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ON HARDY-OPIAL TYPE INTEGRAL INEQUALITIES

B.G. PACHPATTE

Abstract. The aim of the present paper is to establish some new integral inequalities of the Hardy-Opial type involving functions and their derivatives. The analysis used in the proofs is elementary and our results provide new estimates on these types of inequalities.

1. Introduction

This paper is concerned with the integral inequalities of the following type

$$\int_a^b s|u|^p dx \leq \int_a^b r|u'|^p dx, \tag{1}$$

$$\int_a^b s|u|^p|u'|dx \leq \int_a^b r|u'|^{p+1} dx, \tag{2}$$

where s and r will usually positive continuous functions on the open interval (a, b), p is a suitable constant, and the inequalities will hold for $u \in C^1(a, b)$ which satisfy certain other conditions. The inequalities of the forms (1) and (2) are called Hardy and Opial type inequalities, see [3, p.706]. A great many papers have been written dealing with integral inequalities of the type (1) and (2), probably so by the challenge of research in various branches of mathematics, where such inequalities are often the basis for proving various theorems or approximating various functions. Excellent surveys of the work on such inequalities together with many references are contained in the books by Beckenbach and Bellman [2], Hardy, Littlewood and Polya [5], Mitrinovic [6] and the papers by Beesack [3], Shum [12] and the present author [9-11]. In this paper we establish a number of new integral inequalities involving functions and their derivatives which claim their origin to the Hardy and Opial type inequalities given in

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(1) and (2). Our proofs are elementary and the inequalities developed here provide new estimates on these types of inequalities.

2. Statement of results

In this section we state our main results to be proved in this paper. In what follows, we denote by R, the set of real numbers and let $J = [a, b]$, $a < b$ for $a, b \in \mathrm{R}$.

Our first theorem deals with the inequalities in which the constants appearing do not depend on the size of the domain of definition of the function.

Theorem 1. *Let u be a real-valued continuously differentiable function defined on J such that $u(a) = u(b) = 0$.*

(a_1) *If α, p, q be nonnegative real numbers such that $q \geq 1$ and $A = (p + q)/(\alpha + 1)$, then*

$$\int_a^b |t|^{\alpha}|u(t)|^{p+q}dt \leq A^q \int_a^b |t|^{\alpha+q}|u(t)|^p|u'(t)|^q dt, \tag{3}$$

$$\int_a^b |t|^{\alpha}|u(t)|^{p+q}dt \leq A^{p+q} \int_a^b |t|^{\alpha+p+q}|u'(t)|^{p+q} dt. \tag{4}$$

(a_2) *If α, p, q, r be nonnegative real numbers such that $q + e \geq 1$ and $B = (p + q + r)/(\alpha + 1)$, then*

$$\int_a^b |t|^{\alpha+r}|u(t)|^{p+q}|u'(t)|^r dt \leq B^q \int_a^b |t|^{\alpha+q+r}|u(t)|^p|u'(t)|^{q+r} dt, \tag{5}$$

$$\int_a^b |t|^{\alpha+r}|u(t)|^{p+q}|u'(t)|^r dt \leq B^{p+q} \int_a^b |t|^{\alpha+p+q+r}|u'(t)|^{p+q+r} dt. \tag{6}$$

Remark 1. It is interesting to note that the inequalities obtained in (3) and (6) are similar to that of the Opial's type inequality given in (2), see also [7-10]. The inequality (3) yields the lower bound on the integral of the form arising on the left side of (2). The inequality obtained in (4) is analogous to the Hardy's type inequality given in (1) and the inequality established in (5) is different from both the inequalities in (1) and (2).

In the following theorems we establish the inequalities in which the constants appearing depend on the size of the domain of definition of the function.

Theorem 2. *Let u be a real-valued continuously differentiable function defined on J such that $u(a) = u(b) = 0$.*

(b_1) *If $p \geq 0$, $g \geq 1$ be real numbers and $K = (p+q)(b-a)/2$, then*

$$\int_a^b |u(t)|^{p+q} dt \leq K^q \int_a^b |u(t)|^p |u'(t)|^q dt, \tag{7}$$

$$\int_a^b |u(t)|^{p+q} dt \leq K^{p+q} \int_a^b |u'(t)|^{p+q} dt. \tag{8}$$

(b_2) *If p, q, r be nonnegative real numbers such that $q + r \geq 1$ and $L = (p+q+r)(b-a)/2$, then*

$$\int_a^b |u(t)|^{p+q} |u'(t)^r dt \leq L^q \int_a^b |u(t)|^p |u'(t)|^{q+r} dt, \tag{9}$$

$$\int_a^b |u(t)|^{p+q} |u'(t)|^r dt \leq L^{p+q} \int_a^b |u'(t)|^{p+q+r} dt. \tag{10}$$

Remark 2. We note that the inequalities obtained in (7) and (10) are similar to that of the Opial's type inequality given in (2) which in turn yields lower and upper bounds on the integral of the form involved on the left side of (2). The inequality (8) is analogous to the Hardy's type inequality given in (1), while the inequality obtained in (9) is different from those of the inequalities in (1) and (2).

Theorem 3. *Let u be a real-valued twice continuously differentiable function defined on J such that $u(a) = u(b) = 0$.*

(c_1) *If p_0, p_1, p_2, p_3 be nonnegative real numbers such that*

$$p_3 > 1, \quad p_3 \geq p_1, \quad p_1 + p_2 + p_3 - (p_3 - p_1)/(p_3 - 1) \geq 0,$$

and

$$M = [(p_1 + p_2 + p_3 - 1)/(p_0 + 1)]^{p_3} [(p_0 + p_1 + p_2 + p_3)(b-a)/2]^{p_3 - p_1},$$

then

$$\int_a^b |u(t)^{p_0}|u'(t)|^{p_1+p_2+p_3}dt \leq M \int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3}dt, \tag{11}$$

(c_2) *If p_0, p_1, p_2, p_3, p_4 be nonnegative real numbers such that*

$$p_3 > 1, \quad p_0p_3 - p_1p_4 \geq 0, \quad p_3 + p_4 \geq p_1 + (p_1p_4)/p_3,$$

$$\sum_{i=1}^{4} p_i + (p_1p_4)/p_3 - [(p_3 + p_4 - (p_1p_4)/p_3)]/(p_3 + p_4 - 1)] \geq 0,$$

and

$$N = \left[\left(\sum_{i=1}^{4} p_i + (p_1p_4)/p_3 - 1\right)/(p_0 - (p_1p_4)/p_3 + 1\right]^{p_3} \times \left[\left(\sum_{i=0}^{4} p_i\right)(b-a)/2\right]^{(p_3-p_1)},$$

then

$$\int_a^b |u(t)|^{p_0}|u'(t)|^{p_1+p_2+p_3}|u''(t)|^{p_4}dt \leq N \int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3+p_4}dt. \tag{12}$$

Remark 3. It is easy to observe that the inequality obtained in (11) is analogous to the Opial's type inequality given in (2), which yields a new upper bound on the integral of the form arising on the left side of (2). The inequality (12) is different from those of the inequalities given in (1) and (2).

3. Proof of Theorem 1

(a_1) By rewriting the integral on the left side of (3) and making use of the integration by parts, the fact that $u(a) = u(b) = 0$ and the Hölder's inequality with indices $q, q/(q-1)$ we observe that

$$\int_a^b |t|^\alpha |u(t)|^{p+q}dt = \frac{1}{\alpha+1}\int_a^b \frac{d}{dt}(|t|^{\alpha+1} sgn\, t)|u(t)|^{p+q}dt = \tag{13}$$

$$= -A\int_a^b |t|^{\alpha+1} sgn\, t|u(t)|^{p+q-1}u'(t) sgn\, u(t)dt \leq A\int_a^b |t|^{\alpha+1}|u(t)|^{p+q-1}|u'(t)|dt =$$

$$= A\int_a^b [|t|^{\alpha+1-\alpha(q-1)/q}|u(t)|^{p/q}|u'(t)|] \times [|t|^{\alpha(q-1)/q}|u(t)|^{p+q-1-(p/q)}]dt \leq$$

$$\leq A\left[\int_a^b |t|^{\alpha+q}|u(t)|^p|u'(t)|^q dt\right]^{1/q} \times \left[\int_a^b |t|^\alpha |u(t)|^{p+q}dt\right]^{(q-1)/q}.$$

If $\int_a^b |t|^\alpha |u(t)|^{p+q} dt = 0$, then (3) is trivially true, otherwise dividing both sides of (13) by $\left[\int_a^b |t|^\alpha |u(t)|^{p+q} dt\right]^{(q-1)/q}$ and then taking the qth power on both sides of the resulting inequality we get the required inequality in (3).

Rewriting the integral on the right side of (3) and using the Hölder's inequality with indices $(p+q)/p, (p+q)/q$ we observe that

$$\int_a^b |t|^\alpha |u(t)|^{p+q} dt \le A^q \int_a^b [|t|^{(\alpha p)/(p+q)} |u(t)|^p] \times [|t|^{\alpha+q-(\alpha p)/(p+q)} |u'(t)|^q] dt \le \quad (14)$$

$$\le A^q \left[\int_a^b |t|^\alpha |u(t)|^{p+q} dt\right]^{p/(p+q)} \times \left[\int_a^b |t|^{\alpha+p+q} |u'(t)|^{p+q} dt\right]^{q/(p+q)}.$$

Now by following the arguments as in the last part of the proof of inequality (3) with suitable modifications, we get the required inequality in (4).

(a_2) By rewriting the integral on the left side of (5) and using the Hölder's inequality with indices $(q+r)/r, (q+r)/q$ and the inequality (4), we observe that

$$\int_a^b |t|^{\alpha+r} |u(t)|^{p+q} |u'(t)^r dt = \quad (15)$$

$$= \int_a^b [|t|^{\alpha-(\alpha q)/(q+r)} |u(t)|^{(pr)/(q+r)} (|t||u'(t)|)^r] \times [|t|^{(\alpha q)/(q+r)} |u(t)|^{p+q-(pr)/(q+r)}] dt \le$$

$$\le \left[\int_a^b |t|^{\alpha+q+r} |u(t)|^p |u'(t)|^{q+r} dt\right]^{r/(r+q)} \times \left[\int_a^b |t|^\alpha |u(t)|^{p+q+r} dt\right]^{q/(q+r)} \le$$

$$\le \left[\int_a^b |t|^{\alpha+q+r} |u(t)|^p |u'(t)|^{q+r} dt\right]^{r/(q+r)} \times \left[B^{q+r} \int_a^b |t|^{\alpha+q+r} |u(t)|^p |u'(t)|^{q+r} dt\right]^{q/(q+r)} =$$

$$= B^q \int_a^b |t|^{\alpha+q+r} |u(t)|^p |u'(t)|^{q+r} dt.$$

This completes the proof of inequality (5).

Rewriting the integral on the right side of (5) and using the Hölder's inequality with indices $(p+q)/p, (p+q)/q$ we observe that

$$\int_a^b |t|^{\alpha+r} |u(t)|^{p+q} |u'(t)|^r dt \le \quad (16)$$

$$\le B^q \int_a^b [|t|^{(\alpha+r)(p/(p+q))} |u(t)|^p |u'(t)|^{(rp)/(p+q)} \times [|t|^{\alpha+q+r-(\alpha+r)(p/(p+q))} |u'(t)|^{q+r-(rp)/(p+q)}] dt \le$$

$$\leq B^q \left[\int_a^b |t|^{\alpha+r} |u(t)|^{p+q} |u'(t)|^r dt \right]^{p/(p+q)} \times \left[\int_a^b |t|^{\alpha+p+q+r} |u'(t)|^{p+q+r} dt \right]^{q/(p+q)} .$$

Now by following the arguments as in the last part of the proof of inequality (3) with suitable modification, we get the required inequality in (6). The proof is complete.

4. Proof of Theorem 2

(b_1) From the hypothesis of Theorem 2 we have

$$u^{p+q}(t) = (p+q) \int_a^b u^{p+q-1}(s) u'(s) ds = -(p+q) \int_t^b u^{p+q-1}(s) u'(s) ds. \tag{17}$$

From (17) we observe that

$$|u(t)|^{p+q} \leq [(p+q)/2 \int_a^b |u(s)|^{p+q-1} |u'(s)| ds. \tag{18}$$

Integrating both sides of (18) from a to b, and rewriting the right side of the resulting inequality and then using the Hölder's inequality with indices $q, q/(q-1)$ we have

$$\int_a^b |u(t)|^{p+q} dt \leq K \int_a^b |u(t)|^{p+q-1} |u'(t)| dt = \tag{19}$$

$$= K \int_a^b [|u(t)|^{p/q} |u'(t)|][|u(t)|^{p+q-1-(p/q)}] dt \leq$$

$$\leq K \left[\int_a^b |u(t)|^p |u'(t)|^q dt \right]^{1/q} \left[\int_a^b |u(t)|^{p+q} dt \right]^{(q-1)/a} .$$

If $\int_a^b |u(t)|^{p+q} = 0$, then (7) is trivially true, otherwise dividing both sides of (19) by $\left[\int_a^b |u(t)|^{p+q} dt \right]^{(q-1)/q}$ and then taking the qth power on both sides of the resulting inequality, we get the required inequality in (7).

By using the Hölder's inequality with indices $(p+q)/p, (p+q)/q$ on the right side of (7) and following the arguments as in the last part of the proof of inequality (7) with suitable changes, we get the required inequality in (8).

(b_2) Rewriting the integral on the left side of (9) and using the Hölder's inequality with indices $(q+r)/r$, $(q+r)/q$ and the inequality (7), we observe that

$$\int_a^b |u(t)|^{p+q}|u'(t)|^r dt = \int_a^b [|u(t)|^{(pr)/(q+r)}|u'(t)|^r][|u(t)|^{p+q-(pr)/(q+r)}]dt \leq \tag{20}$$

$$\leq \int_a^b |u(t)|^p|u'(t)|^{q+r} dt]^{r/(q+r)} \times \left[\int_a^b |u(t)|^{p+q+r} dt\right]^{q/(q+r)} \leq$$

$$\leq \left[\int_a^b |u(t)|^p|u'(t)|^{q+r} dt\right]^{r/(q+r)} \times \left[K^{q+r}\int_a^b |u(t)|^p|u'(t)|^{q+r} dt\right]^{q/(q+r)} =$$

$$= K^q \int_a^b |u(t)|^p|u'(t)|^{q+r} dt.$$

This is the required inequality in (9).

The details of the proof of inequality (10) are very close to that of the proof of inequality (6) given above with suitable changes and hence we omit it here. The proof is complete.

5. Proof of Theorem 3

(c_1) Be rewriting the integral on the left side of (11) and making use of the integration by parts, the fact that $u(a) = u(b) = 0$, the Hölder's inequality with indices $p_3, p_3/(p_3-1)$ and the inequality (9), we observe that

$$\int_a^b |u(t)|^{p_0}|u'(t)|^{p_1+p_2+p_3} dt = \tag{21}$$

$$= \frac{1}{p_0+1}\int_a^b \left[\frac{d}{dt}(|u(t)|^{p_0+1} sgn\, u(t)\right] \times \left[|u'(t)|^{p_1+p_2+p_3-1} sgn\, u'(t)\right] dt =$$

$$= -\left(\frac{p_1+p_2+p_3-1}{p_0+1}\right)\int_a^b |u(t)|^{p_0+1} sgn\, u(t)|u'(t)|^{p_1+p_2+p_3-1} \times u''(t)(sgn\, u'(t))^2 dt \leq$$

$$\leq \left(\frac{p_1+p_2+p_3-1}{p_0+1}\right)\int_a^b |u(t)|^{p_0+1}|u'(t)|^{p_1+p_2+p_3-2}|u''(t)|dt =$$

$$= \left(\frac{p_1+p_2+p_3-1}{p_0+1}\right)\int_a^b [|u(t)|^{(p_0+p_1)/p_3}|u'(t)|^{p_2/p_3}|u''(t)|]\times$$

$$\times[|u(t)|^{p_0+1-(p_0+p_1)/p_3}|u'(t)|^{p_1+p_2+p_3-2-(p_2/p_3)}]dt \leq$$

$$\leq \left(\frac{p_1+p_2+p_3-1}{p_0+1}\right)\left[\int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3} dt\right]^{1/p_3} \times$$

$$\times \left[\int_a^b |u(t)|^{p_0+(p_3-p_1)/(p_3-1)} \times |u'(t)|^{p_1+p_2+p_3-(p_3-p_1)/(p_3-1)} dt\right]^{(p_3-1)/p_3} \leq$$

$$\leq \left(\frac{p_1+p_2+p_3-1}{p_0+1}\right) \left[\int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3} dt\right]^{1/p_3} \times$$

$$\times[(p_0+p_1+p_2+p_3)(b-a)/2]^{(p_3-p_1)/p_3} \times \left[\int_a^b |u(t)|^{p_0}|u'(t)|^{p_1+p_2+p_3} dt\right]^{(p_3-1)/p_3}.$$

Now by following the arguments as in the last part of the proof of inequality (7) with suitable changes, we get the required inequality in (11).

(c_2) Rewriting the integral on the left side of (12) and using the Hölder's inequality with indices $(p_3+p_4)/p_4, (p_3+p_4)/p_3$ and the inequality (11), we observe that

$$\int_a^b |u(t)|^{p_0}|u'(t)|^{p_1+p_2+p_3}|u''(t)|^{p_4} dt = \tag{22}$$

$$= \int_a^b [|u(t)|^{(p_0+p_1)(p_4/(p_3+p_4))}|u'(t)|^{(p_2p_4)/(p_3+p_4)}|u''(t)|^{p_4}]\times$$

$$\times[|u(t)|^{p_0-(p_0+p_1)(p_4/(p_3+p_4))} \times |u'(t)|^{p_1+p_2+p_3-(p_2p_4)/(p_3+p_4)}] dt \leq$$

$$\leq \left[\int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3+p_4} dt\right]^{p_4/(p_3+p_4)} \times$$

$$\times \left[\int_a^b |u(t)|^{p_0-(p_1p_4)/p_3} \times |u'(t)|^{p_1+(p_1p_4)/p_3)+p_2+(p_3+p_4)} dt\right]^{p_3/(p_3+p_4)} \leq$$

$$\leq \left[\int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3+p_4} dt\right]^{p_4/(p_3+p_4)} \times$$

$$\times N \left[\int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3+p_4} dt\right]^{p_3/(p_3+p_4)} =$$

$$= N \int_a^b |u(t)|^{p_0+p_1}|u'(t)|^{p_2}|u''(t)|^{p_3+p_4} dt.$$

This is the required inequality in (12). The proof is complete.

Remark 4. The multidimensional integral inequalities of the type (1) and (2) and their variants are established by many authors in the literature by using different techniques. In particular, in [4] Dubinskii has established the multidimensional inequalities analogues to the inequalities (7), (9), (11) and (12), see also [1], by using the divergence theorem, Young's inequality and imbedding theorems. Here we note that our proofs of Theorems 1-3 are quite elementary and the constants involved in these inequalities provide precise information.

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DEPARTMENT OF MATHEMATICS, MARATHWADA UNIVERSITY, AURANGABAD 431 004, (MAHARASHTRA) INDIA