SOME QUALITATIVE PROPERTIES OF THE SOLUTIONS TO QUASI-LINEAR DIFFERENTIAL INCLUSIONS

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Abstract. The aim of the present paper is to introduce some recent results on the existence and on some qualitative properties of the solutions to some differential inclusions of evolution.

1. Introduction

By a study of some qualitative properties of the set of solutions to a differential inclusion (similarly to the case of a differential equation) we mean a study of one or several aspects in connection with: the existence of a solution, dependence on the initial value, parameter and/or right hand side, connectedness of the set of solutions, relaxation, periodicity and/or stability of the solution(s), etc. without a straight access to the solution(s).

Let us recall some well-known facts from the theory of differential equations. Consider $I = [0,T]$, $0 < T$, $X = \mathbb{R}^n$. We have the following result

Theorem 1 ([23], p. 10). *Let $x, f \in X$; $f(t,x)$ is continuous on $I \times \{x \mid |x - x_0| \leq b\}$; M is a bound for $|f(t,x)|$ on $I \times \{x \mid |x - x_0| \leq b\}$; $\alpha = \min\{a, b/M\}$. Then*

$$x' = f(t,x), \quad x(0) = x_0 \tag{1}$$

possesses at least one solution $x = x(t)$ on $[0, \alpha]$.

Remarks. In this case each solution is *continuously differentiable* on the interval $]0, \alpha[$. The solution is not unique. For uniqueness it is necessary an extra assumption, e.g. a Kamke condition. The above theorem fails in an infinite dimensional spaces, [6].

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Adding a Lipschitz condition in respect to the second variable, the local existence is guaranteed.

When function f is not longer continuous, but it is continuous in x for almost all t and measurable in t for all x (it is a *Carathéodory function*), then we have

Theorem 2 ([21], p. 4). *For $t \in I$, $\mid x - x_0 \mid \leq b$ let f be a Carathéodory function and $\mid f(t,x) \mid \leq m(t)$, the function m being summable. Then on the closed interval $[0,d]$, where $d > 0$, there exists a solution of the initial value problem* (1). *In this case d satisfies: $d \in (0,T]$, $\phi(t) := \int_0^t m(s)\,ds$, $\phi(t+d) \leq b$.*

Now a solution is an *absolutely continuous* function on I. The assumptions are weaker, the conclusions are weaker, too.

These two situations may be encountered similarly in the case of more general Banach spaces. We have to notice that in a general Banach space X an absolutely continuous function defined on I is not almost everywhere differentiable on $]0,T[$. But if the Banach space X is *reflexive*, thanks to a theorem of Komura, [6, p. 16], we know that every X-valued absolutely continuous function x on I is a.e. differentiable on $]0,T[$ and $x(t) = x(0) + \int_0^t (dx/ds)(s)\,ds$, $t \in I$. Here the integral is considered as a *Bochner integral*, [18] or [17].

Now we take a look to the case of linear differential equations when $X = \mathbb{R}^n$. Consider the following two systems of ordinary differential equations

$$x'(t) = A(t)x(t) \tag{2}$$

$$x'(t) = A(t)x(t) + f(t), \quad x(0) = x_0, \tag{3}$$

where A is a $n \times n$ matrix and f is a function, both continuous on I. If Y is the fundamental matrix of equation (2), then the solution of equation (3) is given by

$$x(t) = Y(t,0)x_0 + \int_0^t Y(t,s)f(s)\,ds, \tag{4}$$

where $Y(t) = Y(t,0)$. Obviously, this solution is continuously differentiable on I. If $f \in \mathcal{L}^1(I)$, then x is absolutely continuous on I.

If X is a Banach space, the function defined by (4) is said to be the *mild solution* of equation (3), if it exists. The study of the equations of the form (3) in infinite dimensional spaces is performed, e.g., [6], [34], [47], [48]; applications [34], [41].

Now we turn for a while to observe some very elementary properties of the differential inclusions. For the beginning we remark that a differential inclusion is a differential equation whose the right-hand side is a set-valued function (multifunction, correspondence, etc.), [21], [3], [16], [31].

Consider the following differential inclusion $x' \in \{-1, +1\}$, $x(0) = 0$, $t \in [0, 1]$. We see that if we require that the solution to be continuously differentiable, then the set of solutions is very poor; but if we permit to a solution to be absolutely continuous, then the set of solutions is rich enough. In this case one can construct a sequence of solutions x_n converging uniformly to the constant function $x = 0$. But $x = 0$ is not a solution, hence the set of solutions is not always a closed set.

A classical way to obtain a differential inclusion, [3], [4], is that, starting from a dynamical system $x'(t) = f(t, x(t), u(t))$, $x(t_0) = x_0$, "controlled" by the parameters $u \in U$, to define $F(t, x(t)) = \{f(t, x(t), u(t))\}_{u \in U}$. For definitions of the solution of a differential inclusion we refer to [21]. The coincidence of the sets of solutions was studied for the first time by Wazewski in [57]. The set-valued functions and differential inclusions are useful tools not only in control problems, but also in economical problems, [31].

Let Z be a linear topological space. We will use the following notations: $\mathrm{P}(Z) = \{A \subset Z \mid A \neq \emptyset\}$, $\mathrm{C}(Z) = \{A \in \mathrm{P}(Z) \mid A \text{ closed}\}$, $\mathrm{CCo}(Z) = \{A \in \mathrm{C}(Z) \mid A \text{ convex}\}$, $\mathrm{KCo}(Z) = \{A \in \mathrm{P}(Z) \mid A \text{ compact and convex}\}$.

The *Hausdorff-Pompeiu metric* of the sets $A, B \in \mathrm{C}(X)$ ((X, ρ) is a metric space) is defined by $\mathrm{D}(A, B) = \max\{\mathrm{d}(A, B), \mathrm{d}(B, A)\}$ where $\mathrm{d}(A, B) = \sup\{\mathrm{d}(x, B) \mid x \in A\}$. Several properties of the Hausdorff-Pompeiu metric may by found in [11], [8].

Let I be the interval $I = [0, T]$, $T > 0$ fixed, and X a Banach space. A family of bounded linear operators $\mathcal{U}(t, s)$, on X, $0 \leq s \leq t \leq T$, depending on two parameters is said to be an *evolution system*, [48], if there are fulfilled the following two conditions $\mathcal{U}(s,s) = 1$, $\mathcal{U}(t,r)\mathcal{U}(r,s) = \mathcal{U}(t,s)$ for $0 \le s \le r \le t \le T$; $(t,s) \to \mathcal{U}(t,s)$ is strongly continuous for $0 \le s \le t \le T$, where by strongly continuity is meant that $\lim_{t \searrow s} \mathcal{U}(t,s)x = x$, for all $x \in X$.

By a *Cauchy problem for a quasi-linear differential inclusion* we mean

$$\frac{dx(t)}{dt} \in A(t,x(t))x(t) + F(t,x(t)), \quad \text{a.e. } t \in I, \quad \text{and } x(0) = x_0, \tag{CP}$$

where $A(t,w)$ is a linear operator from X to X depending on $t \in I$ and $w \in X$, and F is a set-valued map.

If operator A depends on t and w, the differential inclusion in (CP) is said to be *quasi-linear*, if A depends only on t, the differential inclusion is said to be *semi-linear*, and if A depends neither on t nor on w, the differential inclusion is said to be *linear*.

We are interested to study the *mild solutions* of the (CP), i.e. continuous functions having the following representation

$$x(t) = \mathcal{U}_x(t,0)a + \int_0^t \mathcal{U}_x(t,s)f(s)\,ds \quad t \in I, \ f \in S^1_{F(\cdot,x(\cdot))},$$

where $S^1_{F_x} = S^1_{F_x(\cdot)} = S^1_{F(\cdot,x(\cdot))}$ is the set of Bochner integrable selections from $F(\cdot,x(\cdot))$.

We use the following assumptions:

(X_1) X is a separable Banach space;

(X_2) X satisfies (X_1) and, moreover, it is reflexive;

(A) For every $u \in C(I,X)$ the family of linear operators $\{A(t,u) \mid t \in I\}$ generates a unique strongly continuous evolution system $\mathcal{U}_u(t,s)$, $0 \le s \le t \le T$;

(U_1) If $u \in C(I,X)$, the evolution system $\mathcal{U}_u(t,s)$, $0 \le s \le t \le T$ satisfies

(i) there exists a $c_1 \ge 0$ with $\|\mathcal{U}_u(t,s)\| \le c_1$ for $0 \le s \le t \le T$, uniformly in u;

(ii) there exists a $c_2 \geq 0$ such that for any u, $v \in C(I,X)$ and any $w \in X$ we have

$$\|\mathcal{U}_u(t,s)w - \mathcal{U}_v(t,s)w\| \leq c_2\|w\| \int_s^t \|u(\tau) - v(\tau)\|\, d\tau;$$

(U_2) If $u \in C(I,X)$ and $0 \leq s \leq t \leq T$, then $\mathcal{U}_u(t,s)$ is a compact operator, i.e. it transforms bounded sets in relatively compact sets. In this case, [48, p. 48], $\mathcal{U}_u(t,s)$ is continuous in the uniform operatorial topology.

(U_3) If $t, t+\delta \in I$, $\delta > 0$, then $\lim_{\delta \to 0} \mathcal{U}_u(t+\delta, t) = 1$, uniformly in u and t.

Remarks. If operator A does not depend on w, but it depends on t, then the assumption (A) reads as follows: $\{A(t) \mid t \in I\}$ generates a unique strongly continuous evolution system $\mathcal{U}(t,s)$, $0 \leq s \leq t \leq T$. In this case we take $c_2 = 0$ (in (ii) from (U_1)). The dependence that is used in (U_1) (ii) was inspired by [48, p. 202], [32, lemma 2.2].

In connection with the multifunction F we will use the following assumptions:

(F_1) $F : I \times X \to \mathrm{C}(X)$ and for any $x \in X$, $F(\cdot, x)$ is measurable;

(F_2) $F : I \times X \to \mathrm{CCo}(X)$ and for any $x \in X$, $F(\cdot, x)$ is measurable;

(F_3) F satisfies (F_1) and for any $t \in I$, $F(t, \cdot) : X \to \mathrm{C}(X)$ is lower semicontinuous from X in $\mathrm{C}(X)$ and it is upper semicontinuous from X in $\mathrm{C}(w{-}X)$, where $w{-}X$ is X endowed with the weak topology;

(F_4) F satisfies (F_1), it is product-measurable and for all $t \in I$, $F(t, \cdot) : X \to \mathrm{C}(X)$ is upper semicontinuous;

(F_5) F satisfies (F_1) and, moreover, it is $k(t)$-Lipschitz, i.e. exists $k \in \mathcal{L}^1(I, \mathbb{R}_+)$ such that for almost all $t \in I$ and for all x, $y \in X$, $\mathrm{D}(F(t,x), F(t,y)) \leq k(t)\|x - y\|$, D being the Hausdorff-Pompeiu metric;

(F_6) F is integrably bounded by a function $m \in \mathcal{L}^1(I, \mathbb{R}_+)$, that is, for all $x \in C(I,X)$ and $t \in I$ we have $F(t, x(t)) \subset m(t)B$, B is the closed unit ball in X;

(F_7) the function $t \mapsto \mathrm{d}(0, F(t,0))$ is integrable on I.

By an *inclusion of evolution* we mean an inclusion of the following form

$$\frac{dx(t)}{dt} \in A(t, x(t))x(t) + F(t, x(t)), \quad \text{a.e. } t \in I.$$

Remark. The evolution inclusions have been investigated in a series of works: [48], [47], [50], [2], [44], [45], [1], etc. A different approach of evolution inclusions is used in [5], [28]. Their approach is based on the Galerkin approximations (e.g. [15], [33]).

2. Existence of solutions

We need the next assumption (M_1) If A depends on w, then for any $t \in I$, $M_b(t)$ is relatively compact in X, where $b = (\|x_0\| + \|m\|_1)c_1$,

$$M_b = \{x \in C(I, X) \mid x(0) = x_0, \ \|x\| \leq b\}, \quad M_b(t) = \{y(t) \mid y \in \psi(x), x \in M_b\},$$

$$\psi(x) = \{y \mid y(t) = \mathcal{U}_x(t,0)x_0 + \int_0^t \mathcal{U}_x(t,s)f(s)\, ds, \ x(0) = x_0, \ f \in S^1_{F_x}\}.$$

Based on fixed point techniques we have proved the following two existence theorems

Theorem 3 ([40]). *If there are satisfied the following assumptions (X_2), (A), (U_1), (F_2), (F_{5-6}) and if $0 < c_3 < 1$, ($c_3 = c_2T(\|x_0\| + \|m\|_1) + c_1\|k\|_1$), then there exists a mild solution of the problem (CP) in M_b.*

Theorem 4 ([40]). *Suppose there are satisfied the following assumptions*

(i) (X_2), (A), (U_1), (U_3), (F_2), (F_{5-6});

(ii) (M_1) *or* (U_2);

(iii) (F_3) *or* (F_4).

Then there exists a mild solution in M_b of the (CP) problem.

Theorem 3 uses a multivalued version of the Banach fixed point principle, while theorem 4 is based on the Bohnenblust-Karlin fixed point theorem, [16], [49]. More refined existence results may be obtained by the method developed in [19], [20].

3. Filippov-Gronwall theorems

By a Filippov-Gronwall theorem we understand a result which from the existence of a function or a solution of a differential equation ensures the existence of a solution of an other differential equation or inclusion provided a "closeness" condition is satisfied.

First results on this topic have been published by Filippov in [19], [20]. Later there were published more and more papers in connection with Filippov's results, let us mention just a few of them: [25], [29], [42], [26], [3], [16] and [59]. In the case of linear evolution inclusions such a problem has been investigated in [22] and [55]. Tolstonogov, in [55], studied also the case when A is an m-dissipative operator.

In the sequel we consider a new Cauchy problem

$$\frac{dy(t)}{dt} = A(t, y(t))y(t) + g(t), \quad g \in \mathcal{L}^1(I, X), \text{ a.e. } t \in I, \quad \text{and } y(0) = y_0. \tag{5}$$

(S_1) Suppose that problem (5) has a mild solution $y(t) = \mathcal{U}_y(t,0)y_0 + \int_0^t \mathcal{U}_y(t,s)g(s)\,ds$, $t \in I$.

It is shown that if the initial values and the nonlinear parts to (CP) and (5) are sufficiently close and (S_1) it holds, then problem (CP) has a mild solution x whose distance to y does not exceed a certain value.

We consider problems (CP) and (5) under the assumptions (X_1), (A), (F_5) and (F_6). Denote $\delta = \|x_0 - y_0\|$, $p = c_2(\|x_0\| + \|m\|_1)$, $k_\varepsilon(t) = k(t) + \varepsilon$, $\varepsilon > 0$, $K(t) = \int_0^t [p + k_\varepsilon(s)]\,ds$, $E(t) = \exp(K(t))$, $t \in I$. Moreover, we admit assumption (S_1) and let $\gamma(t) = \mathrm{d}(g(t), F(t, y(t)))$, $t \in I$. Based [40, lemma 2.15] we have that $\gamma \in \mathcal{L}^1$ and then consider $n(t) = [\delta + \int_0^t (\gamma(s) + \varepsilon)\,ds]$, $t \in I$.

Theorem 5 ([38]). *Suppose that the following assumptions are fulfilled:* (X_1), (A), (U_1), (F_5), (F_6), (F_7) *and* (S_1). *Then problem (CP) has a mild solution* $x \in C(I, X)$ *such that*

$$\|x(t) - y(t)\| \leq n(t)E(t) = c_1 \left[\delta E(t) + E(t) \int_0^t (\gamma(s) + \varepsilon)\,ds\right], \quad t \in I, \tag{6}$$

$$\|f(t) - g(t)\| \leq \gamma(t) + \varepsilon + n(t)k_\varepsilon(t)E(t) \text{ a.e. } t \in I. \tag{7}$$

The method of the proof (as in [19], [20], [22], [55]) consists in constructing two convergent sequences $(x_n)_{n\geq 1} \subset C(I, X)$ and $(f_n)_{n\geq 1} \subset \mathcal{L}^1(I, X)$ such that x, the limit of $(x_n)_{n\geq 1}$ in the uniform topology from $C(I, X)$, is the mild solution of the problem (CP) and it satisfies (6). f, the limit of the sequence $(f_n)_{n\geq 1}$ in $\mathcal{L}^1(I, X)$, satisfies (7) and appears in the formula of x.

Remark. If problem (CP) is linear, we get a result from [22]. Obviously, in this case the assumption (F_6) is useless and $c_2 = 0$ implies that $p = 0$.

Theorem 6 ([38]). *Suppose there are satisfied all the assumptions of theorem 5. Then problem (CP) has a mild solution $x \in C(I, X)$ such that*

$$\|x(t) - y(t)\| \leq c_1 \left[\delta E(t) + \int_0^t \frac{E(t)}{E(s)}(\gamma(s) + \varepsilon)\, ds\right], \quad t \in I \tag{8}$$

$$\|f(t) - g(t)\| \leq \gamma(t) + \varepsilon + k_\varepsilon(t) c_1 \left[\delta E(t) + \int_0^t \frac{E(t)}{E(s)}(\gamma(s) + \varepsilon)\, ds\right], \quad a.e.\ t \in I.$$

Remark. It is obvious now, comparing (6) and (8), that the estimations in theorem 6 are better than the estimations in theorem 5.

Theorem 7 ([38]). *Suppose there are satisfied all the assumptions of the theorem 6 with the only change that instead of the function n we consider $\overline{n}(t) = c_1 \left[\delta + \int_0^t 2\gamma(s)\, ds\right]$, K is replaced by $\overline{K}(t) = \int_0^t [p + 2c_1 k(s)]\, ds$, and E is replaced by $\overline{E}(t) = \exp(\overline{K}(t))$, $t \in I$. Then problem (CP) has a mild solution $x \in C(I, X)$ such that*

$$\|x(t) - y(t)\| \leq c_1 \left[\delta \overline{E}(t) + \int_0^t \frac{\overline{E}(t)}{\overline{E}(s)} 2\gamma(s)\, ds\right], \quad on\ I$$

$$\|f(t) - g(t)\| \leq 2\gamma(t) + 2k(t) c_1 \left[\delta \overline{E}(t) + \int_0^t \frac{\overline{E}(t)}{\overline{E}(s)} 2\gamma(s)\, ds\right], \quad a.e.\ t \in I.$$

Remark. In [55] it is considered only the case of linear inclusions. This fact implies that the evolution system $\mathcal{U}$ depends only on t. Thus condition (ii) in (U_1) is useless and we may take $c_2 = 0$ (hence $p = 0$). The assumption (ii) in (U_1) gives us the dependence way of the evolution system $\mathcal{U}_u$ in respect to the function $u \in C(I, X)$. This kind of dependence may be met in the so called "hyperbolic" case, [48], [41, p 272]. Theorem 7 contains, for the case of linear inclusion, theorem 3.2 in [55].

4. Continuous dependence results

It is useful to know the manner of dependence of the set of solutions of an initial value problem upon the initial value, a parameter or the right hand side of an equation or inclusion. The continuous dependence results are widely used in numerical methods, too. For ODE dependence results may be found in many books, for instance [23].

Recent dependence results on differential inclusions may be found in: [43], [51], [59], [55], [13] and [14].

The uniqueness of the mild solution of the Cauchy problem for quasi-linear equations follows from

Theorem 8 ([40]). *Let $f,g \in \mathcal{L}^1(I,X)$, $\chi = \|f-g\|_1$ and $\delta = \|x_0 - y_0\|$ such that there are satisfied all the assumptions of theorem 5 taking f instead of F. Denote by x and y two mild solutions of the quasi-linear equations corresponding to f, x_0, respectively g, y_0. Then the following estimation holds*

$$\|x(t)-y(t)\| \leq c_1(\chi+\delta)\exp\left[c_2(\min\{\|x_0\|,\|y_0\|\}+\min\{\|f\|_1,\|g\|_1\})t\right], \quad t \in I.$$

Corollary 1 ([40]). *If the assumptions of the above theorem are satisfied, then*

$$\|x-y\|_{C(I,X)} \leq c_1(\delta+\chi)\exp\left[c_2(\min\{\|x_0\|,\|y_0\|\}+\min\{\|f\|_1,\|g\|_1\})T\right].$$

Corollary 2 ([40]). *If the assumptions of the above theorem are satisfied and if $\delta = \chi = 0$, then $x = y$, hence the mild solution of problem* (5) *is unique.*

Remark. If in problem (5) the operator A is linear, then the uniqueness problem is discussed, for instance, in [48, p. 106].

Let us denote by $\mathcal{S}(x_0)$ the set of mild solutions of (CP). For problem (CP) there holds a Lipschitz dependence upon the initial values:

Theorem 9 ([38]). *Suppose there are satisfied all the assumptions of theorem 7 and, moreover, that* (5) *is a differential inclusion having F instead of g. Then*

$$D(\overline{\mathcal{S}(x_0)}, \overline{\mathcal{S}(y_0)}) \leq L\|x_0 - y_0\|,$$

where $L = c_1\overline{E}(T)$.

Corollary 3 ([38]). *If in the inclusion (CP) A does not depend on* w, *then we obtain theorem 4.1 in* [55].

Corollary 4 ([38]). *If in the above mentioned theorem we consider* $A \equiv 0$, *then we get corollary 1,* [3, p. 121], *and if we suppose, moreover, that* F *is single-valued, then we obtain estimation* (4), *in* [3, p. 119].

Remark. Under the assumptions of the above mentioned theorem the set-valued map $x_0 \mapsto \mathcal{S}(x_0)$ is globally Lipschitz. This set-valued map is studied under various assumptions, for instance, in [55] and [59].

Now we are interested in dependence of the set of solutions upon a parameter.

Definition. Suppose that assumptions (X_1) and (A) are satisfied. Then a function $x(\cdot,\xi) : I \times X \to X$ is said to be a *mild solution* of the problem (CP) with $a = \xi$ if there exists $f(\cdot,\xi) \in \mathcal{L}^1(I,X)$ such that

$$f(t,\xi) \in F(t,x(t,\xi)), \qquad \text{a.e. on } I,$$

$$x(t,\xi) = \mathcal{U}_{x(\cdot,\xi)}(t,0)\xi + \int_0^t \mathcal{U}_{x(\cdot,\xi)}(t,s)f(s,\xi)\,ds, \quad \text{for each } t \in I.$$

We need the following hypothesis (which is (S_1) with $g \equiv 0$):

(S_1'). The next problem

$$\frac{dx(t)}{dt} = A(t,x(t))x(t), \quad \text{a.e. } t \in I, \quad \text{and } x(0) = \xi,$$

has a mild solution, let it be $x_0(t,\xi) = \mathcal{U}_{x_0(\cdot,\xi)}(t,0)\xi$, for all $t \in I$.

Theorem 10 ([38]). *Suppose the following assumptions are satisfied:* (X_1), (A), (U_1), (F_5), (F_6), (F_7) *and* (S_1'). *Denote by* $\mathcal{S}(\xi)$ *the set of solutions of the problem (CP), the initial value being equal to* ξ, $(a = \xi)$. *Then there exists a function* $x(\cdot,\cdot) : I \times X \to X$ *such that*

$$x(\cdot,\xi) \in \mathcal{S}(\xi) \quad \text{for each } \xi \in X,$$

$$\xi \to x(\cdot,\xi) \quad \text{is continuous from } X \text{ in } C(I,X).$$

Partially, the method is the same to the method used in the proof of the theorem 6. Here we do not search integrable selections, but based on theorem 3.1 in [24] and proposition 2.2 in [14] at each iteration we choose a continuous selection.

Remark. If the differential inclusion in (CP) is linear, then we recover a result from [51] or theorem 3.3 in [52]. If the differential inclusion in (CP) is semi-linear, then $c_2 = 0$ and the assumption (F_6) are unnecessary.

5. Connectedness of the set of solutions

The connectedness and the arcwise connectedness of the set of solutions is a topic discussed in several papers such as [51], [52], [53], [54] and [56].

In [56] it is studied problem (CP) with A depending on t only and it is shown that if A is an m-dissipative operator or if it is linear and the infinitesimal generator of a strongly continuous semi-group, then the set of solutions of problem (CP) is connected. The method introduced in [56] may be used also to the case of a quasi-linear inclusion. More exactly we have the following result

Theorem 11. *We suppose that the assumptions (X_1), (A), (U_1), (F_5), (F_6), (F_7) and (S_1) are fulfilled and consider problem (CP). Then $\mathcal{S}(x_0)$ is closed and connected.*

Proof. In addition to the Lebesgue measure on I we consider the measure defined by $d\mu = \exp\left(-2c_1c_{20}\int_0^t k(s)\,ds\right)\,dt$, where $c_{20} = \exp(p)$, $p = c_2(\|x_0\| + \|m\|_1)$. The two measures are equivalent, [7, p. 157]. Also consider the space $\mathcal{L}_1(I,\mu,X)$ of (classes of) Bochner integrable functions in respect to the measure μ and to the norm $\|f\|_{1*} = \int_0^T \|f(t)\|\,d\mu(t)$. Let j be the identity map $j : \mathcal{L}^1(I,X) \to \mathcal{L}^1(I,\mu,X)$. We see that the norms $\|\cdot\|_1$ and $\|\cdot\|_{1*}$ are equivalent. Thus j establishes an homeomorphism between the spaces $\mathcal{L}^1(I,X)$ and $\mathcal{L}^1(I,\mu,X)$.

For $z \in C(I,X)$ from lemma 2.2 in [38] we have that $t \mapsto F(t,z(t))$ is measurable, and by the assumptions we have that it has closed values and it is integrably bounded. It follows that $S^1_{F_z} \neq \emptyset$, and by [24, theorem 3.2], it results that $S^1_{F_z}$ is a bounded set in $\mathcal{L}_1(I,X)$. By theorem [24, theorem 3.1] we know that $S^1_{F_z}$ is a decomposable set. Hence $S^1_{F_z} \in \mathcal{D}$ and $S^1_{F_z}$ is a bounded set in $\mathcal{L}_1(I,X)$.

Take $f \in \mathcal{L}_1(I,X)$. Based on theorem 8 the equation (5) with $g = f$ has a unique mild solution on I. We define the map $d : \mathcal{L}_1(I,X) \to C(I,X)$ such that to a member $f \in \mathcal{L}_1(I,X)$ corresponds the mild solution of the above mentioned quasi-linear equations, namely $d(f) = x$ iff $x(t) = \mathcal{U}_x(t,0)x_0 + \int_0^t \mathcal{U}_x(t,s)f(s)\,ds$, $t \in I$. From the corollary 1 it follows that d is a Lipschitz map, so it is continuous. But d is also one to one.

Let us take the following set-valued map $\Phi : \mathcal{L}_1(I,X) \to \mathrm{C}(\mathcal{L}_1(I,X))$ defined by $\Phi(f) = S^1_{F(\cdot,d(f))}$. It follows that $\Phi(f) \in \mathcal{D}$ and $\Phi(f)$ is bounded in $\mathcal{L}_1(I,X)$, for each $f \in \mathcal{L}_1(I,X)$.

There holds the equality $\mathcal{S}(x_0) = \{d(f) \mid f \in \Phi(f)\}$. If $x \in \mathcal{S}(x_0)$ there exists $f \in S^1_{F_x}$ such that $x(t) = \mathcal{U}_x(t,0)x_0 + \int_0^t \mathcal{U}_x(t,s)f(s)\,ds$, $t \in I$. Then $x = d(f)$. Hence $f \in S^1_{F(\cdot,d(f))}$, that is $f \in \Phi(f)$. Vice versa, we suppose that $x = d(f)$ with $f \in \Phi(f)$. Then $x(t) = \mathcal{U}_x(t,0)x_0 + \int_0^t \mathcal{U}_x(t,s)f(s)\,ds$, $t \in I$ and $f \in S^1_{F(\cdot,d(f))} = S^1_{F_x}$. Thus $x \in \mathcal{S}(x_0)$.

We define the multifunction $\tilde{\Phi} : \mathcal{L}^1(I,\mu,X) \to \mathrm{P}(\mathcal{L}^1(I,\mu,X))$ by $\tilde{\Phi} = j\Phi j^{-1}$. Obviously, $\tilde{\Phi}(f) \in \mathcal{D}$, $\tilde{\Phi}(f)$ is bounded for each $f \in \mathcal{L}^1(I,\mu,X)$, and the sets of fixed points of the two multifunctions Φ and $\tilde{\Phi}$ are equal.

We intend to show that the set of the fixed points of the multifunction $\tilde{\Phi}$ is an absolute retract, [9, p. 85]. Since the map j is an homeomorphism, the set of the fixed points of the multifunction Φ is an absolute retract, [9, p. 86]. Its image by d still remains an absolute retract and coincides with $\mathcal{S}(x_0)$. Hence $\mathcal{S}(x_0)$ being a retract, it is connected, [27, p. 27], and closed.

Take $f, h \in \mathcal{L}_1(I,X)$, $x = d(f)$, $y = d(h)$ and $\alpha_1 = \int_0^T \exp\left[-2c_1c_{20}\int_0^t k(s)\,ds\right] dt$. For each $v \in S^1_{F_x}$ and $\varepsilon > 0$ we choose $u \in S^1_{F_y}$ such that $\|v(t) - u(t)\| \leq \mathrm{d}(v(t), F(t,y(t))) + \varepsilon\alpha_1^{-1} \leq \mathrm{D}(F(t,x(t)), F(t,y(t))) + \varepsilon\alpha_1^{-1} \leq k(t)\|x(t) - y(t)\| + \varepsilon\alpha_1^{-1}$. Then

$$\|u - v\|_{1*} = \int_0^T \exp\left[-2c_1c_{20}\int_0^t k(s)\,ds\right] \|u(t) - v(t)\|\,dt$$
$$\leq \int_0^T \exp\left[-2c_1c_{20}\int_0^t k(s)\,ds\right] k(t)\|x(t) - y(t)\|\,dt + \varepsilon\alpha\alpha^{-1},$$

and by theorem 8 we have

$$\le \int_0^T \exp\left[-2c_1c_{20}\int_0^t k(s)\,ds\right] k(t)c_1c_{20}\int_0^t \|f(s)-h(s)\|\,ds\,dt+\varepsilon$$

$$\le -\frac{1}{2}\int_0^T \left[\exp\left(-2c_1c_{20}\int_0^t k(s)\,ds\right)\right]'\int_0^t \|f(s)-h(s)\|\,ds\,dt+\varepsilon$$

$$= -\frac{1}{2}\exp\left[-2c_1c_{20}\int_0^t k(s)\,ds\right]\int_0^t \|f(s)-h(s)\|\,ds\Big|_0^T$$

$$+\frac{1}{2}\int_0^T \exp\left[-2c_1c_{20}\int_0^t k(s)\,ds\right]\|f(t)-h(t)\|\,dt+\varepsilon$$

$$\le \frac{1}{2}\|f-h\|_{1*}+\varepsilon.$$

Since ε is arbitrary it follows that $\mathrm{d}(v,\tilde{\Phi}(h)) \le \frac{1}{2}\|f-h\|_{1*}$ and $\mathrm{d}(\tilde{\Phi}(f),\tilde{\Phi}(h)) \le \frac{1}{2}\|f-h\|_{1*}$. If we change f by h and vice versa, then we have

$$\mathrm{D}(\tilde{\Phi}(f),\tilde{\Phi}(h)) \le \frac{1}{2}\|f-h\|_{1*}.$$

Based on [10, theorem 1], the set of the fixed points of the multifunction $\tilde{\Phi}$ is an absolute retract. Then the set of the fixed points of the multifunction Φ is an absolute retract. Then $\mathcal{S}(x_0)$ is an absolute retract, too. Thus the theorem is proved. □

6. **Relaxation result**

The *relaxation* theorems (also called Filippov-Wazewski theorems and appeared in [19], [58]) concern with the case when the set of solutions of a differential inclusion (whose right-hand side is not convex) is dense in the set of solutions of a differential inclusion whose right-hand side is, usually, the convex hull of the right-hand side of the first inclusion.

Such theorem may be considered as an existence one for the first inclusion since if it is proved that the convexified inclusion has a solution and the set of solutions of the first inclusion is dense in the set of solutions of the convexified inclusion, then the first inclusion has a solution, too.

The importance of the relaxation theorems in the qualitative theory of the differential inclusions and control theory is emphasized in [3, pp. 123-124]. Recent results on this topic may be found, e.g. in [22], [59], [55] and [30].

Let us consider two Cauchy problems, namely (CP) and the relaxed one

$$\frac{dx(t)}{dt} \in A(t,x(t))x(t) + \overline{\text{co}}F(t,x(t)), \quad \text{a.e. } t \in I, \quad \text{and } x(0) = a. \qquad (\text{CP}_c)$$

Theorem 12 ([39]). *Suppose there are satisfied the following assumptions: (X_1), (A), (U_1), (F_5), (F_6), (F_7) and (S_1). Then*

$$\overline{\mathcal{S}_{S^1_{\overline{\text{co}}F}}(a)} = \overline{\mathcal{S}_{S^1_F}(a)}.$$

Remarks. (a) Theorem 12 is a generalization of the theorems 2.1 and 2.5 in [22]. The generalization concerns the fact that we get the similar results for the corresponding quasi-linear case as well as the fact that in theorem 2.5 in [22] it is supposed the integrable function in the assumption (F_6) is equal to the function k in (F_5), which is not necessary. This last observation was remarked also in [55].

(b) In [55, theorem 3.6] it is proved (for the case of the linear inclusions and under the supplementary assumption that F has weak compact values) the following equality

$$\mathcal{S}_{S^1_{\overline{\text{co}}F}}(a) = \overline{\mathcal{S}_{S^1_F}(a)}.$$

7. Periodic solutions

The last part of this paper exhibits two theorems on a boundary value problem and as a particular case it results sufficient conditions for the existence of a periodic solution.

Consider the following boundary value problem for quasi-linear inclusion

$$\frac{dx(t)}{dt} \in A(t,x(t))x(t) + F(t,x(t)), \quad \text{a.e. } t \in I \qquad \text{and } Lx = 0, \qquad (\text{BP})$$

where L is a linear and continuous operator from $C(I,X)$ in X.

In order to get existence results we reduce the boundary value problem to a fixed point problem. This reduction may be performed by the general method presented in [35], [36].

We need some assumptions

(L) L is a continuous bounded linear operator from the Banach space $C(I,X)$ onto X. Let's take $D = \ker L$. Hence $D \in \text{CCo}(C(I,X))$.

(L_1) For every $v \in D$ we consider the linear mapping $L_{1v} : AC(I,X) \to \mathcal{L}^1(I,X)$ and it is the same with L_1 in [36, p. 18], if A does not depend on w. Otherwise, L_{1v} is the linear and onto mapping defined by $L_{1v}x(\cdot) = \frac{dx(\cdot)}{d\cdot} - A(\cdot,v)x(\cdot)$.

(S_v) For each $v \in D$ S_v is the unique pseudo-inverse of the restriction of L to $\ker L_{1v}$ and it is denoted by S if A does not depend on w. Since we have a set of mappings S_v it is naturally to impose a condition dependence on v. Hence we suppose there are c and $q \in \mathbb{R}_+$ such that $\|S_v\| \leq c$, $\|S_v - S_u\| \leq q\|u-v\|$, $u, v \in D$.

(P) For each $v \in D$ we define the linear and continuous projector P_{1v} on $C(I,X)$ by $P_{1v}(x) = \mathcal{U}_v(\cdot,0)x(0)$. For each $v \in D$ let P_{3v} be the linear and continuous projector from $\ker L_{1v}$ to $\ker L_{1v}$ defined by $P_{3v}(\mathcal{U}_v(\cdot,0)x_c) = \mathcal{U}_v(\cdot,0)x_{c_1}$, x_c and x_{c_1} being two fixed elements in X such that $\operatorname{Im} P_{3v} = \ker(L|_{\ker L_{1v}})$.

(C) Suppose that the following compatibility condition is satisfied $\forall v \in D$, $(1_X - L_{3v}S_v)(L\int_0^t \mathcal{U}_v(t,s)f(s)\,ds) = 0$, $t \in I$, $f \in S^1_{F_x}$, where $L_{3v} = L|_{\ker L_{1v}}$.

Under the assumptions (X_1), (A), (F_1) by a *mild solution* of the boundary value problem (BP) we mean a function $x \in C(I,X)$ which satisfies

$$x(t) = \mathcal{U}_x(t,0)x(0) + \int_0^t \mathcal{U}_x(t,s)f(s)\,ds, \quad Lx = 0, \qquad t \in I, \quad f \in S^1_{F_x}.$$

Remark. From [36] it follows that the set of mild solutions of problem (BP) is contained in the set of the fixed points of the mapping $\psi : D \to \mathrm{P}(D)$, $\psi(v) = C_v(v)$ defined by $C_v(x) = \{y \in D \mid y(t) = P_{3v}(P_{1v}(x)) - S_vL\int_0^t \mathcal{U}_v(t,s)f(s)\,ds + \int_0^t \mathcal{U}_v(t,s)f(s)\,ds,\ t \in I,\ f \in S^1_{F_x}\}$. Taking into account [35] it follows that we may suppose the first term in the formula of y as zero, that is $P_{3v}(P_{1v}(x)) = 0$, for each $x \in D$.

The t-*section* of $\psi(D)$ is $C(t) = \{y(t) \mid y \in C_v(v),\ v \in D\}$. Similarly to ($M_1$) we need the assumption

(M_2) If A depends on w, suppose that for each $t \in I$, $C(t)$ is relatively compact in X.

For an arbitrary positive μ let us denote $c_3 = (c||L||+1)\,[c_1||k_\mu||_1 + c_2T||m||_1]+ qc_1||L||\,||m||_1$.

Theorem 13 ([37]). *If there are satisfied* (X_2), (A), (U_1), (F_2), (F_5), (F_6), (F_3) *or* (F_4), (L), (L_1), (S), (P), (C) *and, moreover,* $0 < c_3 < 1$, *then there exists a mild solution of problem (BP) in* D.

Theorem 14 ([37]). *Suppose that there are satisfied the following assumptions*

(i) (X_2), (A), (F_2), (F_{5-6}), (U_1), (U_3), (L), (L_1), (S_v), (P), (C);

(i) (M_2) *or* (U_2);

(iii) (F_3) *or* (F_4),

then there exists a mild solution of problem (BP) *in* D.

Remarks. (a) If operator A in (BP) does not depend on w, then in [46] it is proved a stronger result.

(b) If in (BP) we take $Lx = x(0) - x(T)$, then, based on theorems 13 and 14, we get sufficient conditions for existence of periodic solutions.

References

[1] AHMED, N. U. and XIANG, X., *Optimal control of infinite-dimensional uncertain systems*, J. Optim. Theory Appl. **80** (1994), 261–272.

[2] ANGURAJ, A. and BALACHANDRAN, K., *Existence of solutions of nonlinear differential inclusions*, Mem. Fac. Sci. Kôchi Univ. Ser. A Math. **13** (1992), 61–66.

[3] AUBIN, J. P. and CELLINA, A., *Differential Inclusions*, Springer, Berlin, 1984.

[4] AUBIN, J. P. and FRANKOWSKA, H, *Set-valued Analysis*, Birkhäuser, Boston-Basel-Berlin, 1990.

[5] AVGERINOS, E. P. and PAPAGEORGIOU, N. S., *Viable solutions for evolution inclusions*, Yokohama Math. J. **37** (1990), 101-108.

[6] BARBU, V., *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Ed, Academiei, Bucureşti; Noordhof, Leyden, 1976.

[7] BERBERIAN, S. K. *Measure and Integration*, MacMillan, New York, 1965.

[8] BORISOVICH, YU. G., GEL'MAN, B. D., MYSHKIS, A. D., and OBUHOVSKIĬ, V. V., *Set-valued mappings*, Itogi Nauki i Tekhniki, Ser. Mat. Anal. **19** (1982), 127–230 (Russian).

[9] BORSUK, K., *Theory of Retracts*, PWN, Warszawa, 1967.

[10] BRESSAN, A., CELLINA, A., and FRYSZKOWSKI, A., *A class of absolute retracts in spaces of integrable function*, Proc. Amer. Math. Soc. **112** (1991), 413–418.

[11] CASTAING, C. and VALADIER, M., *Convex Analysis and Measurable Multifunctions*, Springer, Berlin, 1877.

[12] CELLINA, A., *On the differential inclusion* $x' \in [-1, 1]$, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **69** (1980), no. 1-2, 1–6.

[13] CELLINA, A. and STAICU, V., *Well posedness for differential inclusions on closed sets*, J. Differential Equations **92** (1991), 2–13.

[14] COLOMBO, R. M., FRYSZKOWSKI, A., RZEŽUCHOWSKI, T., and STAICU, V., *Continuous selections of solution sets of Lipschitzean differential inclusions*, Funkcial. Ekvac. **34** (1991), 321–330.

[15] DEIMLING, K., *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[16] DEIMLING, K., *Multivalued Differential Equations*, W. de Gruyter, Berlin, 1992.

[17] DIESTEL, J. and UHL, L., *Vector Measures*, Amer. Math. Soc., Providence, 1977.

[18] DINÇULEANU, N., *Vector Measures*, Pergamon, London, 1972.

[19] FILIPPOV, A. F., *Classical solutions of differential equations with multivalued right-hand side*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **3** (1967), 16–26 (Russian).

[20] FILIPPOV, A. F., *Classical solutions of differential equations with multivalued right-hand side*, SIAM J. Control **5** (1967), 609–621.

[21] FILIPPOV, A. F., *Differential Equations with Discontinuous Righthand Side*, Kluwer, Dordrecht, 1988.

[22] FRANKOWSKA, H., *A priori estimates for operational differential inclusions*, J. Differential Equations **84** (1990), 100-128.

[23] HARTMAN, P., *Ordinary Differential Equations*, Wiley, New York, 1964.

[24] HIAI, F. and UMEGAKI, H., *Integrals, conditional expectations and martingales of multivalued functions*, J. Multivariate Anal. **7** (1977), 149–182.

[25] HIMMELBERG, C. J. and VAN VLECK, F. S., *Lipschitzean generalized differential equations*, Rend. Sem. Mat. Univ. Padova **48** (1973), 159–168.

[26] HIMMELBERG, C. J., *Existence of solutions for generalized differential equations with unbounded right-hand side*, J. Differential Equations **61** (1986), 295–320.

[27] HU, S.-T., *Theory of Retracts*, Wayne State Univ. Press, Detroit, 1965.

[28] HU, S. and PAPAGEORGIOU, N. S., *On the existence of periodic solutions for a class of nonlinear evolution inclusions*, Boll. Un. Mat. Ital. B (7) **7** (1993), 591–605.

[29] KACZYŃSKI, H. and OLECH, C., *Existence of solutions of orientor fields with non-convex right-hand side*, Ann. Polon. Math. **29** (1974), 61–66.

[30] KANDILAKIS, D. A. and PAPAGEORGIOU, N. S., *On the properties of the Aumann integral with applications to differential inclusions and control systems*, Czechoslovak Math. J. **39** (1989), 1–15, and **40** (1990), 695–696.

[31] KLEIN, E. and THOMPSON, A. C., *Theory of Correspondences*, Wiley, New York, 1984.

[32] KOBAYASI, K. and SANEKATA, N., *A method of iteration for quasi-linear evolution equations in nonreflexive Banach spaces*, Hiroshima Math. J. **19** (1989), 521–540.

[33] LIONS, J. L., *Contrôle Optimal de Systemes Gouvernés par des Équations aux Dérivées Partielles*, Dunod, Gauthier-Villars, Paris, 1968.

[34] MARTIN, R. H., *Nonlinear Operators and Differential Equations in Banach Spaces*, Wiley, New York, 1967.

[35] MAWHIN, J., *Equations non linéaires dans les espaces de Banach*, Rap. 39 (1971), Semin. Math. Appl. Mec., Univ. Cath. Louvain.

[36] MAWHIN, J., *Nonlinear operator equations in vector spaces* I. *Reduction theorems to fixed point problems*, Rap. 14 (1972), Semin. Math. Appl. Mec., Univ. Cath. Louvain.

[37] MUREŞAN, M., *On a boundary value problem for quasi-linear differential inclusions of evolution*, Collect. Math. **45** (1994), no. 2, 165–175.

[38] MUREŞAN, M., *Qualitative properties of solutions to quasi-linear inclusions* I., Pure Math. Appl. **5** (1994), 331–353.

[39] MUREŞAN, M., *Qualitative properties of solutions to quasi-linear inclusions* II., Pure Math. Appl. **6** (1995), 331–353.
[40] MUREŞAN, M. and MUREŞAN C., *On the solutions of quasi-linear inclusions of evolution*, (introduced at the 4^{th} Coll. Qualit. Theory Diff. Eqs., August, 17-22, 1993 Hungary).
[41] OKAZAWA, N. and UNAI, A., *Abstract quasilinear evolution equations in a Hilbert space, with applications to symmetric hyperbolic systems*, SUT Jour. Math. **29** (1993), 263–290.
[42] OLECH, C., *Existence of solutions of non-convex orientor fields*, Boll. Un. Mat. Ital. (4), **11** (1975), no. 3. 189–197.
[43] ORNELAS, A., *A continuous version of the Filippov-Gronwall inequality for differential inclusions*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. (9) Mat. Appl. 1 (1990), no. 1, 105–110.
[44] PAPAGEORGIOU. N. K., *On measurable multifunctions with applications to random multivalued equations*, Math. Japon. **32** (1987), 437–464.
[45] PAPAGEORGIOU. N. K., *On multivalued evolution equation and differential inclusions in Banach spaces*, Comment. Math. Univ. St. Paul. **36** (1987), 21–39.
[46] PAPAGEORGIOU. N. K., *Boundary value problems for evolution inclusions*, Comment. Math. Univ. Carolin. **29** (1988), 355–363.
[47] PAVEL, N., *Nonlinear Evolution Operators and Semigroups. Applications to Partial Differential Equations*, Springer, Berlin, 1987.
[48] PAZY, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.
[49] RUS, I. A., *Technique of the fixed point structures for multivalued mappings*, Math. Japon. **38** (1993), 289–296.
[50] SANEKATA, N., *Abstract quasi-linear equations of evolution in nonreflexive Banach spaces*, Hiroshima Math. J. **19** (1989), 109–139.
[51] STAICU V., *Well Posedness for Differential Inclusions*, ISAS, Trieste, Ph. D. thesis, 1989-1990.
[52] STAICU V., *Continuous flows of Lipschitzean differential inclusions*, Ref. S.I.S.S.A. 184/90/M (1990).
[53] STAICU V., *Continuous selections of solution sets to evolution*, Proc. Amer. Math. Soc. **113** (1991), 403–413.
[54] STAICU, V. and WU, H., *Arcwise connectedness of solution sets to Lipschitzean differential inclusions*, Boll. Un. Mat. Ital. (7) **5-A** (1991), 253–256.
[55] TOLSTONOGOV, A. A., *On the solutions of evolution inclusions.* I, Sibirs. Mat. Zh. **33** (1992), no. 3, 161–172 (Russian).
[56] TOLSTONOGOV, A. A. and UMANSKII, Y. I., *On the solutions of evolution inclusions.* II, Sibirs. Mat. Zh. **33** (1992), no. 4, 163–174 (Russian).
[57] WAZEWSKI, T., *Systémes de commande et équations au contingent*, Bull. Polish Acad. Sci. Math. **9** (1961), no. 3, 151–155.
[58] WAZEWSKI, T., *Sur une généralisation de la notion des solutions d'une équation au* contingent, Bull. Polish Acad. Sci. Math. **10** (1962), 11–15.
[59] ZHU, Q. J., *On the solution set of differential inclusions in Banach spaces*, J. Differential Equations **93** (1991), 213–237.

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