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FUZZY SYSTEMATIC SPACES

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Abstract. In this paper fuzzy systematic spaces are considered. In terms of definitions of fuzzy cover and fuzzy sheaf it will be defined fuzzy cohomology in the sense of Čech.

1. Introduction

In philosophy [1,2,3] there are some definitions of systems. These definitions help us to define fuzzy systematic spaces, which are suitable spaces for description of language [6]. It will be shown that on a consistent fuzzy systematic space the notion of fuzzy cover is acquired, so we can define fuzzy sheaf cohomology in the sence of Čech.

Fuzzy relations have been studied by Zadeh [8], Kaufman [5], Rosenfeld [7] and in this paper in the one hand the fuzzy relations are used and in the other hand a fuzzy systematic space is explained.

Definition 1.1. A fuzzy system for the fuzzy set X is a collection $S = \{R_\gamma\}$, $\gamma \in \Gamma$ which satisfies the following conditions:

(*i*) $R_\gamma \subset X \times Y_\gamma$ are fuzzy relations;

(*ii*) for every $x \in X$ there exist $\gamma \in \Gamma$ and y_γ such that $(x, y_\gamma) \in R_\gamma$.

A fuzzy systematic space is an order pair (X, S).

Example 1.2. Let $(C, \subset)$ be a Site [4], for all $a \in C$ define $R_a = \{(a,b) : b \subset a\}$ and $\mu_{R_a}(a,b) := \max\{\mu_C(a), \mu_C(b)\}$. Then $(C, \{R_a\})$ is a fuzzy systematic space. (μ is a membership function.)

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Definition 1.3. Let (X, S) be a fuzzy systematic space. Then an object $R \in S$ is called a speciality of S if for all $\gamma \in \Gamma$, $R \cap R_\gamma \in S$.

Definition 1.4. A consistent fuzzy systematic space is a fuzzy systematic space with the following condition:

For all R_1 and R_2 belong to S, $R_1 \cap R_2 \in S$ and $\mu_{R_1|_{R_1 \cap R_2}} = \mu_{R_2|_{R_1 \cap R_2}}$

An inconsistent fuzzy systematic space is a fuzzy systematic space which is not consistent.

Theorem 1.5. *Let (X, S) be a fuzzy systematic space. Then there are subsets, X_c, X_I of X and S_c, S_I of S such as:*
(i) $X = X_c \cup X_I$;
(ii) $S = S_c \cup S_I$, and $S_c \cap S_I = \emptyset$;
(iii) (X_c, S_c) is a consistent fuzzy systematic space and (X_I, S_I) is an inconsistent fuzzy systematic space that has no speciality.

Proof. Let

$$\begin{aligned} S &= \{R_\gamma\} \text{ put} \\ S_c &= \{R_\gamma : (\forall \beta \in \Gamma)(R_\gamma \cap R_\beta \in S)\}; \\ S_I &= S - S_c, \\ X_I &= \{x \in X : (x, y) \in R_\gamma \text{ for some } R_\gamma \in S_c\}; \\ X_I &= \{x \in X : (x, y) \in R_\gamma \text{ for some } R_\gamma \in S_I\}. \blacksquare \end{aligned}$$

□

Definition 1.6. If $S' \subset S$, then the fuzzy systematic space (X, S') is called a sub-systematic space of (X, S).

2. Fuzzy Sheaf

There must be a definition of fuzzy Grothendieck topology on S, define as a map:

"Element of $S \longmapsto A$ subset of the powerset"

with the following conditions:
(a) $\{R\} \in G(R)$ for all $R \in S$;
(b) If $\{R_i\} \in G(R)$ then $R_i \subset R$ and $\mu_{R_i} = \mu_{R|_{R_i}}$.

A fuzzy cover for $R \in S$ is an element of $G(R)$.

Definition 2.1. A subset U of the fuzzy system S is a fuzzy cover for S if for all $R \in S, U$ contains at least one fuzzy cover of R.

A fuzzy cover U' is a finer fuzzy cover than U, if for all $R' \in U'$ there exists $R \in U$ so that $R' \subset R$ and $\mu_{R'}$ be equal to μ_R on R'.

We define a fuzzy presheaf P on a fuzzy systematic space (X, S) as a map;

$$\begin{array}{rcl} \text{Object in } S & \longrightarrow & \text{Fuzzy abelian groups} \\ R & \longmapsto & \Gamma(R, P) \end{array}$$

together with restriction maps

$$\rho_{RF} : \Gamma(F, P) \longrightarrow \Gamma(R, P) \quad \text{if} \quad R \subset F$$

that satisfies the following properties;

(i) The restrictions are fuzzy group homomorphisms;

(ii) If $R \subset F$ and $G \subset R$ then $\rho_{FF} = id$, $\rho_{GR} o \rho_{RF} = \rho_{GF}$.

A fuzzy presheaf P on systematic space (X, S) is a sheaf if satisfies the following conditions:

(iii) For every fuzzy cover $\{R_i\}$ of R and $a, b \in \Gamma(R, P)$, if $a|_{R_i} = b|_{R_i}$ for all i, then $a = b$;

(iv) For every fuzzy cover $\{R_i\}$ of R, if $a_i \in \Gamma(R_i, P)$ and $a_i|_{R_j} = a_j|_{R_i}$ for all i, j, then there exists $a \in \Gamma(R, P)$ such that $a|_{R_i} = a_i$.

Example 2.2. Suppose that X is an n-dimensional complex fuzzy manifold

$$S = \{R_{UV} = U \times V : U \text{ and } V \text{ are charts of } X\}$$

and $\mu_{R_{UV}}(u, v) = \max\{\mu_X(u), \mu_X(v)\}$ for all $(u, v) \in U \times V$;

(a) If $\Gamma(U \times V, B)$ be the fuzzy group of bounded holomorphic functions on chart $U \times V$ of $X \times X$ then the map, $R_{UV} \longrightarrow \Gamma(U \times V, B)$ is a fuzzy presheaf that is not a fuzzy sheaf.

(b) If $\Gamma(U \times V, O(m))$ be the fuzzy group of homogeneous holomorphic functions of degree m on chart $U \times V$ then the map $R_{UV} \longrightarrow \Gamma(U \times V, O(m))$ is a fuzzy sheaf.

3. Fuzzy Cohomology

Now we define fuzzy cohomology in the sence of Čech for a consistent fuzzy systematic space.

Suppose that U is a fuzzy cover for a consistent fuzzy systematic space (X, S). A q-simplex is a $q+1$ tuple of elements of U. For $\delta = (R_0, R_1, \ldots, R_q)$ define $|\delta| = R_0 \cap R_1 \cap \cdots \cap R_q$. A q-cochain with respect to U with coefficients in a fuzzy sheaf P is a map;

$$\begin{array}{rcl} \{\delta : \delta \text{ is a } q-\text{simplex}\} & \xrightarrow{f} & U\Gamma(\delta, P) \\ \delta & \longmapsto & f(\delta) \end{array}$$

If $\delta = (R_{i_0}, R_{i_1}, \ldots, R_{i_q})$ then we denote $f(\delta)$ by $f_{i_0 i_1 \ldots i_q}$ and $\{f_{i_0 i_1 \ldots i_q} \in \Gamma(R_{i_0} \cap R_{i_1} \cap \cdots \cap R_{i_q}, P)\}$ is a called q-cochain. The set of these q-cochains is denoted by $C^q(U, P)$. The coboundary operator is:

$$\{f_{i_0 i_1 \ldots i_q}\} \overset{\delta_{q+1}}{\longmapsto} \{\rho_{[i_0 f_{i_1} \cdots i_{q+1}]}\}$$

where ρ_{i_0} is restriction to R_{i_0}.

As usual we define $Z^q(U, P) = Ker\delta_{q+1}$ and $B^q(U, P) = I_m\delta_q$. The set $H^q(U, P) = Z^q(U, P)/B^q(U, P)$ is called the fuzzy Čech cohomology of P with respected to U and the set $H^q(S, P) = \lim_{U} \text{ind} H^q(U, P)$ is called the fuzzy Čech cohomology of (X, S) with cofficients in the fuzzy sheaf P, where limind is the inductive limit.

Theorem 3.1. *Let (X, S) be a fuzzy systematic space and suppose that there exists $R \in S$ so that R is the maximal element of S with respected to the inclusion. Then $H^0(S, P) = \Gamma(R, P)$.*

Proof. Let $d \in H^0(S, P) = (U^0 H^0(U, P)/\sim)$ that $\sim$ is the usual equivalence relation. So $d = [(g_i)]$ where $g_i \in \Gamma(R_i, P)$ for some fuzzy cover V of S. By the condition $\delta g = 0$, we have;

$$(\delta g)_{ij} = g_j - g_i = 0 \quad \text{on} \quad R_i \cap R_j.$$

Therefore there is a global section $g \in \Gamma(R, P)$ which agrees with the g_i locally.■ □

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