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A GENERALIZATION OF BECKER'S UNIVALENCE CRITERION

IRINEL RADOMIR

Abstract. In the paper there is presented a sufficient univalence conditions for functions of a complex variable f, verifying the conditions $f(0) = 0$, $f'(0) = 1$. Our condition is a generalization of Becker's univalence criterion.

1. Introduction

Let A be the class of functions f, which are analytic in the unit disk $U = \{z \in C \, , |z| < 1 \}$, with $f(0) = 0$ and $f'(0) = 1$.

Theorem 1.1. ([2]). *Let $f \in A$. If for all $z \in U$*

$$(1 - |z|^2)|\frac{zf''(z)}{f'(z)}| \leq 1 \tag{1}$$

then the function f is univalent in U.

In order to prove the main results we shall need the theory of Loewner chains.

2. Preliminaries

We denote by U_r the disk of z-plane, $U_r = \{ z \in C : |z| < r \}$, where $r \in (0, 1]$ and $I = [0, \infty)$.

Definition. *A function $L(z,t) : U \times I \to C$ is called a Loewner chain if*

$$L(z,t) = e^t z + a_2(t) z^2 + \dots \qquad |z| < 1,$$

is analytic and univalent in U for each $t \in I$ and if $L(z,s) \prec L(z,t)$, $0 \leq s < t < \infty$, where by $\prec$ we denote the relation of subordination.

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Theorem 2.1. (3). *Let r be a real number, $r \in (0,1]$. Let $L(z,t) = a_1(t)z+a_2(t)z^2+\dots$, $a_1(t) \neq 0$ be analytic in U_r for all $t \in I$, locally absolutely continuous in I and locally uniform with respect to U_r. For almost all $t \in I$ suppose*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t} \quad \textit{for all } z \in U_r \, ,$$

where $p(z,t)$ is analytic in U such that $Re\, p(z,t) > 0$ for $z \in U$, $t \in I$.

If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then $L(z,t)$ has, for each $t \in I$, an analytic and univalent extension to the whole disk U.

3. Main results

Theorem 3.1. *Let $f(z) = z + a_2z^2 + \dots$ and $g(z) = z + b_2z^2 + \dots$ be analytic functions in U. If for all $z \in U$*

$$\left| \frac{zf''(z)}{f'(z)} - |z|^2\frac{zg''(z)}{f'(z)} \right| \leq 1 \, , \quad \textit{and} \tag{2}$$

$$\left| \frac{z(\, f(z) - g(z)\,)''}{f'(z)} \right| \leq 1 \, , \tag{3}$$

then the function $g(z) + z(\, f(z) - g(z)\,)'$ is univalent in U.

Proof. We consider the function $L : U \times I \to C$ defined from

$$L(z,t) = (e^t z)f'(e^t z) - \int_0^{e^{-t}z} ug''(u)du \tag{4}$$

Because the functions f and g are analytic in U it results that the function $L(z,t)$ is analytic in U for all $t \in I$. From (4) we obtain

$$L(z,t) = e^t z + a_2(t)z^2 + \dots$$

In order to prove that $\{ \, L(z,t)/e^t \, \}$ forms a normal family in U, it is sufficient to observe that there exist positive numbers k_1, k_2 such that

$$|\, f'(z) \,| \leq k_1 \quad \textit{and} \quad |\, \int_0^z ug''(u)du \,| \leq k_2 \, ,$$

for all $z \in U_r$, $r \in (0,1]$. Therefore we have $| L(z,t)/e^t | \leq k_1 + k_2$ for all $z \in U_r$ and $t \in I$.

We consider the function $p: U_r \times I \to C$ defined by

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} / \frac{\partial L(z,t)}{\partial t} \tag{5}$$

In order to prove that the function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \in I$ it is sufficient to show that the function

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1} \qquad z \in U_r\ , \tag{6}$$

can be continued analytically in U and that

$$| w(z,t) | < 1 \qquad (\forall) z \in U,\ \ t \geq 0.$$

From (4), (5) and (6) we obtain

$$w(z,t) = e^{-t}z\frac{f''(e^{-t}z)}{f'(e^{-t}z)} - e^{-3t}z\frac{g''(e^{-t}z)}{f'(e^{-t}z)} \tag{7}$$

From (3) it results that $f'(z) \neq 0$ for all $z \in U$ and hence the function $w(z,t)$ is analytic in U for all $t \in I$. We have

$$w(z,0) = \frac{z(\ f(z) - g(z)\ )''}{f'(z)}$$

and from (3) it results that $|w(z,0)| \leq 1$ for all $z \in U$. Also we have $w(0,t) = 0 < 1$. If $z \in U$, $z \neq 0$ and $t > 0$, we observe taht the function $w(z,t)$ is analytic in $\bar{U}$,because $|e^{-t}z| \leq e^{-t} < 1$ for all $z \in \bar{U}$. Using the maximum principle, for all $z \in U$ and $t > 0$, we have

$$| w(z,t) | < \max_{|\zeta|=1} |w(\zeta,t)| = |w(e^{i\theta},t)|\ , \tag{8}$$

where $\theta = \theta(t)$ is a real number. Let us denote $u = e^{-t}e^{i\theta}$. Then $|u| = e^{-t}$ and from (7) we obtain

$$w(e^{i\theta},t) = \frac{uf''(u)}{f'(u)} - |u|^2\frac{ug''(u)}{f'(u)} \tag{9}$$

Since $|u| < 1$, from (2), (8) and (9) it results that $|w(z,t)| < 1$ for all $z \in U$, $t \geq 0$. It follows that $L(z,t)$ is a Loewner chain and hence the function $L(z,0) = g(z) + z \cdot (f(z) - g(z))'$ is univalent in U.

Remark. If $g(z) = f(z)$, from Theorem 3.1 we obtain Theorem 1.1.

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DEPARTMENT OF MATHEMATICS, "TRANSILVANIA" UNIVERSITY, BRAŞOV, ROMA-, NIA