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NOTE ON SPREADS AND PARTIAL SPREADS

DĂNUŢ MARCU

Abstract. The aim of this note is to give an answer to a question of [1].

1. Introduction

In this note, we show the existence of a spread, which is not a dual spread, thus answering to a question in [1]. We also obtain some related results on spreads and partial spreads.

Let $\mathbf{P} = PG(2t-1, F)$ be a projective space of odd dimension $(2t-1,\ t \geq 2)$ over the field F. In accordance with [1], we use the following definitions. A partial spread S of $\mathbf{P}$ is a collection of $(t-1)$-dimensional projective subspaces of $\mathbf{P}$, which are pairwise disjoint. S is maximal, if it is not properly contained in any other partial spread. In particular, if every point of $\mathbf{P}$ is contained in some member of S, then S is a spread. If each $(2t-2)$-dimensional projective subspace of $\mathbf{P}$ contains exactly one member of S, then S is called a dual spread.

2. Main results

In the sequel, $|S|$ will denote the number of subspaces in S.

Theorem 1. *If F is finite, then S is a spread if and only if S is a dual spread.*

Proof. Suppose that S is a spread, which is not a dual spread of $\mathbf{P}$. Let δ be any correlation of $\mathbf{P}$ (for the existence of such a δ, see [2, p.41]). Then, S^δ, the image of S under δ, ia a partial spread, which is not a spread. But, $|S^\delta| = |S|$ and F is finite. So, we obtain a contradiction. Similarly, every dual spread is a spread. □

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For simplicity, we now specialize to the case $t = 2$ and we assume that F is commutative, to facilitate the notion of regulus.

We say that a spread S is regular provided that, for every line l of $\mathbf{P}$ which is not in S, the lines of S meeting l form a regulus R of $\mathbf{P}$.

Not al spreads are regular. We can obtain a new non-regular spread S' from S, by the process of replacing some regulus R by its opposite regulus R'. If S' can be obtained from a regular spread S by finitely many iterations of such a process, then S is called subregular.

Theorem 2. *Every regular spread S of |bfP is a dual spread.*

Proof. Let π be any plane of $\mathbf{P}$. Then, π contains at most one line of S. To show that there must be one, let l be any line of π, which is not in S. The lines of S, meeting l, form a regulus R. Let p and q be any two lines of the opposite regulus R', different from l. Then, p and q meet π in distinct point P and Q, not on l. The line PQ of π meets l and, hence, meets three lines of R'. Thus, PQ is a line of R, that is, of S. □

A straightforward extension of this argument yields the following

Theorem 3. *Let S be a spread, which is a dual spread. Suppose that S contains a regulus R. Then, the spread S', obtained from S by replacing the regulus R by its opposite regulus R', is also a dual spread.*

Corollary 1. *Every subregular spread is a dual spread.*

Theorem 4. *There exists a spread S of $\mathbf{P}$, such that S is not a dual spread and no four lines of S are contained in a regulus.*

Proof. Let F be infinite and countable. Choose any plane π and list the points in $\pi(P_1, P_2, P_3, \ldots)$ and the points not in $\pi(Q_1, Q_2, Q_3, \ldots)$. Through P_1, construct the line $l_1 = P_1Q_1$. Suppose that $l_1, l_2, \ldots, l_n$ have been constructed, such that:

(a) no l_i is in π,

(b) no two l_i intersect and

(c) no four l_i are in a regulus.

We now show that l_{n+1} can be constructed in such a way, that (a)-(c) are satisfied also by $\{l_1, l_2, \ldots, l_{n+1}\}$.

If n is odd, let $X = P_j$ be the first pint in π, which is on none of the lines $l_1, l_2, \ldots, l_n$ and $Y = Q_k$ the first point not in π, such that:

(d) Y is on none of the n planes Xl_i, $i = 1, 2, \ldots, n$ and

(e) XY does not belong to any one of the (n_3) reguli determined by $l_1, l_2, \ldots, l_n$.

Then, put $l_{n+1} = XY = P_jQ_k$.

If n is even, let $X = Q_s$ be the first point not in π, which is on none of the l_i, $i = 1, 2, \ldots, n$ and $Y = P_t$ the first point in π, such that (d) and (e) are satisfied. Then, put $l_{n+1} = XY = Q_sP_t$.

Clearly, $l_1, l_2, \ldots, l_{n+1}$ satisfy the conditions (a)-(c). Furthermore, our construction guarantees that each point of $\mathbf{P}$ is on a line of S. Thus, the theorem is proved. □

There is an interesting consequence of the Theorem 4, that is,

Corollary 2. *Maximal partial spreads W, which are not spreads, exist in* $\mathbf{P}$.

Proof. Consider the image W of S, under any correlation of $\mathbf{P}$. □

Remark. the above corollary is also true if F is finite (for an example in $PG(3,4)$, see [3]).

We end this note with the following

Conjecture. *There exist such maximal partial spreads W, with $q^2 - q + 1 \leq |W| \leq q^2 - q + 2$ in $PG(3,q)$, for any q.*

References

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STR. PASULUI 3, SECT.2, 70241 BUCHAREST, ROMANIA