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ABOUT AN INTEGRAL OPERATOR PRESERVING THE UNIVALENCE

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Abstract. In this work an integral operator is studied and the author determines conditions for the univalence of this integral operator.

1. Introduction

Let A be the class of the functions f which are analytic in the unit disc $U = \{z \in C; |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by S the class of the function $f \in A$ which are univalent in U.

Many authors studied the problem of integral operators which preserve the class S. In this sense an important result is due to J. Pfaltzgraff [4].

Theorem A ([4]). *If f(z) is univalent in U, α a complex number and $|\alpha| \leq \frac{1}{4}$, then the function*

$$G_\alpha(z) = \int_o^z [f'(\xi)]^\alpha d\xi \tag{1}$$

is univalent in U.

Theorem B ([3]). *If the function $g \in S$ and α is a complex number, $|\alpha| \leq \frac{1}{4n}$, then the function defined by*

$$G_{\alpha,n}(z) = \int_o^z [g'(u^n)]^\alpha du \tag{2}$$

is univalent in U for all positive integer n.

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2. Preliminaries

For proving our main result we will need the following theorem and lemma.

Theorem C ([1]). *If the function f is regular in the unit disc U, $f(z) = z + a_2 z^2 + \ldots$ and*

$$\left(1 - |z|^2\right) \left| \frac{z f''(z)}{f'(z)} \right| \leq 1 \tag{3}$$

for all $z \in U$, then the function f is univalent in U.

Lema Schwarz ([2]). *If the function g is regular in U, $g(0) = 0$ and $|g(z)| \leq 1$ for all $z \in U$, then the following inequalities hold*

$$|g(z)| \leq |z| \tag{4}$$

for all $z \in U$, and $|g'(0)| \leq 1$, the equalities (in inequality (4) for $z \neq 0$) hold only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

3. Main result

Theorem 1. *Let γ be a complex number and the function $g \in A$, $g(z) = z + a_2 z^2 + \ldots$. If*

$$\left| \frac{g''(z)}{g'(z)} \right| \leq \frac{1}{n} \tag{5}$$

for all $z \in U$ and

$$|\gamma| \leq \frac{1}{\left(\frac{n}{n+2}\right)^{\frac{n}{2}} \frac{2}{n+2}} \tag{6}$$

then the function

$$G_{\gamma,n}(z) = \int_0^z \left[g'(u^n)\right]^\gamma du \tag{7}$$

is univalent in U for all $n \in N^* - \{1\}$.

Proof. Let us consider the function

$$f(z) = \int_0^z \left[g'(u^n)\right]^\gamma du. \tag{8}$$

The function

$$h(z) = \frac{1}{\gamma}\frac{f''(z)}{f'(z)}, \tag{9}$$

where the constant γ satisfies the inequality (6) is regular in U.

From (9) and (8) it follows that

$$h(z) = \frac{\gamma}{|\gamma|}\left[\frac{nz^{n-1}g''(z^n)}{g'(z^n)}\right]. \tag{10}$$

Using (10) and (5) we have

$$|h(z)| \leq 1, \tag{11}$$

for all $z \in U$. From (10) we obtain $h(0) = 0$ and applying Schwarz-Lemma we have

$$\frac{1}{|\gamma|}\left|\frac{f''(z)}{f'(z)}\right| \leq |z|^{n-1} \leq |z| \tag{12}$$

for all $z \in U$, and hence, we obtain

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq |\gamma|\left(1-|z|^2\right)|z|^n. \tag{13}$$

Let us consider the function Q:[0, 1] $\to R$, $Q(x) = \left(1-x^2\right)x^n$; $x = |z|, z \in U$, which has a maximum at a point $x = \sqrt{\frac{n}{n+2}}$, and hence

$$Q(x) < \left(\frac{n}{n+2}\right)^{\frac{n}{2}}\frac{2}{n+2} \tag{14}$$

for all $x \in (0,1)$. Using this result and (13) we have

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq |\gamma|\left(\frac{n}{n+2}\right)^{\frac{n}{2}}\frac{2}{n+2}. \tag{15}$$

From (15) and (6) we obtain

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{16}$$

for all $z \in U$. From (16) and (8) and Theorem C it follows that $G_{\gamma,n}$, is in the class S. □

Remark. For $n = 2$, we obtain $|\gamma| \leq 4$ and the function $G_{\gamma,2}$ is in the class S.

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