

$$i_H(\ker p) = i_H(f(Z)) = f'(i_Z(Z)) \subseteq f'(Z^b).$$

Since $f'(Z^b) = \ker q$, one obtains from the above relation that

$$(q \circ i_H)(\ker p) \subseteq \{1\},$$

i.e., (2.7) holds. Applying Lemma 2.3, there exists a continuous homomorphism $\phi\colon G \to K$ such that (2.6) is commutative.

Since $G^b = \{1\}$, we must have that

$$\phi(g) = 1, \text{ for all } g \in G.$$

Thus

$$(\phi \circ p)(h) = 1, \text{ for all } h \in H.$$

From the commutative diagram (2.6) we now get that

$$(q \circ i_H)(h) = 1, \text{ for all } h \in H,$$

i.e.,

$$i_H(H) \subseteq \ker q = f'(Z^b).$$

Since $\ker q$ is closed and since i_H is dense, it follows that $H^b = f'(Z^b)$. According to condition (ii) of the hypotheses we know that Z is abelian. Then so are Z^b and $f'(Z^b)$. Thus H^b is abelian. □

We recall that for a group G, the commutator subgroup is denoted by G'.

Lemma 2.7. *Let G be a connected Lie group. Then G^b is abelian if and only if $\ker i_G = \overline{G'}$.*

Proof. First suppose that $\ker i_G = \overline{G'}$. The inclusion $G' \subseteq \ker i_G$ implies that $i_G(G)$ is abelian. Then so is $\overline{i_G(G)} = G^b$.

Now assume that G^b is abelian. This implies the inclusion

$$\overline{G'} \subseteq \ker i_G. \tag{2.8}$$

To prove the converse inclusion, consider $T := G/\overline{G'}$. Thus T is a connected abelian Lie group. According to Korollar III.3.25 of [3] there are natural numbers m and n such that T is both algebraically and topologically isomorphic to the direct product $\mathbb{R}^m \times (\mathbb{R}/\mathbb{Z})^n$. It follows that $i_T : T \to T^b$ is injective.

Denote by $q : G \to T$ the canonical quotient map. The universality of (i_G, G^b) implies the existence of a continuous homomorphism $\bar{q} : G^b \to T^b$ which makes the diagram

$$\begin{array}{ccc} G & \xrightarrow{i_G} & G^b \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \bar{q}} \\ T & \xrightarrow[i_T]{} & T^b \end{array}$$

commutative. Now consider an arbitrary element $g \in \ker i_G$. Thus $\bar{q} \circ i_G(g) = 1$, or, by the commutativity of the above diagram, $i_T(q(g)) = 1$. Since i_T is injective, it follows that $q(g) = 1$, i.e., $g \in \ker q = \overline{G'}$. Thus

$$\ker i_G \subseteq \overline{G'}. \tag{2.9}$$

By (2.8) and (2.9) one obtains the desired conclusion. □

The following result is a consequence of Theorem 2.6 and Lemma 2.7.

Corollary 2.8. *Add to the hypotheses of* Theorem 2.6 *that H is a connected Lie group. Then* $\ker i_H = \overline{H'}$.

We define now a special type of topological groups, which will enable us to give an answer to the problem of the triviality of the Bohr-compactification presented in the previous section.

Definition 2.9. A topological group G is called *topologically perfect* if $G = \overline{G'}$.

The next result gives a class of Lie groups for which the converse of Corollary 2.2 is true.

Theorem 2.10. *Let G be a connected Lie group satisfying the property that the simply connected covering group $\widetilde{G}$ of it is topologically perfect. Then $G^b = \{1\}$ if and only if $(\widetilde{G})^b = \{1\}$.*

Proof. If $(\widetilde{G})^b = \{1\}$, then Corollary 2.2 yields that $G^b = \{1\}$. For the converse statement let $p \colon \widetilde{G} \to G$ be a covering morphism and denote by $Z := \ker p$. It is known that Z is an abelian subgroup of $\widetilde{G}$. Denote by $i \colon Z \to \widetilde{G}$ the inclusion map. The map p is a quotient map since it is a covering morphism. Thus the sequence

$$Z \xrightarrow{i} \widetilde{G} \xrightarrow{p} G$$

satisfies the conditions (i)–(iv) of Theorem 2.6. Applying Corollary 2.8, one obtains that

$$\ker i_{\widetilde{G}} = \overline{(\widetilde{G})'}.$$

Since $\widetilde{G}$ is topologically perfect, one concludes that $\ker i_{\widetilde{G}} = \widetilde{G}$, i.e., $(\widetilde{G})^b = \{1\}$. □

Connected semisimple Lie groups are common examples of topologically perfect groups. Thus the following result is a direct consequence of Theorem 2.10.

Corollary 2.11. *Let G be a connected semisimple Lie group and $\widetilde{G}$ the simply connected covering group of it. Then $G^b = \{1\}$ if and only if $(\widetilde{G})^b = \{1\}$.*

3. The structure of the Bohr-compactification of Lie groups

As we shall see, Corollary 2.11 implies statements about the structure of the Bohr-compactification of connected simple Lie groups and of connected semisimple Lie groups. For this we also need the following theorem, which is a consequence of a deep result by RUPPERT.

Theorem 3.1. *The Bohr-compactification of a non-compact connected simple Lie group G with finite center is trivial.*

Proof. This statement follows from two results of [7], namely from Theorem III.1.19 and assertion (i) of Theorem III.6.3. □

Theorem 3.2. *Let $\mathfrak{g}$ be a simple non-compact Lie algebra and let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then $G^b = \{1\}$.*

Proof. Let $\widetilde{G}$ be a simply connected covering group of G. According to Satz I.5.19 and Satz II.7.1 of [3] there is a connected linear Lie group G^* possessing a Lie algebra isomorphic to $\mathfrak{g}$. Since G^* is a simple linear Lie group, Proposition 5.1 of Chapter 1 of [6] implies that it has finite center. Applying Theorem 3.1, we get that $(G^*)^b = \{1\}$. Now consider a simply connected covering group $\widetilde{G^*}$ of G^*. Corollary 2.11 yields that $(\widetilde{G^*})^b = \{1\}$. On the other hand, the Lie groups $\widetilde{G^*}$ and $\widetilde{G}$ are isomorphic having isomorphic Lie algebras. Thus $(\widetilde{G})^b = \{1\}$ and so the assertion follows from Corollary 2.2. □

Now we turn our attention to connected semisimple Lie groups. A first step in determining the structure of their Bohr-compactification is contained in the following result:

Lemma 3.3. *Let $n \geq 1$ be a natural number and let $\mathfrak{s}$ be a semisimple Lie algebra with the property that*

$$\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n,$$

where $\mathfrak{s}_i$ are simple and non-compact ideals of $\mathfrak{s}$. If S is a connected Lie group with Lie algebra $\mathfrak{s}$, then $S^b = \{1\}$.

Proof. For $i \in \{1, \ldots, n\}$ let $\widetilde{S}_i$ be a simply connected Lie group with Lie algebra $\mathfrak{s}_i$. Put

$$\widetilde{S} := \prod_{i=1}^{n} \widetilde{S}_i.$$

Then $\widetilde{S}$ is a simply connected Lie group whose Lie algebra is isomorphic to $\mathfrak{s}$. Denote by $f_i : \widetilde{S}_i \to \widetilde{S}$ $(i = \overline{1, n})$ the canonical injections. The subgroup $f_i(\widetilde{S}_i)$ $(i \in \{1, \ldots, n\})$ of $\widetilde{S}$ is a connected Lie group whose Lie algebra is isomorphic to $\mathfrak{s}_i$. Thus, according to Theorem 3.2, the group $f_i(\widetilde{S}_i)$ has trivial Bohr-compactification for each $i \in \{1, \ldots, n\}$. It follows that

$$i_{\widetilde{S}}(f_i(\widetilde{S}_i)) = \{1\} \text{ for each } i \in \{1, \ldots, n\}.$$

Since

$$\widetilde{S} = f_1(\widetilde{S}_1) \ldots f_n(\widetilde{S}_n),$$

one obtains that $i_{\widetilde{S}}(\widetilde{S}) = \{1\}$, i.e., $(\widetilde{S})^b = \{1\}$. Since the group $\widetilde{S}$ is a simply connected covering group of S, Corollary 2.2 yields that $S^b = \{1\}$. □

In stating the structure theorem for the Bohr-compactification of connected semisimple Lie groups we need the following result about the structure of connected semisimple Lie groups. In the proof of this result one uses the structure theorem of semisimple Lie algebras (see Satz II.3.7 of [3]).

Lemma 3.4. *Let G be a connected semisimple Lie group and $\mathfrak{g}$ its Lie algebra. Then the following statements hold:*

(1) There are two finite sets I and J and simple ideals $\mathfrak{k}_i$ $(i \in I)$ and $\mathfrak{s}_j$ $(j \in J)$ of $\mathfrak{g}$ satisfying the properties that $\mathfrak{k}_i$ is compact for each $i \in I$, $\mathfrak{s}_j$ is non-compact for each $j \in J$, and

$$\mathfrak{g} = \bigoplus_{i \in I} \mathfrak{k}_i \oplus \bigoplus_{j \in J} \mathfrak{s}_j.$$

(By definition, if $I = \emptyset$, then $\oplus_{i \in I} \mathfrak{k}_i := \{0\}$ and similarly, if $J = \emptyset$, then $\oplus_{j \in J} \mathfrak{s}_j := \{0\}$.)

(2) *There is a compact connected normal subgroup K of G and there is a closed connected normal subgroup S of G such that*

$$\mathbf{L}(K) = \bigoplus_{i \in I} \mathfrak{k}_i, \quad \mathbf{L}(S) = \bigoplus_{j \in J} \mathfrak{s}_j, \text{ and } G = KS.$$

Moreover, $K \cap S$ is a discrete subgroup of G.

We are now prepared for the structure theorem of the Bohr-compactification of a connected semisimple Lie group.

Theorem 3.5. *Let G be a connected semisimple Lie group and $\mathfrak{g}$ its Lie algebra. There is a compact connected normal subgroup K of G and there is a closed connected normal subgroup S of G such that the following assertions hold:*

(i) $G = KS$.

(ii) *The groups G/S and $K/K \cap S$ are algebraically and topologically isomorphic.*

(iii) *If $q\colon G \to G/S$ denotes the canonical quotient map, then $(q, G/S)$ is the Bohr-compactification of G.*

Proof. Let K and S be the subgroups of assertion (2) of Lemma 3.4. Then (i) obviously holds.

(ii) This assertion follows from (i) and Corollary 2.5.

(ii) According to (ii) the pair $(q, G/S)$ is a topological group compactification of G. Consider an arbitrary continuous homomorphism $f\colon G \to T$ of G into a compact Hausdorff topological group T. We know by Lemma 3.3 that $S^b = \{1\}$, hence $f(S) = \{1\}$, i.e., $\ker q = S \subseteq \ker f$. Applying Lemma 2.3, we find a continuous homomorphism $\bar{f}\colon G/S \to T$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{q} & G/S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ T & = & T \end{array}$$

is commutative. This means that $(q, G/S)$ is the universal topological group compactification of G, hence also the Bohr-compactification of G. □

For the sake of completeness we finish this section with some considerations on the Bohr-compactification of another important class of Lie groups, namely the solvable Lie groups. We determine even the structure of the Bohr-compactification of solvable connected topological groups. For this we state first the following useful result:

Proposition 3.6. *A compact connected Hausdorff topological group G which is solvable is abelian.*

Proof. Proposition 9.4 of [4] implies that $G'' = G'$. Since G is solvable, it follows that $G' = \{1\}$. Thus G is abelian. □

We are now able to give the structure of the Bohr-compactification of solvable topological groups. For a topological group G denote by T the quotient group $G/\overline{G'}$ and by $q\colon G \to T$ the canonical quotient map.

Theorem 3.7. *Let G be a solvable connected topological group. Then $(i_T \circ q, T^b)$ is the Bohr-compactification of G.*

Proof. It is clear that $(i_T \circ q, T^b)$ is a topological group compactification of G. Now consider an arbitrary continuous and dense homomorphism $f\colon G \to K$ of G into a compact Hausdorff topological group K. Since G is connected and solvable, so is $f(G)$. Thus the group $\overline{f(G)} = K$ is also connected and solvable. Hence Proposition 3.6 yields that K is abelian. It follows that

$$\ker q = \overline{G'} \subseteq \ker f.$$

In view of Lemma 2.3 there exists a continuous homomorphism $f'\colon T \to K$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{q} & T \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ K & = & K \end{array} \qquad (3.1)$$

commutes. The universality of (i_T, T^b) implies the existence of a continuous homomorphism $\bar{f}\colon T^b \to K$ such that the diagram

$$\text{(3.2)}\qquad \begin{array}{ccc} T & \xrightarrow{i_T} & T^b \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle \bar{f}} \\ K & = & K \end{array}$$

commutes. The diagrams (3.1) and (3.2) yield

$$\bar{f} \circ i_T \circ q = f' \circ q = f,$$

i.e., the following diagram

$$\begin{array}{ccc} G & \xrightarrow{i_T \circ q} & T^b \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \bar{f}} \\ K & = & K \end{array}$$

is commutative. This shows that $(i_T \circ q, T^b)$ is the universal topological group compactification of G, hence also the Bohr-compactification of G. □

Remark. Suppose in addition to the hypotheses of Theorem 3.7 that G is locally compact. In view of assertion (iii) of Theorem 7.57 of [4] the connected locally compact abelian group T is both algebraically and topologically isomorphic to the direct product $\mathbb{R}^n \times C$ with a compact connected group C. Since the Bohr-compactification of this direct product is known, it is now clear what the Bohr-compactification of a solvable connected locally compact topological group looks like.

4. An example

We give now the example promised in the remark after Corollary 2.2 for a connected topological group satisfying the conditions that it has a trivial Bohr-compactification and the simply connected covering group of it has a non-trivial Bohr-compactification. Let $\widetilde{G} = \mathbb{R} \times \widetilde{\mathrm{Sl}}(2, \mathbb{R})$ be the direct product of the additive group of real numbers (endowed with the usual topology) and the simply connected covering group of the special linear group. Let $T = \mathbb{Z} + \sqrt{2}\mathbb{Z}$. We know by Lemma I.3.14 of [3] that T is a dense subgroup of $\mathbb{R}$. It is known (see, for example, Theorem V.4.37 of [2]) that $\widetilde{\mathrm{Sl}}(2, \mathbb{R})$ has a discrete center which is isomorphic to $\mathbb{Z}$. Denote by $z \in \widetilde{\mathrm{Sl}}(2, \mathbb{R})$

the generator of this center. Now consider the subgroup Z of $\widetilde{G}$ generated by the elements $(1,1)$ and $(\sqrt{2}, z)$. Then

$$Z = \{(m + n\sqrt{2}, z^n) \mid m, n \in \mathbb{Z}\}.$$

The subgroup Z of $\widetilde{G}$ is discrete and normal in $\widetilde{G}$. We consider the quotient group $G := \widetilde{G}/Z$. For this group we can state the following proposition:

Proposition 4.1. *The topological group G has a trivial Bohr-compactification. The Bohr-compactification of its simply connected covering group $\widetilde{G}$ satisfies $(\widetilde{G})^b \simeq \mathbb{R}^b$.*

Proof. Denote by $q\colon \widetilde{G} \to G$ the canonical quotient map. Since q is a homomorphism, we have

$$q\left((\widetilde{G})'\right) = G'.$$

Thus $G' = (\widetilde{G})'Z/Z$. Applying Corollary 2.4, there is a continuous isomorphism $\phi\colon (\widetilde{G})'/((\widetilde{G})' \cap Z) \to (\widetilde{G})'Z/Z$. Hence $\phi\colon (\widetilde{G})'/((\widetilde{G})' \cap Z) \to G'$ is a continuous isomorphism. On the other hand, since $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ is simple, we have the following equality

$$(\widetilde{G})' = \{0\} \times \widetilde{\mathrm{Sl}}(2,\mathbb{R}).$$

Thus $(\widetilde{G})' \cap Z = \{(0,1)\}$. It follows that $(\widetilde{G})'/((\widetilde{G})' \cap Z)$ is both algebraically and topologically isomorphic to $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$. Thus $(\widetilde{G})'/((\widetilde{G})' \cap Z)$ has a trivial Bohr-compactification by Theorem 3.2. Applying Lemma 2.1 to the map ϕ, we conclude that $(G')^b = \{1\}$. Let us observe that

$$G' = q\left((\widetilde{G})'Z\right).$$

Since

$$T \times \widetilde{\mathrm{Sl}}(2,\mathbb{R}) \subseteq (\widetilde{G})'Z$$

and since $\overline{T} = \mathbb{R}$, we deduce that

$$\overline{(\widetilde{G})'Z} = \widetilde{G}.$$

The continuity of q and the above relations yield that

$$G = q(\widetilde{G}) \subseteq \overline{q((\widetilde{G})'Z)} = \overline{G'}.$$

Thus $\overline{G'} = G$. Taking into account that $(G')^b = \{1\}$, it follows that $i_G(G') = \{1\}$. Since $\overline{G'} = G$, the continuity of i_G implies that $i_G(G) = \{1\}$, i.e., $G^b = \{1\}$. Since q is open and since $\ker q = Z$ is discrete, it follows that q is a covering morphism. Since $\widetilde{G}$ is simply connected, it is a simply connected covering group of G. On the other hand, since $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ has a trivial Bohr-compactification, one has that $(\widetilde{G})^b \simeq \mathbb{R}^b$. $\square$

Note. This paper is a part of the author's doctoral dissertation written at Darmstadt University of Technology under the direction of Professor Karl H. Hofmann.

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