ON α-TYPE UNIFORMLY CONVEX FUNCTIONS

IOANA MAGDAŞ

Abstract. We determine necessary and sufficient condition for a function f with negative coefficients to be n-uniformly starlike of type α and we obtain a conection between the class of all such functions $UT_n(\alpha)$ and the class of the functions n-starlike of order α and type β with negative coefficients $T_n(\alpha, \beta)$. Distortion bounds and extreme points are also obtained.

1. Introduction

Denote by S the family of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic and univalent in the unit disk $U = \{z : |z| < 1\}$ and by S^*, respectively $S^c(\alpha)$ the usual class of starlike functions, respectively convex functions of order α, $\alpha \geq 0$.

Definition 1. A function f is said to be uniformly convex in U if f is in S^c and has the property that for every circular arc γ contained in U, with center ζ also in U, the arc $f(\zeta)$ is a convex arc.

Let be UCV or US^c denote the class of all such functions.

Goodman gave the following two-variable analytic characterizations of this class, then Ma and Minda [1] and Rønning [2] independently found a one variable characterization for US^c.

Theorem A. *Let f have the form (1). Then the following are equivalent:*

1991 *Mathematics Subject Classification:* 30C45.
Key words and phrases: starlikeness, α-convexity.

(i) $f \in US^c$

(ii) $\operatorname{Re}\left\{1+\frac{(z-\zeta)f''(z)}{f'(z)}\right\} \geq 0$ *for all pairs* $(z,\zeta) \in U \times U$

(iii) $\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|$, *for all* $z \in U$

(iv) $1+\frac{zf''(z)}{f'(z)} \prec q$, *where* $q(z) = 1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$ *is a Riemann mapping function from* U *to* $\Omega = \{w = u+iv : \; v^2 < 2u-1\} = \{w : \; \operatorname{Re} w > |w-1|\}$.

Note that Ω is the interior of a parabola in the right half-plane which is symmetric about the real axis and has vertex at $(1/2, 0)$.

Denote by T the subclass of S consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \; (n \in \mathbb{N} \setminus \{0,1\}), \; z \in U \tag{2}$$

and denote by $T^*(\alpha)$ and $T^c(\alpha)$ the class of functions of the form (2) that are, respectively, starlike of order α and convex of order α, $\alpha \in [0,1)$, and denote by $UT^c = US^c \cap T$ the class of functions uniformly convex with negative coefficients.

Definition 2. A function f of the form (1) is said to be uniformly convex of α-type, $\alpha \geq 0$ if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \alpha \left|\frac{zf''(z)}{f'(z)}\right|, \tag{3}$$

for all $z \in U$.

We let $US^c(\alpha)$ denote the class of all such functions.

Note that $US^c(0) = S^c$, $US^c(1) = US^c$ and $US^c(\alpha) \subset US^c$ for $\alpha > 1$.

Remark. A function f of the form (1) is in $US^c(\alpha)$ if and only if $1+zf''(z)/f'(z) \in D$ for all $z \in U$, where D is:

i) for $\alpha > 1$ bounded by the ellipse

$$\frac{\left(u-\frac{\alpha^2}{\alpha^2-1}\right)^2}{\frac{\alpha^2}{(\alpha^2-1)^2}} + \frac{v^2}{\frac{1}{\alpha^2-1}} = 1$$

ii) for $\alpha = 1$ bounded by the parabola

$$v^2 = 2u - 1$$

iii) for $\alpha \in (0, 1)$ bounded by the positive branch of the hyperbole

$$\frac{\left(u + \frac{\alpha^2}{1-\alpha^2}\right)^2}{\frac{\alpha^2}{(1-\alpha^2)^2}} - \frac{v^2}{\frac{1}{1-\alpha^2}} = 1$$

iv) for $\alpha = 0$ the half-plane $u \geq 0$

In conclusion $US^c(\alpha) \subset S^c(\alpha/(\alpha+1))$ for $\alpha \geq 0$.

In [5] is defined $UT^c(\alpha) = US^c(\alpha) \cap T$ and it is given a coefficient characterization for this class.

Theorem A. *Let f have the form (1) and $\alpha \geq 0$. f is in $UT^c(\alpha)$ if and only if*

$$\sum_{j=2}^{\infty} j[j(\alpha+1) - \alpha]a_j \leq 1, \tag{4}$$

hence $UT^c(\alpha) = T^c(\alpha/(\alpha+1))$.

Sălăgean [4] introduced the differential operator

$$D^n : A \to A, \quad n \in \mathbb{N},\ A = \{f \in H(U) :\ f(0) = f'(0) - 1 = 0\}$$

defined by $D^0f(z) = f(z)$, $D^1f(z) = Df(z) = zf'(z)$, $D^nf(z) = D(D^{n-1}f(z))$, for $n \geq 2$ and it is easy to prove that

$$D^nf(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j. \tag{5}$$

He also defined the class $S_n(\alpha, \beta)$ of n-starlike functions of order α and type β by

$$S_n(\alpha, \beta) = \{f \in A : \ |J(f, n, \alpha; z)| < \beta\}, \quad \alpha \in [0, 1),\ \beta \in (0, 1],\ n \in \mathbb{N}$$

where

$$J(f, n, \alpha; z) = \frac{D^{n+1}f(z) - D^nf(z)}{D^{n+1}f(z) + (1 - 2\alpha)D^nf(z)}, \quad z \in U. \tag{6}$$

Denote by $T_n(\alpha, \beta) = S_n(\alpha, \beta) \cap T$ the class of functions n-starlike of order α and type β with negative coefficients.

Sălăgean [4] gave a coefficient characterization for this class.

Theorem B. *Let f have the form (2), $\alpha \in [0, 1)$, $\beta \in (0, 1]$. f is in $T_n(\alpha, \beta)$ if and only if*

$$\sum_{j=2}^{\infty} j^n[j - 1 + \beta(j + 1 - 2\alpha)]a_j \leq 2\beta(1 - \alpha). \tag{7}$$

The result is exactly and the extremal functions are

$$f_j(z) = z - \frac{2\beta(1 - \alpha)}{j^n[j - 1 + \beta(j + 1 - 2\alpha)]}z^j, \quad j \in \mathbb{N}_2 = \mathbb{N} \setminus \{0, 1\}. \tag{8}$$

Definition 3. A function f of the form (1) is said to be n-uniformly starlike of type α, $\alpha \geq 0$ and $n \in \mathbb{N}$ if

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^nf(z)}\right\} \geq \alpha \left|\frac{D^{n+1}f(z)}{D^nf(z)} - 1\right| \tag{9}$$

for all $z \in U$.

We let $US_n(\alpha)$ denote the class of all such functions.

Note that $US_0(1) = S_p$ introduced in [3], $US_1(1) = US^c$ and because $US_n(\alpha) \subset S_n(0, 1) \subset S^*$ follow that the uniformly functions of type α are univalents.

Remark. f is in $US_n(\alpha)$ if and only if $D^{n+1}f(z)/D^n f(z) \in D$ for all $z \in U$.

Denote by $UT_n(\alpha) = US_n(\alpha) \cap T$ the class of n-uniformly starlike functions of type α with negative coefficients.

We will give a coefficient characterization for this class.

2. Main results

Theorem 1. *Let f have the form (2), $\alpha \geq 0$, $n \in \mathbb{N}$. Then f is in $UT_n(\alpha)$ if and only if*

$$\sum_{j=2}^{\infty} j^n[j(\alpha+1)-\alpha]a_j \leq 1. \tag{10}$$

The result is exactly and the extremal functions are

$$f_j(z) = z - \frac{1}{j^n[j(\alpha+1)-\alpha]}z^j, \quad j \in \mathbb{N}_2 = \mathbb{N}\setminus\{0,1\}.$$

Proof. Assume that $f \in UT_n(\alpha)$, then

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} \geq \alpha\left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| \tag{11}$$

for all $z \in U$.

For $z \in [0,1)$ the inequality become

$$\frac{1-\sum_{j=2}^{\infty} j^{n+1}a_j z^{j-1}}{1-\sum_{j=2}^{\infty} j^n a_j z^{j-1}} \geq \alpha\left|\frac{\sum_{j=2}^{\infty} j^n(j-1)a_j z^{j-1}}{1-\sum_{j=2}^{\infty} j^n a_j z^{j-1}}\right|. \tag{12}$$

Since $UT_n(\alpha) \subset T_n(0,1)$ we have:

$$\sum_{j=2}^{\infty} j^{n+1}a_j < 1$$

then

$$\sum_{j=2}^{\infty} j^n a_j z^{j-1} < 1.$$

Inequality (13) reduce to

$$1-\sum_{j=2}^{\infty} j^{n+1}a_j z^{j-1} \geq \alpha\sum_{j=2}^{\infty} j^n(j-1)a_j z^{j-1}$$

and letting $z \to 1^-$ along the real axis, we obtain the desired inequality

$$\sum_{j=2}^{\infty} j^n [j(\alpha+1)-\alpha] a_j \leq 1.$$

Conversely we assume the inequality (11) and it suffices to show that:

$$\alpha \left| \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right| - \operatorname{Re}\left\{ \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right\} \leq 1.$$

We have

$$\alpha \left| \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right| - \operatorname{Re}\left\{ \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right\} \leq (\alpha+1) \left| \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right| \leq$$

$$\leq (\alpha+1) \frac{\sum\limits_{j=2}^{\infty} j^n (j-1) a_j |z|^{j-1}}{1 - \sum\limits_{j=2}^{\infty} j^n a_j |z|^{j-1}} \leq (\alpha+1) \frac{\sum\limits_{j=2}^{\infty} j^n (j-1) a_j}{1 - \sum\limits_{j=2}^{\infty} j^n a_j} \leq 1$$

according to (11), and the proof is complete. □

Remark. For $n = 1$ we obtain the Theorem A.

Corollary 1. *Let f have the form (2). If f is in $UT_n(\alpha)$ then*

$$a_j \leq \frac{1}{j^n[j(\alpha+1)-\alpha]}, \quad j \in \mathbb{N}_2. \tag{13}$$

Corollary 2. *For $\alpha \geq 0$ and $n \in \mathbb{N}$, $UT_n(\alpha) = T_n(\alpha/\alpha+1, 1)$.*

Proof. Replacing α with $\alpha/\alpha+1$, β with 1 in the necessary and sufficient coefficient conditions in Theorem B, we obtain the corresponding coefficient condition of Corollary 2. □

Theorem 2. *If $f \in UT_n(\alpha)$, $\alpha \geq 0$ then*

$$\begin{matrix} r - \dfrac{1}{2^n(\alpha+2)} r^2 \leq |f(z)| \leq r + \dfrac{1}{2^n(\alpha+2)} r^2 \\ 1 - \dfrac{1}{2^{n-1}(\alpha+2)} r \leq |f'(z)| \leq 1 + \dfrac{1}{2^{n-1}(\alpha+2)} r^2 \end{matrix}, \quad |z| = r.$$

The results are the best possible.

Let f and g be two functions of the form (2)

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j \quad \text{and} \quad g(z) = z - \sum_{j=2}^{\infty} b_j z^j$$

then we define the (modified) Hadamard product or convolution of f and g by

$$(f * g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j.$$

Theorem 3. *If $f, g \in UT_n(\alpha)$, $\alpha \geq 0$ then $f * g \in UT_n\left(\frac{\rho}{1-\rho}\right)$, where*

$$\rho = 1 - \frac{1}{2^{2n}(\alpha+2)^2 - 1}.$$

This result is sharp.

References

[1] W. Ma and D. Minda, *Uniformly convex functions*, Ann. Polon. Math., 57(1992), 166-175.

[2] F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., 118(1992), 449-454.

[3] F. Rønning, *Integral representations of bounded starlike functions*, Ann. Polon. Math., LX3(1995), 289-297.

[4] G.S. Sălăgean, *Subclasses of univalent functions*, Lect. Notes in Math., 1013, Springer Verlag, 1983, 362-372.

[5] K.G. Subramanian, G. Murugusundaramoorthy, P. Balasubrahmanyam and H. Silverman, *Subclasses of uniformly convex and uniformly starlike functions*, Math. Japonica, No.3(1996), 517-522.

"BABEŞ-BOLYAI" UNIVERSITY, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, 3400 CLUJ-NAPOCA, ROMANIA.