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ON A CLASS OF VOLTERRA INTEGRAL EQUATIONS WITH DEVIATING ARGUMENT

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Abstract. Existence and data dependence results for some Volterra integral equations with linear deviating of the argument are given.

1. Introduction

Differential-functional equations with linear deviating of the argument have been studied in many papers ([1]-[10], [18], [19],...).

In [9], by using the Picard operators' technique and a suitable Bielecki norm, we have given existence and uniqueness theorems for some Volterra integral equations which contain a linear deviating of the argument.

In this paper we study the existence and the data dependence for the solutions of the following Volterra integral equation with linear deviating of the argument:

$$x(t) = x(0) + \int_0^t f(s, x(\lambda s))ds, \quad t \in [0, b],\ 0 < \lambda < 1.$$

We use the weakly Picard operators' technique, a fixed point theorem given by Rus in [12] and some data dependence results given by Rus and Mureşan in [17].

2. A fixed point theorem

Let (X, d) be a metric space and $A : X \to X$ an operator. We denote by F_A the fixed point set of A, that is

$$F_A := \{x \in X |\ A(x) = x\}.$$

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We have:

Theorem 2.1. (Rus [12]) *Let (X, d) be a complete metric space and $A : X \to X$ a continuous operator. We suppose that there exists $\alpha \in [0, 1[$ such that*

$$d(A^2(x), A(x)) \leq \alpha d(x, A(x)), \text{ for all } x \in X.$$

Then:

a) $F_A \neq \emptyset$;

b) $A^n(x) \to x^*(x)$ *as* $n \to \infty$, *for all* $x \in X$, *and* $x^*(x) \in F_A$.

3. Volterra integral equations with deviating argument

We consider the following Volterra integral equation with deviating argument:

$$x(t) = x(0) + \int_0^t f(s, x(\lambda s))ds, \quad t \in [0, b], \ 0 < \lambda < 1, \tag{3.1}$$

where $f \in C([0, b] \times \mathbf{R})$.

We have

Theorem 3.1. *We suppose that there exists $L > 0$ such that*

$$|f(s, u) - f(s, v)| \leq L|u - v|, \text{ for all } s \in [0, b] \text{ and all } u, v \in \mathbf{R}.$$

Then the equation (3.1) has solutions in $C[0, b]$.

Proof. Let $(C[0, b], ||\cdot||_B)$ be, where

$$||x||_B = \max_{t\in[0,b]}(|x(t)|e^{-\tau t}), \quad \tau > 0.$$

We consider the operator

$$A : (C[0, b], ||\cdot||_B) \to (C[0, b], ||\cdot||_B),$$

defined by

$$A(x)(t) := x(0) + \int_0^t f(s, x(\lambda s))ds, \quad t \in [0, b], \ 0 < \lambda < 1, \tag{3.2}$$

which is a continuous operator.

This operator is not a contraction.

We have

$$A^2 : (C[0,b], ||\cdot||_B) \to (C[0,b], ||\cdot||_B),$$

$$A^2(x)(t) := x(0) + \int_0^t f\left(s, x(0) + \int_0^{\lambda s} f(u, x(\lambda u))du\right) ds.$$

It follows that

$$|A^2(x)(t) - A(x)(t)| \leq L \int_0^t \left| x(0) - x(\lambda s) + \int_0^{\lambda s} f(u, x(\lambda u))du \right| ds.$$

By denoting $\lambda s = v$, we obtain

$$|A^2(x)(t) - A(x)(t)| \leq \frac{L}{\lambda} \int_0^{\lambda t} \left| x(0) - x(v) + \int_0^{v} f(u, x(\lambda u))du \right| dv =$$

$$= \frac{L}{\lambda} \int_0^{\lambda t} |A(x)(v) - x(v)| e^{-\tau v} e^{\tau v} dv \leq$$

$$\leq \frac{L}{\lambda \tau} ||A(x) - x||_B (e^{\tau \lambda t} - 1) \leq \frac{L}{\lambda \tau} ||A(x) - x||_B e^{\tau t}.$$

Therefore,

$$|A^2(x)(t) - A(x)(t)| e^{-\tau t} \leq \frac{L}{\lambda \tau} ||A(x) - x||_B, \text{ for all } t \in [0,b].$$

So, we have that

$$||A^2(x) - A(x)||_B \leq \frac{L}{\lambda \tau} ||A(x) - x||_B, \text{ for all } x \in C[0,b].$$

We can choose τ so that $\frac{L}{\lambda \tau} < 1$. Let $\tau = \frac{L}{\lambda} + 1$ be.

We denote

$$\frac{\frac{L}{\lambda}}{\frac{L}{\lambda} + 1} = \alpha.$$

Thus

$$||A^{n+1}(x) - A^n(x)||_B \leq \alpha^n ||A(x) - x||_B$$

and

$$||A^{n+p}(x) - A^n(x)||_B \leq \frac{\alpha^n}{1-\alpha} ||A(x) - x||_B, \text{ for all } n \in \mathbf{N} \text{ and all } p \in \mathbf{N},\ p \geq 2.$$

So $(A^n(x))_{n\in\mathbb{N}^*}$ is a Cauchy sequence, for all $x \in C[0,b]$. Because $(C[0,b],d)$, where $d(x,y) = ||x-y||_B$, is a complete metric space, we have that $(A^n(x))_{n\in\mathbb{N}^*}$ is a convergent sequence, for all $x \in C[0,b]$.

We denote $A^\infty(x) = \lim\limits_{n\to\infty} A^n(x)$. From $A^{n+1}(x) = A(A^n(x))$ and the continuity of the operator A we have that $A^\infty(x) \in F_A$, that is $F_A \neq \emptyset$.

So, the equation (3.1) has solutions in $C[0,b]$. □

4. An example of weakly Picard operator

We have

Definition 4.1. (Rus [16]) Let (X,d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator if the sequence $(A^n(x))_{n\in\mathbb{N}^*}$ converges for all $x \in X$ and its limit, denoted by $A^\infty(x)$, is a fixed point of A.

For more details about the Picard operators and the weakly Picard operators see [13]-[16].

Let $(C[0,b], ||\cdot||_C)$ be, where $||x||_C = \max\limits_{t\in[0,b]} |x(t)|$.

We consider the following operator:

$$A : (C[0,b], ||\cdot||_C) \to (C[0,b], ||\cdot||_C),$$

defined by

$$A(x)(t) := x(0) + \int_0^t f(s, x(\lambda s))ds, \quad t \in [0,b],\ 0 < \lambda < 1, \tag{4.1}$$

where f is as in the Theorem 3.1.

We have

Theorem 4.1. *The operator A defined by (4.1) is a weakly Picard operator.*

Proof. We consider $(C[0,b], ||\cdot||_B)$, where

$$||x||_B = \max_{t\in[0,b]} \left(|x(t)|e^{-\left(\frac{L}{\lambda}+1\right)t}\right).$$

From the proof of the Theorem 3.1 we have that the operator

$$A : (C[0,b], ||\cdot||_B) \to (C[0,b], ||\cdot||_B),$$

$$A(x)(t) := x(0) + \int_0^t f(s, x(\lambda s))ds, \quad t \in [0,b],\ 0 < \lambda < 1,$$

is a weakly Picard operator.

But $\|\cdot\|_C$ on $C[0,b]$ is metric equivalent with $\|\cdot\|_B$ on $C[0,b]$. Therefore, the operator A defined by (4.1) is a weakly Picard operator. □

Remark 4.1. The operator

$$A : (C[0,b], \|\cdot\|_C) \to (C[0,b], \|\cdot\|_C),$$

defined by

$$A(x)(t) := \int_0^t f(s, x(\lambda s))ds, \quad t \in [0,b],\ 0 < \lambda < 1,$$

is a Picard operator (F_A has a unique fixed point).

So the integral equation

$$x(t) = \int_0^t f(s, x(\lambda s))ds, \quad t \in [0,b],\ 0 < \lambda < 1,$$

has a unique solution in $C[0,b]$ (Theorem 3.1.1, [9]).

5. Data dependence of the solutions set

Let (X,d) be a metric space. We use the following notations:

$$P(X) = \{Y \subseteq X \mid Y \neq \emptyset\},$$

$$P_{b,cl}(X) = \{Y \in P(X) \mid Y \text{ is bounded and closed}\}$$

and

$$O_A(x) = \{x, A(x), A^2(x), \ldots, A^n(x), \ldots\} \text{ (the orbit of } x \in X).$$

Then we have

$$\delta(Y) = \sup\{d(a,b) \mid a,b \in Y\}, \text{ the diameter of } Y \in P(X)$$

and

$$H : P_{b,cl}(X) \times P_{b,cl}(X) \to \mathbf{R}_+,$$
$$H(Y,Z) = \max\left(\sup_{a\in Y}\inf_{b\in Z} d(a,b), \sup_{b\in Z}\inf_{a\in Y} d(a,b)\right),$$

the Hausdorff-Pompeiu distance on $P_{b,cl}(X)$ set.

Let $A, B : (X,d) \to (X,d)$ two operators for which there exists $\eta > 0$ such that $d(A(x), B(x)) < \eta$, for all $x \in X$. The data dependence problem of the solutions set is to estimate the "distance" between the two fixed point sets F_A and F_B of these operators.

In order to study the data dependence of the solutions set of the equation (3.1), we need the following result:

Theorem 5.1. (Th.2.4, [17]) *Let (X, d) be a complete metric space and $A, B : X \to X$ two orbitally continuous operators. We suppose that:*

(i) there exists $\alpha \in [0, 1[$ such that $d(A^2(x), A(x)) \leq \alpha d(x, A(x))$, for all $x \in X$

and

$d(B^2(x), B(x)) \leq \alpha d(x, B(x))$, for all $x \in X$;

(ii) there exists $\eta > 0$ such that $d(A(x), B(x)) \leq \eta$, for all $x \in X$.

Then

$$H(F_A, F_B) \leq \frac{\eta}{1-\alpha}.$$

Now we consider the following Volterra integral equations with deviating argument:

$$x(t) = x(0) + \int_0^t f(s, x(\lambda s))ds, \quad t \in [0, b],\ 0 < \lambda < 1, \tag{5.1}$$

$$x(t) = x(0) + \int_0^t g(s, x(\lambda s))ds, \quad t \in [0, b],\ 0 < \lambda < 1, \tag{5.2}$$

in which λ is the same and $f, g \in C([0, b] \times \mathbf{R})$.

We have

Theorem 5.2. *We suppose that*

(i) there exists $L > 0$ such that

$$|f(s, u) - f(s, v)| \leq L|u - v|, \text{ for all } s \in [0, b] \text{ and all } u, v \in \mathbf{R},$$

and

$$|g(s, u) - g(s, v)| \leq L|u - v|, \text{ for all } s \in [0, b] \text{ and all } u, v \in \mathbf{R};$$

(ii) there exists $\eta_1 > 0$ such that

$$|f(s, u) - g(s, u)| \leq \eta_1, \text{ for all } s \in [0, b] \text{ and all } u \in \mathbf{R};$$

(iii) $Lb < 1$.

Then

(a) $F_A \neq \emptyset$ *and* $F_B \neq \emptyset$;

(b) $H_{\|\cdot\|_C}(F_A, F_B) \leq \frac{\eta_1 b}{1 - Lb}$, *where by* $H_{\|\cdot\|_C}$ *we denote the Hausdorff-Pompeiu metric with respect to* $\|\cdot\|_C$ *on* $C[0, b]$.

Proof. (a) By using the results of the Theorem 3.1 we have that $F_A \neq \emptyset$ and $F_B \neq \emptyset$.

(b) We consider the operators

$$A, B : (C[0, b], \|\cdot\|_C) \to (C[0, b], \|\cdot\|_C),$$

defined by

$$A(x)(t) := x(0) + \int_0^t f(s, x(\lambda s))ds, \quad t \in [0, b],\ 0 < \lambda < 1,$$

$$B(x)(t) := x(0) + \int_0^t g(s, x(\lambda s))ds, \quad t \in [0, b],\ 0 < \lambda < 1,$$

in which λ is the same.

Then

$$|A^2(x)(t) - A(x)(t)| \leq \frac{L}{\lambda} \int_0^{\lambda t} |A(x)(v) - x(v)|dv \leq$$

$$\leq Lb\|A(x) - x\|_C, \text{ for all } t \in [0, b].$$

Therefore,

$$\|A^2(x) - A(x)\|_C \leq Lb\|A(x) - x\|_C, \text{ for all } x \in C[0, b].$$

Similarly,

$$\|B^2(x) - B(x)\|_C \leq Lb\|B(x) - x\|_C, \text{ for all } x \in C[0, b].$$

From (ii) we obtain that

$$\|A(x) - B(x)\|_C \leq \eta_1 b, \text{ for all } x \in C[0, b].$$

By applying the Theorem 5.1 we have that

$$H_{\|\cdot\|_C}(F_A, F_B) \leq \frac{\eta_1 b}{1 - Lb}.$$

□

References

[1] G.M. Dunkel, *On nested functional-differential equations*, SIAM J. Appl. Math., vol.18, nr.2(1970), 514-525.

[2] A. Elbert, *Asumptotic behaviour of the analytic solution of the differential equation* $y'(t) + y(qt) = 0$ *as* $q \to 1^-$, J. Comput. Appl. Math. 41(1992), nr.1-2, 5-22.

[3] A. Iserles, *On the generalized pantograph functional-differential equation*, European J. Appl. Math. 4(1992), 1-38.

[4] T. Kato, J.B. McLeod, *The functional-differential equation* $y'(x) = ay(\lambda x) + by(x)$, Bull. Amer. Math. Soc., 77, 6(1971), 891-937.

[5] M.R.S. Kulenović, *Oscillation of the Euler differential equation with delay*, Czech. Math. J., 45(120), nr.1(1995), 1-6.

[6] E.B. Lim, *Asymptotic behaviour of solutions of the functional differential equation* $x'(t) = Ax(\lambda t) + Bx(t)$, $\lambda > 0$, J. Math. Anal. Appl. 55(1976), 794-806.

[7] E.B. Lim, *Asymptotic bounds of solutions of the functional-differential equation* $x'(t) = ax(\lambda t) + bx(t) + f(t)$, $0 < \lambda < 1$, SIAM J. Math. Anal. 9(1978), no.5, 915-920.

[8] H. Melvin, *A family of solutions of the IVP for the equation* $x'(t) = ax(\lambda t)$, $\lambda > 1$, Aequationes Math. 9(1973), 273-280.

[9] V. Mureşan, *Differential equations with afine deviating of the argument*, Transilvania Press, Cluj-Napoca, 1997 (in romanian).

[10] V. Mureşan, D. Trif, *Newton's method for nonlinear differential equations with linear deviating of the argument*, Studia Univ. "Babeş-Bolyai", Mathematica, vol. XLI, nr.4(1996), 89-96.

[11] I.A. Rus, *The fixed point theory. (II). The fixed point theory in functional analysis*, "Babeş-Bolyai" Univ. Cluj-Napoca, 1973.

[12] I.A. Rus, *The principles and applications of the fixed point theory*, Ed. Dacia, Cluj-Napoca, 1979 (in romanian).

[13] I.A. Rus, *Generalized contractions*, "Babeş-Bolyai" Univ. Cluj-Napoca, Preprint 3, (1983), 1-30.

[14] I.A. Rus, *Picard mappings: results and problems*, "Babeş-Bolyai" Univ. Cluj-Napoca, Preprint 6, (1987), 55-64.

[15] I.A. Rus, *Weakly Picard mappings*, Comment. Math. Univ. Carolinae, 34, 4(1993), 769-773.

[16] I.A. Rus, *Picard operators and applications*, "Babeş-Bolyai" Univ. Cluj-Napoca, Preprint 3(1996).

[17] I.A. Rus, S. Mureşan, *Data dependence of the fixed points set of weakly Picard operators*, Studia Univ. "Babeş-Bolyai" Cluj-Napoca (to appear).

[18] L. Pandolfi, *Some observations on the asymptotic behaviour of the solutions of the equation* $x'(t) = A(t)x(\lambda t) + B(t)x(t)$, $\lambda > 0$, J. Math. Anal. Appl. 67(1979), no.2, 483-489.

[19] J. Terjéki, *Representation of the solutions to linear pantograph equations*, Acta Sci. Math. (Szeged), 60(1995), 705-713.

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