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PERIHELION ADVANCE AND MANEFF'S FIELD

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Abstract. We compute the advance of the perihelion of a planetary orbit as predicted by the Maneff's gravitational law and we compare the result with the results of the general relativity theory, as well as with the observational data for Mercury and for the binary pulsar PSR 1913+16. The effects resulting from the adoption of the Maneff's potential are analysed both in the classical and the relativistic case. For the relatistic analysis we propose a new form of the metric associated to Maneff's gravitational potential.

The results show that in the classical case the advance of the perihelion (periastron) predicted in the Maneff's model is exactly half of the observed one, while putting the prediction of this model in accord with the prediction of general relativity requires a modification of the perturbating factor in Maneff's potential with a factor of 2.

The computation made in the relativistic case with the Maneff potential give a result which is not in concordance with the observational data, because in this case the advance of the perihelion is a superposition of the value due to the relativistic effect and that resulting from the modification of the potential in the Maneff case.

1. Introduction

G. Maneff considered a post-Newtonian nonrelativistic law of gravitation, assuming that the gravitational interaction between two masses m_1 and m_2 is given

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by the "force function" ([5]):

$$U = \frac{Gm_1 m_2}{r}\left[1 + \frac{3G(m_1 + m_2)}{2c^2 r}\right], \qquad (1)$$

where r is the distance between m_1 and m_2, G is the Newtonian gravitational constant, and c is the speed of light.

Recently, several *theoretical* approaches considered that the study of the consequences of adopting Maneff's potential, would be an ideal method to investigate the susceptibility to generalization of the mathematical models and techniques developed in Celestial Mechanics ([6],[1]) and Stellar Astrophysics ([9], [10]).

In this paper, we analyze the way in which the Maneff's gravitational interaction responds to one of the most important *observational* facts that have become a milestone in the evolution of the theory of gravitation: the advance of the planetary perihelion.

This phenomenon was discovered by Le Verrier in 1859 as a discrepancy between the observations and the theoretical predictions for the shift of Mercury's perihelion. Present-day measurements indicate that Mercury exhibits an excess motion in the perihelion shift of about 43" per century. The attempts to explain this phenomenon have to consider that either a hidden planet or some sort of diffuse material should orbit in the neighborhood of the Sun - or Newton's theory of gravity should suffer some adjustments. All the models involving hidden mass within Newton's theory of gravitation have constantly failed, while the excellent correlation between the observations and the theoretical predictions of Einstein's General Relativity became one of the great successes of this theory.

This paper analyses the problem of perihelion advance in a potential-independent fashion, i.e. we infer the expression for the perihelion advance as a functional of the potential expression. In section 2, we develop the Binet-like differential equation for the orbit of a body moving in a central spherical symmetric field. The form of the potential $\Phi(r)$ is not specified, so the equation explicitly depends on Φ. The potential is then particularized to Maneff's expression and the perihelion advance is computed in this case. Section 3, after introducing a general relativistic metric to be associated with a spherical symmetric potential function $\Phi(r)$, proceeds in a similar fashion to derive the perihelion advance in the relativistic framework. The problem of periastron advance for the binary pulsar is considered in Section 4. In section 5 we summarize the results, compare the observational values for the Mercury's orbit and for binary pulsar PSR 1913+16 with those theoretically predicted and drop the conclusions.

2. The classical framework

2.1. The Binet-like equation. We shall consider a massive body of mass M and a test particle of mass $m \ll M$ in the gravitational field of M. The effects of this field on m can be descried by the following potential Φ, which is attached to U from (1) $(m_1 = M,\, m_2 = m)$:

$$\Phi(r) = -\frac{GM}{r} - \frac{3G^2M^2}{2c^2r^2} \tag{2}$$

The spherical symmetry implies that the orbit is planar, so we restrict our considerations to the two-dimensional problem, i.e. finding the equation of the orbit in the form $r = r(\theta)$ We shall start the derivation of the differential equation of the trajectory from the laws of conservation for energy and angular momentum:

$$\begin{cases} \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 + 2\Phi(r) = h \\ r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = C \end{cases} \tag{3}$$

After the change of unknown function to $u = 1/r$ we obtain from (3) the Binet-like equation:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -u - \frac{1}{C^2}\frac{\mathrm{d}\Phi}{\mathrm{d}u} \tag{4}$$

2.2. Solution for Newtonian potential. If the potential is Newtonian we find the well-known conic solution:

$$u = \frac{GM}{C^2}\left[1 + e\cos(\theta - \omega)\right]. \tag{5}$$

For the adequate values of h and C this orbit will be an ellipse.

2.3. **Solution for Maneff case.** For Maneff potential the equation (4) becomes:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} = -u\left(1 - 3\frac{G^2M^2}{C^2c^2}\right) - \frac{GM}{C^2} \tag{6}$$

One observes that this equation has an exact analytic solution. If we use the notation:

$$\alpha = 3\frac{G^2M^2}{C^2c^2}$$

with $\alpha < 1$ for the realistic astrophysical situations (noncollisional orbits), the solution of (6) is:

$$u = \frac{GM}{C^2(1-\alpha)}\left[1 + e\cos(\sqrt{1-\alpha}\,\theta - \omega)\right]. \tag{7}$$

For $0 < e < 1$ and $\alpha \ll 1$ this represents approximately an allipse. If we try to identify the perihelion advance in (7) by putting it in the form:

$$u = \frac{GM}{C^2(1-\alpha)}\left[1 + e\cos(\theta - \omega - \delta(\theta))\right]. \tag{8}$$

and if we take into account the fact that usually $\alpha \ll 1$, we get:

$$\delta(\theta) = \frac{1}{2}\alpha\theta = \frac{3}{2}\frac{G^2M^2}{C^2c^2}\theta. \tag{9}$$

The perihelion advance predicted by Maneff's field is proportional with the value obtained by Einstein's relativity (see below eq. (17)), i.e. Einstein's expression for perihelion advance is twice as big as Maneff's. This problem can easily be solved by scaling the "perturbative" term in Maneff's formula. Thus, if we took the potential of the form:

$$\Phi(r) = -\frac{GM}{r} - 3\frac{G^2M^2}{c^2r^2} \tag{10}$$

we would obtain the exact relativistic formula for the perihelion advance within the classical framework.

3. The relativistic framework

3.1. **The potential-dependant metric.** Let $x^0 = ct$ be the temporal coordinate and $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$ the spherical (Schwarzschild) coordinates. Then, we shall associate to the potential Φ the following general relativistic metric ([4])

$$\mathrm{d}s^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2\,\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 + \frac{2\Phi}{c^2}} - r^2\,\mathrm{d}\theta^2 - r^2 \sin^2\theta\,\mathrm{d}\varphi^2, \tag{11}$$

where $\mathrm{d}s$ is the elementary interval.

One should note that the metric given by eq. (11) does not satisfy Einstein's field equations ([8], [2]) unless the potential Φ is Newtonian, i.e. it has an expression of the form:

$$\Phi(r) = -\frac{a}{r}.$$

Therefore, any attempt to extend this approach beyond the problem of the motion in a central field (e.g. modeling massive relativistic bodies as it is required in astrophysical or cosmological applications) should start from defining a proper adjustment to the field equations. Fortunately, it is not the case for the matter of perihelion advance, since the equation of the orbit will be straightly inferred from the equations of the geodesics.

3.2. **The Binet-like equation.** Once the relativistic metric of the field is specified the derivation of the Binet-like equation for the trajectory proceeds by computing Christoffel's symbols and then writing the equations of the geodesics. One should refer to Tolman ([8]) for the details of this derivation for the general Schwartzschild metric:

$$\mathrm{d}s^2 = e^{\nu(r)}\,\mathrm{d}t^2 - e^{\lambda(r)}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2 \sin^2\theta\,\mathrm{d}\varphi^2, \tag{12}$$

noting that our metric (3) is a particular case of (12).

The system of 10 geodesic equations for the metric (12), finally reduces to the following two equations:

$$\begin{cases} e^{\lambda}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 - e^{-\nu}K^2 + 1 = 0 \\ \\ r^2\frac{\mathrm{d}\theta}{\mathrm{d}s} = H, \end{cases} \tag{13}$$

where K is a dimensionless constant and Hc is the relativistic equivalent of C constant in the classical approach.

For the metric (3), eqs. (13) become:

$$\begin{cases} \frac{1}{1+\frac{2\Phi}{c^2}}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 - \frac{K^2}{1+\frac{2\Phi}{c^2}} + 1 = 0 \\ \\ r^2\frac{\mathrm{d}\theta}{\mathrm{d}s} = H, \end{cases} \tag{14}$$

Taking the new unknown function $u = 1/r$ we obtain the Binet-like equation:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -u\left(1 + \frac{2\Phi}{c^2}\right) - \frac{u^2}{c^2}\frac{\mathrm{d}\Phi}{\mathrm{d}u} - \frac{1}{c^2H^2}\frac{\mathrm{d}\Phi}{\mathrm{d}u} \tag{15}$$

3.3. **Newtonian potential.** In the case of Newtonian potential, eq. (15) becomes:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -u + \frac{GM}{c^2H^2} + \frac{3GM}{c^2}u^2. \tag{16}$$

This equation cannot be integrated in terms of elementary functions. The approximate approach in solving equation (16) takes into account the fact the the bounded (noncollisional) orbit is quasi-elliptical, so the solution can be put in the following form ([8]):

$$u = \frac{GM}{C^2}\left[1 + e\cos(\theta - \omega - \delta(\theta)\right]$$

where $\delta(\theta) \ll \theta$.

In the first-order analysis, one obtains for $\delta(\theta)$ the expression:

$$\delta(\theta) = 3\frac{G^2M^2}{C^2c^2}\theta. \tag{17}$$

Then, the Einstein's formula for the perihelion advance for one period will be:

$$\Delta\omega = 6\pi\frac{G^2M^2}{C^2c^2} = \frac{24\pi^3a^2}{c^2P^2(1-e^2)} \tag{18}$$

It is well-known that equation (18) accounts for the observed orbits with an excellent accuracy.

Taking into account the Kepler's third law, the expression (18) can be put in the form ([3]):

$$\Delta\omega = \frac{6\pi GM}{c^2 a(1-e^2)} \tag{19}$$

or in the terms of Schwarzschild radius:

$$R_S = \frac{2GM}{c^2} \tag{20}$$

we have

$$\Delta\omega = \frac{3\pi}{1-e^2}\frac{R_S}{a}. \tag{21}$$

3.4. **Maneff potential.** For Maneff's potential, eq. (15) becomes:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} = -u\left(1 - 3\frac{G^2M^2}{H^2c^4}\right) + \frac{GM}{H^2c^2} + \frac{3GM}{c^2}u^2 + \frac{6G^2M^2}{c^4}u^3. \tag{22}$$

In this case, the computation of the *small* perihelion advance gives the following result:

$$\delta(\theta) = \frac{9}{2}\frac{G^2M^2}{C^2c^2}\theta. \tag{23}$$

One observes that the value of the perihelion advance is a sum of the values given by eqs. (9) and (17). The effect of changing the potential in the relativistic framework is a simple superposition of the relativistic effect and the effect of changing the potential.

The prediction of eq. (23) is not in agreement with the observational data and seems not to justify the intricacies which a relativistic Maneff approach implies (such as adjusting Einstein's field equations).

4. **Periastron advance of binary pulsar**

The obtained reults can also be applied for the binary systems having the two components of comparable masses M_1 and M_2. In this case the product GM

must to be changed in $\mu = G(M_1 + M_2)$. Then, using the Kepler's third law, from the equation (18) we obtain the rate of the periastron advance ([11])

$$\dot{\omega} = 3\left(\frac{2\pi}{P}\right)^{5/3} \frac{G^{2/3}}{c^2(1-e^2)}(M_1 + M_2)^{2/3}. \tag{24}$$

The equation (21) shows that in the binary pulsars, where the semimajor axis a is small, the periastron advance is large. For the binary pulsar PSR 1913+16, taking $M_1 = M_p = 1.4M_\odot$, $M_2 = M_c = 1.4M_\odot$, $P = 27\,907$ sec, the observed rate of periastron advance will be obtained, namely $\dot{\omega} = 4.23$ deg yr^-1.

If we shall use the equation (9), only half of the observed value will be obtained. This means that the Maneff gravitational field explains the periastron advance in the binary pulsar PSR 1913+16 only *qualitatively* but not *quantitatively*, as this was considered in a recent paper ([7]). We observe that, if the equation (7c) from the cited paper will be used (noncollisional orbits) the same expression (9) from our paper will be obtained.

In conclusion, if we take the Maneff gravitational field as an alternative post-Newtonian nonrelativistic law of gravitation, the "perturbative" term in the Maneff potential (2) must to be scaled by the factor 2 as in the expression (10). In this way the Maneff gravitational field will explain the perihelion advance of the planetary orbits as well as the periastron advance for the binary pulsars.

5. Conclusions

Analyzing the Binet-like equations (6), (16), (22) and the perihelion advance formulae (9), (17), (23), we come to the following conclusions:

- The theoretical results of Einstein's relativity are in perfect agreement with the observational evidence. No corrections are *necessary* for this theory. It explains the planetary perihelion advance, as well as the periastron advance of binary pulsars.
- In the classical framework, Maneff's field *can* explain the phenomenon of perihelion advance qualitatively. A scaling of the second term in Maneff's formula would lead to the exact relativistic result for the angular advance

formula. The main strength of Maneff's formalism is, in our opinion, its simplicity that makes relativistic effects such as the perihelion advance amenable to the analysis of Celestial Mechanics.

- Within the relativistic framework, Maneff's potential predicts a result which is in disagreement with the observed data. It seems that there is no need to change Einstein's equations in a manner that would affect Schwartzschild solution.
- One should note, however, that the derivation of the formula for the perihelion advance was carried out within the first-order analysis. This is perfectly justified at the scale of the Solar System as well as for the binary pulsars, where the components are close to the mass points.

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