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SOME APPLICATIONS OF ORE'S GENERALIZED THEOREMS IN THE FORMATION THEORY

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Abstract. Ore's theorems [9] are a powerful tool in the formation theory of finite solvable groups. In [4] we obtained a generalization of some of these theorems on finite π-solvable groups, where π is an arbitrary set of primes. The present paper applies Ore's generalized theorems to prove the existence and conjugacy of covering subgroups in finite π-solvable groups.

1. Preliminaries

All groups considered in the paper are finite. We denote by π an arbitrary set of primes and by π' the complement to π in the set of all primes.

Definition 1.1.

1. A class $\underline{X}$ of groups is a *homomorph* if $\underline{X}$ is closed under homomorphisms.
2. A group G is *primitive* if G has a *stabilizer*, i.e. a maximal subgroup H with $\mathrm{core}_G H = 1$, where $\mathrm{core}_G H = \cap \{ H^g \ / \ g \in G \}$.
3. A homomorph $\underline{X}$ is a *Schunck class* if $\underline{X}$ is *primitively closed*, i.e. if any group G, all of whose primitive factor groups are in $\underline{X}$, is itself in $\underline{X}$.

4. If $\underline{X}$ is a class of groups and G is a group, a subgroup E of G is called an $\underline{X}$-*covering subgroup* of G if: (i) $E \in \underline{X}$; (ii) $E \leq V \leq G$, $V_0 < V$, $V/V_0 \in \underline{X}$ imply $V = E V_0$.

Definition 1.2. a) A group G is π-*solvable* if every chief factor of G is either a solvable π-group or a π'-group. When π is the set of all primes, we obtain the notion of

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solvable group.

b) A class $\underline{X}$ of groups is said to be π-*closed* if:

$G/O_{\pi'}(G) \in \underline{X} \Rightarrow G \in \underline{X}$,

where $O\pi'(G)$ denotes the largest normal π'-subgroup of G. We shall call π-*homomorph* a π-closed homomorph and π-*Schunck class a* π-closed Schunck class.

Let $\underline{X}$ be a homomorph. The following properties given in [8] are also true for any finite group:

Proposition 1.3. *If E is an $\underline{X}$-covering subgroup of G and $E \leq H \leq G$, then E is an $\underline{X}$-covering subgroup of H.*

Proposition 1.4. *Let E be an $\underline{X}$-covering subgroup of G and N a normal subgroup of G. Then EN/N is an $\underline{X}$-covering subgroup of G/N.*

Proposition 1.5. *If N is a normal subgroup of G, E^*/N is an $\underline{X}$-covering subgroup of G/N and E is an $\underline{X}$-covering subgroup of E^*, then E is an $\underline{X}$-covering subgroup of G.*

Finally, we shall use a result of R. Baer [1] which we give below:

Theorem 1.6. *A solvable minimal normal subgroup of a group is abelian.*

2. Ore's generalized theorems

In [3] we gave some properties of finite primitive groups, among which we remind the following:

Proposition 2.1. *If G is a primitive group and W is a stabilizer of G, then for any minimal normal subgroup M of G we have $MW = G$.*

In [4] we proved the following theorems generalizing Ore's theorems from [9]:

Theorem 2.2. *Let G be a primitive π-solvable group. If G has a minimal normal subgroup which is a solvable π-group, then G has one and only one minimal normal subgroup.*

Corollary 2.3. *If G is a primitive π-solvable group, then G has at most one minimal normal subgroup which is a solvable π-group.*

Corollary 2.4. *If a primitive π-solvable group G has a minimal normal subgroup which is a solvable π-group, then G has no minimal normal subgroups which are π'-groups.*

Theorem 2.5. *If G is a primitive π-solvable group and N is a minimal normal subgroup of G which is a solvable π-group, then $C_G(N) = N$.*

Theorem 2.6. *Let G be a π-solvable group such that:*
(i) there is a minimal normal subgroup M of G which is a solvable π-group and $C_G(M) = M$;
(ii) there is a minimal normal subgroup L/M of G/M such that L/M is a π'-group.
Then G is primitive.

Theorem 2.7. *If G is a π-solvable group satisfying (i) and (ii) from 2.6., then any two stabilizers W and W^* of G are conjugate in G.*

Theorem 2.8. *If G is a primitive π-solvable group, $V < G$ such that there is a minimal normal subgroup M of G which is a solvable π-group and $MV = G$, then V is a stabilizer of G.*

3. Existence and conjugacy of covering subgroups in finite π-solvable groups

We give here a new proof of the existence and conjugacy theorems of covering subgroups in finite π-solvable groups [2]. The proof from [2] is based on some R. Baer's theorems (see [1]). According to the importance of Ore's theorems in the formation theory of finite solvable groups, we put the question if Ore's generalized theorems could not be used to prove the existence and conjugacy theorems of covering subgroups in finite π-solvable groups. The answer is affirmative as we show below.

Theorem 3.1. *If $\underline{X}$ is a π-homomorph and G is a π-solvable group, then any two $\underline{X}$-covering subgroups of G are conjugate in G.*

Proof. By induction on $|G|$. Let E and F be two $\underline{X}$-covering subgroups of G.
If $G \in \underline{X}$, using 1.1.d) we obtain $E = F = G$ and so E and F are conjugate in G. Let now $G \notin \underline{X}$. If N is a minimal normal subgroup of G, by 1.4. we have that EN/N and FN/N are $\underline{X}$-covering subgroups of G/N. By the induction, EN/N and FN/N are conjugate in G/N and so $EN/N = (FN/N)^{xN}$, where $x \in G$. But this imply $EN = F^x N$.
We distinguish two cases:
1) There is a minimal normal subgroup M of G such that $EM \neq G$. We put $N = M$. By 1.3., E and F^x are $\underline{X}$-covering subgroups of EM, hence by the induction E and F^x are conjugate in EM and so E and F are conjugate in G.
2) For any minimal normal subgroup N of G we have $EN = G = FN$. We prove that any minimal normal subgroup N of G is a solvable π-group. Indeed, since G is π-solvable, N is either a solvable π-group or a π'-group. Supposing that N is a π'-group, we obtain $N \leq O\pi'(G)$. From

$$G/\ O\pi'(G) \cong (G/N)/(\ O\pi'(G)/N)$$

and

$$G/N = EN/N \cong E/E \cap N \in \underline{X}$$

it follows $G/\ O\pi'(G) \in \underline{X}$. By the π-closure of $\underline{X}$ we obtain the contradiction $G \in \underline{X}$.. Thus N is a solvable π-group and by 1.6. N is abelian.
Now E is a stabilizer of G. Indeed, E is a maximal subgroup of G since $E \neq G$ ($E \in \underline{X}$ but $G \notin \underline{X}$) and if $E \leq H < G$ then $E = H$, because otherwise let $h \in H-E$, $h = en$, $e \in E$, $n \in N$ and $n = e^{-1}h \in N \cap H = 1$ (N being abelian) which means the contradiction $h = e \in E$. Further $core_G E = 1$, for supposing $core_G E \neq 1$ we have a minimal normal subgroup M of G with $M \leq core_G E$, hence $G = EM = E\, core_G E = E$, in contradiction with $E \in \underline{X}$ and $G \notin \underline{X}$. So E is a stabilizer of G and G is a primitive π-solvable group. Since $F < G$
($F \in \underline{X}$ but $G \notin \underline{X}$) and since, for any minimal normal subgroup N of G, N is a solvable π-group and $FN = G$, applying 2.8. we obtain that F is also a stabilizer of G.
By 2.2., G has one and only one minimal normal subgroup N. By 2.5., $C_G(N) = N$. So condition (i) from 2.6. is valid. Further, we shall prove below that condition (ii) from 2.6. is also true. Indeed, let us suppose that (ii) is not valid. It means that there is

not a minimal normal subgroup L/N of G/N such that L/N is a π'-group. G/N being π-solvable, we deduce that any minimal normal subgroup L/N of G/N is a solvable π-group. But N being a solvable π-group, it follws that L is a solvable π-group. L being normal in G, we have two possibilities, both leading to a contradiction:

a) L is a minimal normal subgroup of G. But G having one and only one minimal normal subgroup N, we deduce that L = N, a contradiction with $L/N \neq 1$.

b) L is not a minimal normal subgroup of G. Then N < L, hence

$G = EN < EL \leq G$,

a contradiction.

We proved that G is a π-solvable group satisfying conditions (i) and (ii) from 2.6. Then, by theorem 2.7., we obtain that the two stabilizers E and F are conjugate in G. □

Theorem 3.2. *Let $\underline{X}$ be a π-homomorph. $\underline{X}$ is a Schunck class if and only if any π-solvable group G has $\underline{X}$-covering subgroups.*

Proof. Let $\underline{X}$ be a π-Schunck class. We prove by induction on $|G|$ that any π-solvable group G has $\underline{X}$-covering subgroups. Two cases are considered:

1) There is a minimal normal subgroup M of G such that $G/M \notin \underline{X}$. By the induction, G/M has an $\underline{X}$-covering subgroup H^*/M. Since $G/M \notin \underline{X}$ we have $H^* < G$. By the induction, H^* has an $\underline{X}$-covering subgroup H. Applying now 1.5., H is an $\underline{X}$-covering subgroup of G.

2) For any minimal normal subgroup M of G we have $G/M \in \underline{X}$. Two possibilities can be considered again:

a) G is not primitive. Let G/K be a primitive factor of G. Since $K \neq 1$, there is a minimal normal subgroup M of G such that $M \subseteq K$. We have $G/M \in \underline{X}$. Hence

$G/K \cong (G/M)/(K/M) \in \underline{X}$.

By the primitively closure of $\underline{X}$, $G \in \underline{X}$. So G is its own $\underline{X}$-covering subgroup.

b) G is primitive. Let S be a stabilizer of G. If $G \in \underline{X}$, then G is its own $\underline{X}$-covering subgroup. Let now $G \notin \underline{X}$. We shall prove that S is an $\underline{X}$-covering subgroup of G. First

$S \in \underline{X}$. Indeed, let M be a minimal normal subgroup of G. Since G is primitive and S is a stabilizer of G, by 2.1. we have MS = G. On the other side, G being π-solvable, M is either a solvable π-group or a π'-group. But if we suppose that M is a π'-group we have

$$M \leq O_{\pi'}(G)$$

and

$$G/O_{\pi'}(G) \cong (G/M)/(O_{\pi'}(G)/M) \in \underline{X},$$

hence by the π-closure of $\underline{X}$ we deduce that $G \in \underline{X}$, a contradiction. Thus M is a solvable π-group. Applying 1.6., M is abelian. This and G = MS lead to $M \cap S = 1$. Then

$$S \cong S/1 = S/M \cap S \cong MS/M = G/M \in \underline{X}.$$

So $S \in \underline{X}$. Further if $S \leq V \leq G$, $V_0 < V$, $V/V_0 \in \underline{X}$, we shall prove that $V = SV_0$. Because S is a maximal subgroup of G, two possibilities can happen: V = S or V = G. If V = S, we have $V = VV_0 = S\ V_0$. If V = G, we notice that V_0 is a normal subgroup of G and $V_0 \neq 1$ (else, $G = V \cong V/1 = V/V_0 \in \underline{X}$, a contradiction). Then let M_0 be a minimal normal subgroup of G such that $M_0 \subseteq V_0$. Applying 2.1., M_0S = G. Hence

$V = G = M_0S = V_0S = SV_0$.

Conversely, let $\underline{X}$ be a π-homomorph such that any π-solvable group has $\underline{X}$-covering subgroups. We prove that $\underline{X}$ is primitively closed. Suppose that $\underline{X}$ is not primitively closed and let G be a π-solvable group of minimal order with respect to the conditions: any primitive factor of G is in $\underline{X}$ but $G \notin \underline{X}$. Let M be a minimal normal subgroup of G. By the minimality of G we have $G/M \in \underline{X}$. G being π-solvable, G has an $\underline{X}$-covering subgroup H. From $H \leq G = G$, $M < G$, $G/M \in \underline{X}$ follows G = MH. By the π-closure of $\underline{X}$, M is a solvable π-group and so by 1.6. M is abelian. From this and from G=MH we obtain $M \cap H = 1$. Like in the proof of theorem 3.1., we obtain that H is a maximal subgroup of G. Two cases are possible:

1) G is primitive. Then $G \cong G/1$ is a primitive factor of G and by the choice of G, we obtain $G \cong G/1 \in \underline{X}$, in contradiction with $G \notin \underline{X}$. So this case leads to a

contradiction.

2) G is not primitive. Then $\mathrm{core}_G H \neq 1$, else H is a stabilizer of G and G is primitive. By the minimality of G we have $G/\mathrm{core}_G H \in \underline{X}$. By 1.4., $H/\mathrm{core}_G H$ is an $\underline{X}$-covering subgroup of $G/\mathrm{core}_G H$. It follows that $H/\mathrm{core}_G H = G/\mathrm{core}_G H$, hence $H = G$, in contradiction with $H \in \underline{X}$ but $G \notin \underline{X}$. This case leads also to a contradiction.

It follows that $\underline{X}$ is primitively closed and so $\underline{X}$ is a Schunck class. □

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