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ON CERTAIN CLASSES OF GENERALIZED CONVEX FUNCTIONS WITH APPLICATIONS, I.

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Abstract. The aim of this series of papers is to introduce certain new concepts of generalized convex functions with applications. The first part contains results related to the η-invex functions first introduced by the author in 1988. Here are studied also η-cvazi-invexity and generalized η-pseudo-invexity with connections to well known classes of functions as subadditive or Jensen-convex functions. The second part treats the so-called A-convex functions, due to the author. Finally the part III, on Λ-invex functions, leads to a generalization of the Banach-Steinhaus theorem of condenzation of singularities.

Invex functions

A. A differentiable function $f : \mathbf{R}^n \to \mathbf{R}$ is called **invex**, if there exists a vector function $\eta(x,u) \in \mathbf{R}^n$ such that

$$f(x) - f(u) \geq (\eta(x,u))^t \nabla f(u). \tag{1}$$

This concept has been introduced by Hanson [1], who proved that, if in place of the usual convexity conditions the functions involved in a nonlinear optimization problem, all satisfy condition (1) for the same function η, then there hold true weak duality results, and that the sufficiency of the Kuhn-Tucker conditions are true. Hanson's paper was the source of inspiration for many later researches. Craven [3] has introduced the name of "invex" functions, and obtained duality theorems for fractional programming. Mond and Hanson [2] have extended the concept of invexity to polyhedral cones, Craven and Glover [5] proved that the class of invex functions

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is equivalent with the class of functions having all stationary points as global minimum points. Martin [7] has defined the Kuhn-Tucker invexity, while Jeyakumar [6] introduced weak and strong invexity.

Examples. 1) Let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x^3$. This function is not invex, as the critical point $x = 0$ is not a global minimum point. But, as it is well known, f is cvazi-convex.

2) Let $f : \mathbf{R}^2 \to \mathbf{R}$, $f(x, y) = x^3 - 10y^3 + x - y$. It is easy to see that f is invex, but it is not cvazi-convex.

In the same way, f is called **cvazi-invex**, if

$$f(y) - f(x) \leq 0 \Rightarrow (\eta(x, y))^t \nabla f(x) \leq 0 \tag{2}$$

and **pseudo-invex**, if

$$(\eta(x, y))^t \nabla f(x) \geq 0 \Rightarrow f(y) - f(x) \geq 0. \tag{3}$$

Clearly, in example 1) f is cvazi-invex; and that by a theorem of Craven and Glover [5] we have that there is no difference between the class of invex and pseudo-invex functions.

Definition 1. Let $S \subset \mathbf{R}^n$, open, and $f : S \to \mathbf{R}$ a differentiable function. If relation (1) is valid for all $x, u \in S$, we will say that f is **invex related** to η on S.

The following propositions give certain connextions between functions invex related to η and other classes of functions.

Proposition 1. Let $S \subset \mathbf{R}^n$ be open, convex and f invex related to η on S. Let us assume that the following condition is true:

$$f(x) < f(y) \Rightarrow (x - y)^t \nabla f(y) \gtrless (\eta(x, y))^t \nabla f(y) \text{ for all } x, y \in S. \tag{4}$$

Then f is pseudo-convex (strictly pseudo-convex).

Proof. Let $x, y \in S$ and $f(x) < f(y)$. Then, in view of (1) we can write

$$(x - y)^t \nabla f(y) = [(x - y) - \eta(x, y)]^t \nabla f(y) + (\eta(x, y))^t \nabla f(y) \leq$$

$$\leq (x - y - \eta(x, y))^t \nabla f(y) + f(x) - f(y) < [(x - y) - \eta(x, y)]^t \nabla f(y) < 0.$$

If $x, y \in S$ and $f(x) \leq f(y)$, then from

$$(x - y)^t \nabla f(y) = (x - y - \eta(x, y))^t \nabla f(y) + (\eta(x, y))^t \nabla f(y) \leq$$

$$\leq (x - y - \eta(x,y))^t \nabla f(y) + f(x) - f(y) \leq (x - y - \eta(x,y))^t \nabla f(y) < 0,$$

so we have strict pseudo-convexity if there is strict inequality in (4).

Proposition 2. *Let f be invex related to η on the open, convex set S. If the implication*

$$(x-y)^t \nabla f(y) > 0 \;\Rightarrow\; (\eta(x,y))^t \nabla f(y) \geq (x-y)^t \nabla f(y) \quad (x, y \in S) \tag{5}$$

is true, then f is cvazi-convex (and thus, cvazi-invex, too).

Proof. We can write successively

$$f(x) - f(y) \geq (\eta(x,y))^t \nabla f(y) = (\eta(x,y) - (x-y))^t \nabla f(y) + (x-y)^t \nabla f(y) >$$

$$> (\eta(x,y) - (x-y))^t \nabla f(y) > 0,$$

thus $(x-y)^t \nabla f(y) > 0 \;\Rightarrow\; f(x) - f(y) > 0$, implying the cvazi-convexity of f, in case of differentiable function f.

The following proposition gives a simple method of construction of new invex functions.

Proposition 3. *Let $g : \mathbf{R} \to \mathbf{R}$ be a differentiable, increasing and convex function. Let $f : \mathbf{R}^n \to \mathbf{R}$ be invex related to η. Then $g \circ f : \mathbf{R}^n \to \mathbf{R}$ is invex related to η, too.*

Proof. It is known that $g(x+t) \geq g(x) + g'(x)t$ for all $x, t \in \mathbf{R}$. Thus we have

$$g(f(x)) \geq g[f(y) + (\eta(x,y))^t \nabla f(y)] \geq g[f(y)] + g'(f(y)) \nabla[\eta(x,y) f(y)] =$$

$$= g[f(y)] + (\eta(x,y))^t \nabla (g \circ f)(y),$$

which means that $g \circ f$ is invex related to η.

Definition 2. ([8]) Let $f : \mathbf{R}^n \to \mathbf{R}$, and $\eta : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be a given function. We say that f is **η-invex** (incave) if the following inequality is true:

$$f(y + \lambda \eta(x,y)) \lesseqgtr \lambda f(x) + (1-\lambda) f(y), \text{ for all } x, y \in \mathbf{R}^n, \text{ all } \lambda \in [0,1]. \tag{6}$$

Proposition 4. *If f is differentiable and η-invex, then f is invex related to η (on $\mathbf{R}^n$).*

Proof. Relation (6) can be rewritten in the form

$$f(y + \lambda \eta(x,y)) - f(y) \leq \lambda [f(x) - f(y)].$$

Let $\lambda > 0$, so by division with λ and by taking $\lambda \to 0+$, in case of differentiable f, one easily obtains

$$(\eta(x,y))^t \nabla f(y) \leq f(x) - f(y)$$

what means that f is invex related to η.

Clearly, the converse of this property is not generally true, but there exist conditions when this converse is valid, too.

We obtain first certain connections to the class of subadditive functions.

Proposition 5. *Let $f : \mathbf{R}_+ \to \mathbf{R}$ be an η-incave function, and let us suppose that $f(0) \geq 0$ and $\eta(0,x) = -x$ for all $x \in \mathbf{R}$. Then the function f is subadditive.*

Proof. From the η-incavity of f (see Definition 2) and $f(0) \geq 0$ we have

$$f(v + \lambda\eta(0,v)) \geq \lambda f(0) + (1-\lambda)f(v) \geq (1-\lambda)f(v).$$

Let $v := x + y$ and $\lambda := \dfrac{y}{x+y} \in [0,1]$ in this relation. From the equality $x = x + y + \dfrac{x}{x+y}\eta(0, x+y)$ we immediately get $f(x) \geq \dfrac{x}{x+y} f(x+y)$. Replacing x by y we get $f(y) \geq \dfrac{x}{x+y} f(x+y)$, so by addition it results $f(x) + f(y) \geq f(x+y)$, i.e. the subadditivity of f.

Proposition 6. Let $f : (0,\infty) \to \mathbf{R}$ be a subadditive function which is η-invex, and satisfies the condition

$$f(y) \leq f\left(x + y + \frac{x}{y}\eta(x, x+y)\right) \quad \textit{for all } x, y \in (0,\infty). \tag{7}$$

Let $g : (0,\infty) \to \mathbf{R}$ be defined by $g(x) = \dfrac{f(x)}{x}$. Then the function g is a decreasing function.

Proof. In $f(v + \lambda\eta(u,v)) \leq \lambda f(u) + (1-\lambda) f(v)$ let us put (with $x < y$, $x, y \in (0,\infty)$)

$$\lambda := \frac{x}{y}, \quad u := x, \quad v := x + y.$$

We can obtain the relation

$$f\left(x + y + \frac{x}{y}\eta(x, x+y)\right) \leq \frac{x}{y} f(x) + \left(1 - \frac{x}{y}\right) f(x+y) \leq$$

$$\leq \frac{x}{y} f(x) + \left(1 - \frac{x}{y}\right)[f(x) + f(y)] = f(x) + f(y) - \frac{x}{y} f(y).$$

From condition (7) we can deduce the inequality

$$f(y) \leq f(x) + f(y) - \frac{x}{y} f(y), \text{ or } \frac{f(y)}{y} \leq \frac{f(x)}{x} \text{ for } x < y.$$

Thus $g(y) \leq g(x)$ for $x < y$.

Remark. For $\eta(a, b) = a - b$ (when f is convex), relation (7) becomes $f(y) \leq f(y)$, which is always true.

B. In Definition 2 we have introduced the notion of η-invexity on the entire space $\mathbf{R}^n$. In many circumsctances, it will be important to consider such functions on a subset $S \subset \mathbf{R}^n$. Then the necessity of generalization of convex sets arises.

Definition 3. ([8]) Let $S \subset \mathbf{R}^n$ and $\eta : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be given. We say that the set S is η-invex, if the implication

$$x, y \in S, \quad \lambda \in [0, 1] \Rightarrow y + \lambda \eta(x, y) \in S \tag{8}$$

is true.

Remarks. Clearly, all subset S is η-invex to $\eta \equiv 0$. The definition essentially says that there exists a curve in S beginning from y. It is not required that x is one of the final points of this curve for all x, y.

Examples. 1) Let $\eta_1 : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $\eta_1(x, y) = x - y$ for $x, y \geq 0$, $x - y$ for $x \leq 0, y \leq 0$; $-7-y$ for $x \geq 0, y \leq 0$; $2-y$ for $x \leq 0, y \geq 0$. Let $S_1 = [-7, -2] \cup [2, 10]$. Then S_1 is η_1-invex; as an easy verification applies.

2) Let $S_1 \subset \mathbf{R}$ be η_1-invex and $S_2 \subset \mathbf{R}$ be η_2-invex, and define $S = S_1 \times S_2 \subset \mathbf{R}^n$. Let $\eta : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$\eta(x, y) = (\eta_1(x), \eta_2(y)),$$

where $x = (x_1, y_2)$, $y = (x_2, y_2)$.

Then S is η-invex. This follows immediately.

Definition 4. Let $S \subset \mathbf{R}^n$ be η-invex set, where η is given. We say that f is **η-invex** on S ($f : S \to \mathbf{R}$) if relation (6) is valid for all $x, y \in S$.

If S is open, and f is differentiable, we can state the following proposition, similar to Proposition 4.

Proposition 7. *If $S \subset \mathbf{R}^n$ is open, invex, and $f : S \to \mathbf{R}$ is differentiable, and η-invex, then f is invex related to η.*

Remark. The converse of this proposition is not true. Let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x^2$. Since all critical points are global minima, Craven and Glover's theorem implies that f is invex related to $\eta(x,y) = \dfrac{x^2 - y^2}{2y}$ $(y \neq 0)$; 0 $(y = 0)$.

On the other hand, inequality (6) is transformed into (we omit the simple algebraic manipulations) $\lambda^2(x^4 - 2x^2y^2 + y^4) \leq 0$, which is valid only if $\lambda = 0$ or $x = y = 0$. Thus f is not η-invex.

Proposition 8. *Let η be given, and let $S \subset \mathbf{R}^n$ be η-invex. Let $X \supset S$ be an open set and $f : X \to \mathbf{R}$ invex related to η, and differentiable. Let us suppose that η satisfies the following conditions*

$$\begin{aligned} &\eta(x, x + \lambda\eta(y,x)) = -\lambda\eta(y,x), \\ &\eta(y, x + \lambda\eta(y,x)) = (1-\lambda)\eta(y,x) \quad (x, y \in \mathbf{R}^n;\ \lambda \in [0,1]). \end{aligned} \tag{9}$$

Then f is η-invex on S.

Proof. Let $y, x \in S$. Let $0 < \lambda < 1$ be given and consider $z = x + \lambda\eta(y,x)$. Clearly $z \in S$. From the invexity of f related to η we have

$$f(y) - f(z) \geq (\eta(y,z))^t \nabla f(x). \tag{10}$$

In the same manner,

$$f(x) - f(z) \geq (\eta(x,z))^t \nabla f(z). \tag{11}$$

From (10) and (11) we can derive

$$\lambda f(y) + (1-\lambda) f(x) - f(z) \geq [\lambda(\eta(y,z))^t + (1-\lambda)(\eta(x,z))^t] \nabla f(z).$$

But from the given condition (9) we have

$$\lambda(\eta(y,z))^t + (1-\lambda)(\eta(x,z))^t = [\lambda(1-\lambda) - \lambda(1-\lambda)](\eta(y,x))^t = 0$$

and the theorem is proved.

Remark. Condition (9) is not trivial. Let $S = [-7,-2] \cup [2,10]$, given in example 1) from A. Then the η-function given there verifies (9). Thus $\eta(x,y) \neq x - y$.

Definition 5. Let η be given and $S \subset \mathbf{R}^n$ an η-invex set. We say that f is **η-cvazi-invex**, if

$$f(y + \lambda\eta(x,y)) \le \max\{f(x), f(y)\}, \quad x, y \in S,\ \lambda \in [0,1] \tag{12}$$

is true.

Theorem 1. *A function $f : S \to \mathbf{R}$ is η-cvazi-invex iff all level sets $S(f,\alpha)$ of f are η-invex sets.*

Proof. Let f be η-cvazi-invex on S, and let $x, y \in S(f,\alpha) = \{z : \ f(z) \le \alpha\}$, $\alpha \in \mathbf{R}$. We have

$$f(y + \lambda\eta(x,y)) \le \max\{f(x), f(y)\} = f(y), \text{ if } f(x) \le f(y).$$

Supposing $y \in S(f,\alpha)$, we get $f(y) \le \alpha$, so $f(x) \le \alpha$ (thus $x \in S(f,\alpha)$), it results that $f(y+\lambda\eta(x,y)) \le \alpha$, yielding $y+\lambda\eta(x,y) \in S(f,\alpha)$. Thus the sets $S(f,\alpha)$ are η-invex.

Let us now assume that these level sets are η-invex for all $\alpha \in \mathbf{R}$, and put $\alpha = \max\{f(x), f(y)\}$. Then $x \in S(f,\alpha)$, $y \in S(f,\alpha)$. By invexity of $S(f,\alpha)$ we have $x + \lambda\eta(g,x) \in S(f,\alpha)$, thus $f[x + \lambda\eta(y,x)] \le \alpha = \max\{f(x), f(y)\}$, which means the η-cvazi-invexity of f.

Theorem 2. *Let $S \subset \mathbf{R}^n$ be η-invex, and let $f : S \to \mathbf{R}$ be η-cvazi-invex function. Let us suppose that the function η has the following property:*

$$x \ne y \ \Rightarrow \ \eta(x,y) \ne 0. \tag{13}$$

Then all strict-local minimum point of f is a strict global minim point of f.

Proof. Let y be a strict local minim point, which is not global, then there exists $x^* \in S$ with $f(x^*) < f(y)$. But f being η-cvazi-invex, we have $f(y+\lambda\eta(x^*,y)) \le f(y)$, where $y + \lambda\eta(x^*,y) \in S$. This gives a contradiction with the assumption on y.

We now introduce a new class of functions, namely the class of η-pseudo-invex functions.

Definition 6. ([9]) Let $S \subset \mathbf{R}^n$ be an η-invex set. We say that the function $f : S \to \mathbf{R}$ si **η-pseudo-invex** if for all $x, y \in S$ with $f(y) < f(x)$ there exists $c > 0$ and $\alpha \in (0,1]$

such that for all $a \in (0, \alpha)$ we have the inequality

$$f(x + a\eta(y, x)) \leq f(x) - ac. \tag{14}$$

Proposition 9. *If f is η-invex, then it is η-pseudo-invex, too.*

Proof. Indeed, one has

$$f(x + \lambda\eta(y, x)) \leq \lambda f(y) + (1 - \lambda)f(x) = f(x) - \lambda[f(x) - f(y)].$$

Let now $c := f(x) - f(y) > 0$ and put $\alpha := 1$. Then inequality (14) is valid.

We can consider functions $f : M \to \mathbf{R}$ with $M \subset \mathbf{R}^n$ a nonvoid set, and $S \subset M$. We say that $x^0 \in S$ is a **local-minim** point of f **relatively** to S if there exists a vecinity $V \in \mathcal{V}(x^0)$ such that for all $x \in S \cap V$ we have $f(x^0) \leq f(x)$.

Theorem 3. Let $f : M \to \mathbf{R}$, ($S \subset M$ defined as above) an η-pseudo-invex function on the invex set S. If $x^0 \in S$ is a local-minim point of f relatively to S, then x^0 is a global-minim point of f relatively to S.

Proof. There exists $V \in \mathcal{V}(x^0)$ such that for all $x \in S \cap V$ we have $f(x^0) \leq f(x)$. Let $B(x^0, r)$ be a ball inscribed in V, with $r > 0$. Thus for all $x \in S \cap B(x^0, r)$ we have

$$f(x^0) \leq f(x). \tag{15}$$

Let us suppose now, on the contrary, that there exists $y \in S$ with $f(y) < f(x^0)$. The function f being η-pseudo-invex, there exists $c > 0$ and $\alpha \in (0, 1]$ such that for all $a \in (0, \alpha)$ we have

$$f(x^0 + a\eta(y, x^0)) < f(x^0) \tag{16}$$

where $y \neq x^0$. Let a_0 be selected such that

$$0 < a_0 < \frac{r}{||\eta(y, x^0)||}$$

(which is possible since $\eta(y, x^0) \neq 0$ for $y \neq x^0$). Put $z := x^0 + a_0\eta(y, x^0)$. From (11) we get $f(z) < f(x^0)$. On the other hand, we have

$$||z - x^0|| = a_0||\eta(y, x^0)|| < r,$$

thus $z \in B(x^0, r)$. Clearly $z \in S$ (which is η-invex), so via (15) we obtain that $f(x_0) < f(z)$, a contradiction to $f(z) < f(x^0)$. This contradiction finishes the proof of the theorem.

C. We now introduce the **Jensen-invex** sets and functions.

Definition 7. The set $U \subset \mathbf{R}^n$ will be called **η-Jensen-invex** (where, as usual, $\eta : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is given) if for all $x, y \in U$ we have $y + \frac{1}{2}\eta(x,y) \in U$.

If U is η-Jensen-invex set, then the function $f : U \to \mathbf{R}$ will be called **η-Jensen-invex function** (or η-J-invex) if

$$f\left(y + \frac{1}{2}\eta(x,y)\right) \leq \frac{f(x)+f(y)}{2} \text{ for all } x, y \in U. \tag{17}$$

Remark. The η-J-invex sets (or functions) could also be named $\frac{1}{2} - \eta$-invex, as we have selected from $\lambda \in [0,1]$ the set $\lambda \in \left\{\frac{1}{2}\right\}$. The J-convex functions are J-invex for $\eta(x,y) = x - y$.

Proposition 10. *Let* $f : \mathbf{R}^n \to \mathbf{R}$ *be invex related to* η *and let us suppose that* η *satisfies the functional equation*

$$\eta\left(x, \frac{x+y}{2}\right) + \eta\left(y, \frac{x+y}{2}\right) = 0 \quad (x, y \in \mathbf{R}^n). \tag{18}$$

Then f *is J-convex function.*

Proof. $f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) = \left[f(x) - f\left(\frac{x+y}{2}\right)\right] + \left[f(y) - f\left(\frac{x+y}{2}\right)\right] \geq$

$$\geq \left(\eta\left(x, \frac{x+y}{2}\right)\right)^t \nabla f\left(\frac{x+y}{2}\right) + \left(\eta\left(y, \frac{x+y}{2}\right)\right)^t \nabla f\left(\frac{x+y}{2}\right) = 0$$

on base of (1) and (18). Thus we can deduce that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2},$$

i.e. the J-convexity of f.

The following proposition can be proved in the same manner, and we omit its proof.

Proposition 11. *Let f be as in Proposition 10, and let us assume that η satisfies the functional equation*

$$\eta\left(x+\frac{1}{2}\eta(x,y)\right)+\eta\left(y+\frac{1}{2}\eta(x,y)\right)=0 \quad (x,y\in\mathbf{R}^n). \tag{19}$$

Then the function f is η-J-invex.

The J-invexity and the continuity of functions of a variable is contained in the following:

Proposition 12. *Let $f:\mathbf{R}\to\mathbf{R}$ be η-J-invex, where $\eta:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ satisfies the following conditions:*

$$\eta(x,x_n)\nearrow 0 \text{ if } x_n\nearrow x,$$

and

$$\eta(x,x_n)\searrow 0 \text{ if } x_n\searrow x \quad (n\to\infty).$$

Let us suppose that there exist two sequences (x_n), (y_n), where $x_n\nearrow x_0$, $y_n\searrow x_0$ $(n\to\infty)$, $x_0\in\mathbf{R}$ such that

$$\lim_{n\to\infty} f\left[x_n+\frac{1}{2}\eta(x_n,y_n)\right]=f(x_0).$$

If there exist the (lateral) limits $f(x_0-)$ and $f(x_0+)$, then f is continuous at x_0.

Proof. In the inequality

$$f\left(x+\frac{1}{2}\eta(x,y)\right)\le\frac{f(x)+f(y)}{2}$$

put $x:=x_n$, $y:=y_n$. From the given conditions we obtain

$$f(x_0)\le\frac{f(x_0-)+f(x_0+)}{2}. \tag{20}$$

Let now $x:=x_0$, $y:=x_n$ in relation (17). From $x_0+\frac{1}{2}\eta(x_0,x_n)\nearrow x_0$ we get

$$f(x_0-)\le\frac{f(x_0)+f(x_0-)}{2},$$

or

$$f(x_0-)\le f(x_0). \tag{21}$$

Let now $y := y_n$, $x := x_0$ in (17). As above, we can deduce:

$$f(x_0+) \leq f(x_0). \tag{22}$$

From (20), (21), (22) we can deduce $f(x_0) = f(x_0-) = f(x_0+)$, yielding the continuity of f at x_0.

D. Lastly, we deal with **almost-invex** functions.

Definition 8. Let $f : S \subset \mathbf{R}^n \to \mathbf{R}$, where S is an η-invex set.

We say that the function f is **η-almost invex** if

$$f(y + \lambda\eta(x,y)) \leq \lambda f[y + \eta(x,y)] + (1-\lambda)f(y) \tag{23}$$

holds true for all $x, y \in S$.

Remark. The name "almost invex" follows from the observation that (23) may be written also as

$$f(x + \lambda\eta(x,y)) \leq \lambda f(x) + (1-\lambda)f(y) + \lambda g(x,y)$$

where $g(x,y) = f(y + \eta(x,y)) - f(x)$.

In the case of $g(x,y) \equiv 0$ we obtain the classical η-invex functions.

The convex functions are almost invex, as we shall see.

Definition 9. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a differentiable function. We say that ∇f is **η-increasing** function, if

$$(\eta(x,y))^t(\nabla f(x) - \nabla f(y)) \geq 0, \ \forall \ x, y \in \mathbf{R}^n. \tag{24}$$

Proposition 13. *If the function $f : \mathbf{R}^n \to \mathbf{R}$ is invex related to η, and if η is an antisymmetric function, then ∇f is η-increasing.*

Proof. We have $f(x) - f(y) \geq (\eta(x,y))^t\nabla f(y)$ and $f(y) - f(x) \geq (\eta(y,x))^t\nabla f(x)$. From $\eta(y,x) = -\eta(x,y)$, and by addition we easily get $0 \geq (\eta(x,y))^t[\nabla f(y) - \nabla f(x)]$, so (24) follows.

We now prove:

Theorem 4. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be differentiable, and ∇f an η-increasing function. Then f is η-almost-invex function.*

Proof. Let us introduce the function $\phi : [0,1] \to \mathbf{R}$ by $\phi(x) = f(y + \lambda\eta(x,y))$. If $0 \le \lambda_1 < \lambda_2 \le 1$, put $u_1 = y + \lambda_1\eta(x,y)$ and $u_2 = y + \lambda_2\eta(x,y)$. Thus $(u_2 - u_1) = (\lambda_2 - \lambda_1)\eta(x,y)$. From the η-monotonicity of ∇f we can write $0 \le (u_2 - u_1)^t(\nabla f(u_2) - \nabla f(u_1)) = (\lambda_2 - \lambda_1)(\eta(x,y))^t[\nabla f(u_2) - \nabla f(u_1)]$. On the other hand we have $\phi'(\lambda_1) = h^t\nabla f(u_1) \le h^t\nabla f(u_2)$, where $h = \eta(x,y)$. Thus the application ϕ' is increasing, so this function of a single variable is convex. Therefore we have $\phi(\lambda \cdot 1 + (1-\lambda)0) \le \lambda\phi(1) + (1-\lambda)\phi(0) = \lambda f(y + \eta(x,y)) + (1-\lambda)f(y)$, i.e. the η-almost-invexity of f.

Remark. The application $\phi(\lambda) = f(y + \lambda\eta(x,y))$ introduced above has the properties $\phi(0) = f(y)$, $\phi(1) = f(y+\eta(x,y))$. Thus if $g : [0,1] \to \mathbf{R}$ is convex, then f is η-almost-invex. If the application ϕ is cvaziconvex, i.e. $g(x) \le \max\{g(0), g(1)\}$, $\forall\ x \in [0,1]$, we can obtain the notion of **almost-cvazi-invexity**. Thus $f : S \to \mathbf{R}$ will be η-**almost-cvazi-invex** if

$$f(y + \lambda\eta(x,y)) \le \max\{f(y), f(y + \eta(x,y))\}.$$

However, we do not study this class of functions here.

References

[1] M.A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. **80**(1981), 545-550.

[2] M.A. Hanson, B. Mond, *Further generalizations of convexity in mathematical programming*, J. Inform. Optim. Sci. **3**(1982), 25-32.

[3] B.D. Craven, *Invex functions and constrained local minima*, Bull. Austral. Math. Soc. **24**(1981), 357-366.

[4] B.D. Craven, *Duality for generalized convex fractional programs*, Generalized concavity in optimization and economics, 473-489 Proc. NATO Conf., Univ. of British Columbia, Vancouver, 1980, Academic Press, 1981.

[5] B.D. Craven, B.M. Glover, *Invex functions and duality*, J. Austral. Math. Soc. Ser. A **39**(1985), 1-20.

[6] V. Jeyakumar, *On optimality conditions in nonsmooth inequality constrained minimization*, Numer. Funct. Anal. Optim. **9**(1987), 535-546.

[7] D.H. Martin, *The essence of invexity*, J. Optim. Theory Appl. **47**(1985), 65-76.

[8] J. Sándor, *Despre funcţiile invexe*, Simpozion cu probleme de optim. în înv., cercetare, prod., matematică, informatică, inventică, 2-3 iunie 1989, Odorheiu-Secuiesc.

[9] J. Sándor, *Generalized invexity and its applications in optimization theory*, First Joint Conf. on Modern Applied Math. Ilieni, 12-17 June 1995.

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