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# Certain sufficient conditions for $\phi-$ like functions in a parabolic region

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**Abstract.** To obtain the main result of the present paper we use the technique of differential subordination. As special cases of our main result, we obtain sufficient conditions for $f \in \mathcal{A}$ to be $\phi-$like, starlike and close-to-convex in a parabolic region.

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## 1. Introduction

Let us denote the class of analytic functions in the unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ by $\mathcal{H}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of the functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots.$$

Let $\mathcal{A}$ be the class of functions $f$, analytic in the unit disk $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.
Let $\mathcal{S}$ denote the class of all analytic univalent functions $f$ defined in the open unit disk $\mathbb{E}$ which are normalized by the conditions $f(0) = f'(0) - 1 = 0$. The Taylor series expansion of any function $f \in \mathcal{S}$ is

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots.$$

Let the functions $f$ and $g$ be analytic in $\mathbb{E}$. We say that $f$ is subordinate to $g$ written as $f \prec g$ in $\mathbb{E}$, if there exists a Schwarz function $\phi$ in $\mathbb{E}$ (i.e. $\phi$ is regular in $|z| < 1$,

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$\phi(0) = 0$ and $|\phi(z)| \le |z| < 1$) such that

$$f(z) = g(\phi(z)),\ |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, $p$ an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and $h$ be univalent in $\mathbb{E}$. Then the function $p$ is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z),\ \Phi(p(0), 0; 0) = h(0). \tag{1.1}$$

A univalent function $q$ is called dominant of the differential subordination (1.1) if $p(0) = q(0)$ and $p \prec q$ for all $p$ satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants $q$ of (1.1), is said to be the best dominant of (1.1). The best dominant is unique up to the rotation of $\mathbb{E}$.
A function $f \in \mathcal{A}$ is said to be starlike in the open unit disk $\mathbb{E}$, if it is univalent in $\mathbb{E}$ and $f(\mathbb{E})$ is a starlike domain. The well known condition for the members of class $\mathcal{A}$ to be starlike is that

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{E}.$$

Let $\mathcal{S}^*$ denote the subclass of $\mathcal{S}$ consisting of all univalent starlike functions with respect to the origin.
A function $f \in \mathcal{A}$ is said to be close-to-convex in $\mathbb{E}$, if there exists a convex function $g$ (not necessarily normalized) such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0,\ z \in \mathbb{E}.$$

In addition, if $g$ is normalized by the conditions $g(0) = 0 = g'(0) - 1$, then the class of close-to-convex functions is denoted by $\mathcal{C}$.
A function $f \in \mathcal{A}$ is called parabolic starlike in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|,\ z \in \mathbb{E}, \tag{1.2}$$

and the class of such functions is denoted by $S_P$.
A function $f \in \mathcal{A}$ is said to be uniformly close-to-convex in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \left|\frac{zf'(z)}{g(z)} - 1\right|,\ z \in \mathbb{E}, \tag{1.3}$$

for some $g \in \mathcal{S}_P$. Let UCC denote the class of all such functions. Note that the function $g(z) \equiv z \in \mathcal{S}_P$. Therefore, for $g(z) \equiv z$, condition (1.3) becomes:

$$\Re\left(f'(z)\right) > \left|f'(z) - 1\right|,\ z \in \mathbb{E}. \tag{1.4}$$

Ronning [6] and Ma and Minda [2] studied the domain $\Omega$ and the function $q(z)$ defined below:

$$\Omega = \left\{u + iv : u > \sqrt{(u-1)^2 + v^2}\right\}.$$

Clearly the function

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$

maps the unit disk $\mathbb{E}$ onto the domain $\Omega$. Hence the conditions (1.2) and (1.4) are equivalent to

$$\frac{zf'(z)}{f(z)} \prec q(z), \ z \in \mathbb{E},$$

and

$$f'(z) \prec q(z).$$

Let $\phi$ be analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0$ and $Re(\phi'(0)) > 0$. Then, the function $f \in \mathcal{A}$ is said to be $\phi-$ like in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > 0, \ z \in \mathbb{E}.$$

This concept was introduced by Brickman [1]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if $f$ is $\phi-$ like for some analytic function $\phi$. Later, Ruscheweyh [7] investigated the following general class of $\phi-$like functions:
Let $\phi$ be analytic in a domain containing $f(\mathbb{E})$, where $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi(w) \neq 0$ for some $w \in f(\mathbb{E})\backslash\{0\}$, then the function $f \in \mathcal{A}$ is called $\phi-$like with respect to a univalent function $q$, $q(0) = 1$, if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \ z \in \mathbb{E}.$$

A function $f \in \mathcal{A}$ is said to be parabolic $\phi-$ like in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > \left|\frac{zf'(z)}{\phi(f(z))} - 1\right|, \ z \in \mathbb{E}. \tag{1.5}$$

Equivalently, condition (1.5) can be written as:

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2.$$

In 2005, Ravichandran et al. [5] proved the following result for $\phi$-like functions:
Let $\alpha \neq 0$ be a complex number and $q(z)$ be a convex univalent function in $\mathbb{E}$.
Suppose $h(z) = \alpha q^2(z) + (1-\alpha)q(z) + \alpha z q'(z)$ and

$$\Re\left\{\frac{1-\alpha}{\alpha} + 2q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0, \ z \in \mathbb{E}.$$

If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\phi(f(z))}\left(1 + \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha(f'(z) - (\phi(f(z)))'}{\phi(f(z))}\right) \prec h(z),$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \ z \in \mathbb{E},$$

and $q(z)$ is best dominant. Later on, Shanmugam et al. [8] and Ibrahim [4] also obtained the results for $\phi$-like functions similar to the above mentioned results of

Ravichandran [5].
In this paper, we investigate the differential operator

$$\left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(g(z))}+b\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta}$$

where $f,\ g \in \mathcal{A}$ and $\beta,\ \gamma$ be complex numbers such that $\beta \neq 0$. Also $\phi$ is an analytic function in a domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\setminus\{0\}$, for real numbers $a, b\ (\neq 0)$. As consequences of our main results, we obtain sufficient conditions for $\phi$-like, parabolic $\phi$-like, starlike, parabolic starlike, close-to-convex and uniformly close-to-convex functions.
We shall need the following lemma to prove our main result.

**Lemma 1.1.** ([3], Theorem 3.4h, p. 132) *Let $q$ be univalent in $\mathbb{E}$ and let $\theta$ and $\varphi$ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\varphi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set*

$$Q_1(z) = zq'(z)\varphi[q(z)],\ \ h(z) = \theta[q(z)] + Q_1(z)$$

*and suppose that either*
*(i) $h$ is convex, or*
*(ii) $Q_1$ is starlike.*
*In addition, assume that*
*(iii) $\Re\left(\dfrac{zh'(z)}{Q_1(z)}\right) > 0$ for all $z \in \mathbb{E}$.*
*If $p$ is analytic in $\mathbb{E}$, with $p(0) = q(0),\ p(\mathbb{E}) \subset \mathbb{D}$ and*

$$\theta[p(z)] + zp'(z)\varphi[p(z)] \prec \theta[q(z)] + zq'(z)\varphi[q(z)],\ \ z \in \mathbb{E},$$

*then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.*

## 2. Main results

**Theorem 2.1.** *Let $\beta$ and $\gamma$ be complex numbers such that $\beta \neq 0$. Let $q(z) \neq 0$, be a univalent function in $\mathbb{E}$, such that*

$$\Re\left[1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}\right] > max\left\{0,\ -\frac{a}{b}\left(1+\frac{\gamma}{\beta}\right)\Re(q(z))\right\} \tag{2.1}$$

*where $a$ and $b(\neq 0)$ are real numbers. Let $\phi$ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\setminus\{0\}$. If $f,\ g \in \mathcal{A},\ \dfrac{zf'(z)}{\phi(g(z))} \neq 0,\ z \in \mathbb{E}$, satisfy the differential subordination*

$$\left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(g(z))}+b\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta}$$

$$\prec (q(z))^{\gamma}\left[aq(z)+b\frac{zq'(z)}{q(z)}\right]^{\beta} \tag{2.2}$$

*then*

$$\frac{zf'(z)}{\phi(g(z))} \prec q(z),\ z \in \mathbb{E},$$

*and $q(z)$ is the best dominant.*

*Proof.* On writing $\frac{zf'(z)}{\phi(g(z))} = p(z)$ in (2.2), we obtain:

$$(p(z))^{\gamma}\left(ap(z)+b\frac{zp'(z)}{p(z)}\right)^{\beta} \prec (q(z))^{\gamma}\left(aq(z)+b\frac{zq'(z)}{q(z)}\right)^{\beta}$$

or

$$a(p(z))^{\frac{\gamma}{\beta}+1}+b(p(z))^{\frac{\gamma}{\beta}-1}zp'(z) \prec a(q(z))^{\frac{\gamma}{\beta}+1}+b(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

Let us define the functions $\theta$ and $\phi$ as follows:

$$\theta(w) = aw^{\frac{\gamma}{\beta}+1} \text{ and } \phi(w) = bw^{\frac{\gamma}{\beta}-1}$$

Obviously, the functions $\theta$ and $\phi$ are analytic in domain $\mathbb{D} = \mathbb{C}\backslash\{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$.
Therefore,

$$Q(z) = \phi(q(z))zq'(z) = b(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

and

$$h(z) = \theta(q(z)) + Q(z) = a(q(z))^{\frac{\gamma}{\beta}+1} + b(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

On differentiating, we obtain

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}$$

and

$$\frac{zh'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)} + \frac{a}{b}\left(1+\frac{\gamma}{\beta}\right)q(z).$$

In view of the given condition (2.1), we see that $Q$ is starlike and
$\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$. Therefore, the proof, now follows from the Lemma [1.1]. □

On taking $g(z) = f(z)$ in Theorem 2.1, we have the following result:

**Theorem 2.2.** *Let $\beta$ and $\gamma$ be complex numbers such that $\beta \neq 0$ and $q(z) \neq 0$, be a univalent function in $\mathbb{E}$, satisfying the condition (2.1) of Theorem 2.1 for real numbers $a, b\ (\neq 0)$. Let $\phi$ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfy the differential subordination*

$$\left(\frac{zf'(z)}{\phi(f(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(f(z))} + b\left(1+\frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right)\right]^{\beta}$$
$$\prec (q(z))^{\gamma}\left[aq(z)+b\frac{zq'(z)}{q(z)}\right]^{\beta}$$

*then*

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \; z \in \mathbb{E},$$

*and $q(z)$ is the best dominant.*

On taking $\phi(z) = z$, $g(z) = f(z)$ in Theorem 2.1, we have the following result:

**Theorem 2.3.** *Let $\beta$ and $\gamma$ be complex numbers such that $\beta \neq 0$ and $q(z) \neq 0$, be a univalent function in $\mathbb{E}$, and satisfies the condition (2.1) of Theorem 2.1 for real numbers $a, \; b(\neq 0)$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0, \; z \in \mathbb{E}$, satisfies*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma} \left[(a-b)\frac{zf'(z)}{f(z)} + b\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\beta} \prec (q(z))^{\gamma} \left[aq(z) + b\frac{zq'(z)}{q(z)}\right]^{\beta}$$

*then*

$$\frac{zf'(z)}{f(z)} \prec q(z), \; z \in \mathbb{E},$$

*and $q(z)$ is the best dominant.*

On selecting $a = 1$ and $b = \alpha$ in Theorem 2.3, we get the following result for the class of $\alpha$-convex functions.

**Theorem 2.4.** *Let $\beta$ and $\gamma$ be complex numbers such that $\beta \neq 0$. Let $\alpha$ be a non-zero real number and $q(z) \neq 0$, be a univalent function in $\mathbb{E}$, and satisfies the condition (2.1) of Theorem 2.1. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0 \; z \in \mathbb{E}$, satisfies*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma} \left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\beta} \prec (q(z))^{\gamma} \left[q(z) + \alpha\frac{zq'(z)}{q(z)}\right]^{\beta}$$

*then*

$$\frac{zf'(z)}{f(z)} \prec q(z), \; z \in \mathbb{E},$$

*and $q(z)$ is the best dominant.*

By defining $\phi(z) = g(z) = z$ in Theorem 2.1, we obtain the following result:

**Theorem 2.5.** *Let $\beta$ and $\gamma$ be complex numbers such that $\beta \neq 0$ and $q(z) \neq 0$, be a univalent function in $\mathbb{E}$, and satisfies the condition (2.1) of Theorem 2.1 for real numbers $a, \; b(\neq 0)$. If $f \in \mathcal{A}$, $f'(z) \neq 0, \; z \in \mathbb{E}$, satisfies*

$$(f'(z))^{\gamma} \left[af'(z) + b\frac{zf''(z)}{f'(z)}\right]^{\beta} \prec (q(z))^{\gamma} \left(aq(z) + b\frac{zq'(z)}{q(z)}\right)^{\beta}$$

*then*

$$f'(z) \prec q(z), \; z \in \mathbb{E},$$

*and $q(z)$ is the best dominant.*

## 3. Applications

**Remark 3.1.** When we select the dominant

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,$$

we observed that the condition (2.1) of Theorem 2.1 holds, for real numbers $a, b\ (\neq 0)$ such that $\frac{a}{b} > 0$ and real numbers $\beta(\neq 0),\ \gamma$ such that $\frac{-3}{4} < \frac{\gamma}{\beta} < \frac{3}{2}$. Consequently, we get:

**Theorem 3.2.** *Let $\beta(\neq 0)$ and $\gamma$ be real numbers such that $\frac{-3}{4} < \frac{\gamma}{\beta} < \frac{3}{2}$ and $a, b$ $(\neq 0)$ be real numbers having same sign. Let $\phi$ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A},\ \frac{zf'(z)}{\phi(g(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta}$$

$$\prec \left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^{\gamma}$$

$$\left\{a + \frac{2a}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{\frac{4b\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^{\beta}$$

*then*

$$\frac{zf'(z)}{\phi(g(z))} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z \in \mathbb{E}.$$

On taking $g(z) = f(z)$ in above theorem, we obtain:

**Corollary 3.3.** *Let $\beta(\neq 0)$ and $\gamma$ be real numbers such that $\frac{-3}{4} < \frac{\gamma}{\beta} < \frac{3}{2}$ and $a, b$ $(\neq 0)$ be real numbers having same sign. Let $\phi$ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A},\ \frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\left(\frac{zf'(z)}{\phi(f(z))}\right)^{\gamma}\left[a\frac{zf'(z)}{\phi(f(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right)\right]^{\beta}$$

$$\prec \left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^{\gamma}$$

$$\left\{a+\frac{2a}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4b\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^{\beta}$$

*then*

$$\frac{zf'(z)}{\phi(f(z))}\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E},$$

*and hence $f(z)$ is parabolic $\phi$-like.*

For $\phi(z)=z$ and $g(z)=f(z)$ in Theorem 3.2, we obtain the following result:

**Corollary 3.4.** *Let $\beta(\neq 0)$ and $\gamma$ be real numbers such that $\frac{-3}{4}<\frac{\gamma}{\beta}<\frac{3}{2}$ and $a, b$ $(\neq 0)$ be real numbers having same sign. If $f\in\mathcal{A}$, $\frac{zf'(z)}{f(z)}\neq 0$, $z\in\mathbb{E}$, satisfy*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left[(a-b)\frac{zf'(z)}{f(z)}+b\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\beta}\prec\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^{\gamma}$$

$$\left\{a+\frac{2a}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4b\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^{\beta}$$

*then*

$$\frac{zf'(z)}{f(z)}\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E},$$

*and hence $f(z)$ is parabolic starlike.*

Selecting $a=1$ and $b=\alpha$ in above corollary, we get the following result for the class of $\alpha$-convex functions:

**Corollary 3.5.** *Let $\beta(\neq 0)$ and $\gamma$ be real numbers such that $\frac{-3}{4}<\frac{\gamma}{\beta}<\frac{3}{2}$ and $\alpha$ be a non-zero real number. If $f\in\mathcal{A}$, $\frac{zf'(z)}{f(z)}\neq 0$, $z\in\mathbb{E}$, satisfies*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\beta}\prec\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^{\gamma}$$

$$\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^{\beta}$$

*then*

$$\frac{zf'(z)}{f(z)}\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,\ z\in\mathbb{E},$$

*and hence $f(z)$ is parabolic starlike.*

On taking $\phi(z)=g(z)=z$ in Theorem 3.2, we have:

**Corollary 3.6.** *Let $\beta(\neq 0)$ and $\gamma$ be real numbers such that $\frac{-3}{4} < \frac{\gamma}{\beta} < \frac{3}{2}$ and $a, b$ $(\neq 0)$ be real numbers having same sign. If $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies*

$$(f'(z))^{\gamma}\left[af'(z) + b\left(\frac{zf''(z)}{f'(z)}\right)\right]^{\beta} \prec \left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^{\gamma}$$

$$\left\{a + \frac{2a}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{\frac{4b\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\}^{\beta}$$

*then*

$$f'(z) \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, \; z \in \mathbb{E},$$

*and hence $f(z)$ is uniformly close-to-convex.*

**Remark 3.7.** It is easy to verify that the dominant $q(z) = \frac{1+z}{1-z}$, satisfies the condition (2.1) of Theorem 2.1, for real numbers $a, b$ $(\neq 0)$ having same sign and real numbers $\gamma$ and $\beta(\neq 0)$ such that $\gamma = \beta$ or $\gamma = 0$.
For $\gamma = \beta$, Theorem 2.1 yields:

**Theorem 3.8.** *Let $\phi$ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f, \; g \in \mathcal{A}$, $\frac{zf'(z)}{\phi(g(z))} \neq 0$, $z \in \mathbb{E}$, and for real numbers $a, b$ $(\neq 0)$ having same sign, satisfies*

$$a\left(\frac{zf'(z)}{\phi(g(z))}\right)^2 + b\left(\frac{zf'(z)}{\phi(g(z))}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right) \prec a\left(\frac{1+z}{1-z}\right)^2 + \frac{2bz}{(1-z)^2}$$

*then*

$$\frac{zf'(z)}{\phi(g(z))} \prec \frac{1+z}{1-z}, \; z \in \mathbb{E}.$$

On taking $g(z) = f(z)$ in above theorem, we obtain:

**Corollary 3.9.** *Let $\phi$ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, and for real numbers $a, b$ $(\neq 0)$ having same sign, satisfies*

$$a\left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + b\left(\frac{zf'(z)}{\phi(f(z))}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right) \prec a\left(\frac{1+z}{1-z}\right)^2 + \frac{2bz}{(1-z)^2}$$

*then*

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+z}{1-z}, \ z \in \mathbb{E},$$

*i.e. $f(z)$ is $\phi-$like function.*

For $\phi(z) = z$ and $g(z) = f(z)$ in Theorem 3.8, we obtain the following result:

**Corollary 3.10.** *Let $a, b\ (\neq 0)$ be real numbers having same sign. If $f \in \mathcal{A}$,*

$$\frac{zf'(z)}{f(z)} \neq 0, \ z \in \mathbb{E},$$

*satisfy*

$$(a-b)\left(\frac{zf'(z)}{f(z)}\right)^2 + b\left(\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec a\left(\frac{1+z}{1-z}\right)^2 + \frac{2bz}{(1-z)^2}$$

*then*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \ z \in \mathbb{E},$$

*and hence $f(z)$ is starlike.*

Selecting $a = 1$ and $b = \alpha$ in above corollary, we get the following result for the class of $\alpha$-convex functions:

**Corollary 3.11.** *Let $\alpha$ be a non-zero real number. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies*

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)^2 + \alpha\left(\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^2 + \frac{2\alpha z}{(1-z)^2}$$

*then*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \ z \in \mathbb{E}.$$

*Hence $f(z)$ is starlike.*

On taking $\phi(z) = g(z) = z$ in Theorem 3.8, we have:

**Corollary 3.12.** *Let $a, b\ (\neq 0)$ are real numbers with same sign. If $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies*

$$a\left(f'(z)\right)^2 + bzf''(z) \prec a\left(\frac{1+z}{1-z}\right)^2 + \frac{2bz}{(1-z)^2}$$

*then*

$$f'(z) \prec \left(\frac{1+z}{1-z}\right), \ z \in \mathbb{E},$$

*and hence $f(z)$ is close-to-convex.*

For $\gamma = 0$, Theorem 2.1 yields:

**Theorem 3.13.** *Let $\phi$ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \neq 0,\ z \in \mathbb{E}$, and for real numbers $a, b\ (\neq 0)$ with same sign, satisfies*

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right) \prec a\left(\frac{1+z}{1-z}\right) + \frac{2bz}{(1-z)^2}$$

*then*

$$\frac{zf'(z)}{\phi(g(z))} \prec \frac{1+z}{1-z},\ z \in \mathbb{E}.$$

On taking $g(z) = f(z)$ in above theorem, we obtain:

**Corollary 3.14.** *Let $\phi$ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, and for real numbers $a, b\ (\neq 0)$ with same sign, satisfies*

$$a\frac{zf'(z)}{\phi(f(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right) \prec a\left(\frac{1+z}{1-z}\right) + \frac{2bz}{(1-z)^2}$$

*then*

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

*i.e. $f(z)$ is $\phi-$like function.*

For $\phi(z) = z$ and $g(z) = f(z)$ in Theorem 3.13, we obtain the following result:

**Corollary 3.15.** *Let $a, b\ (\neq 0)$ are real numbers with same sign. If $f \in \mathcal{A}$,*

$$\frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E},$$

*satisfy*

$$(a-b)\frac{zf'(z)}{f(z)} + b\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec a\left(\frac{1+z}{1-z}\right) + \frac{2bz}{(1-z)^2}$$

*then*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

*and hence $f(z)$ is starlike.*

Selecting $a = 1$ and $b = \alpha$ in above corollary, we get the following result for the class of $\alpha$-convex functions:

**Corollary 3.16.** *Let $\alpha$ be a non-zero real number. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{1+z}{1-z} + \frac{2\alpha z}{(1-z)^2}$$

*then*
$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \ z \in \mathbb{E}.$$
*Hence $f(z)$ is starlike.*

On taking $\phi(z) = g(z) = z$ in Theorem 3.13, we have:

**Corollary 3.17.** *Let $a, b$ $(\neq 0)$ are real numbers with same sign. If $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies*
$$af'(z) + b\frac{zf''(z)}{f'(z)} \prec a\left(\frac{1+z}{1-z}\right) + \frac{2bz}{(1-z)^2}$$
*then*
$$f'(z) \prec \frac{1+z}{1-z}, \ z \in \mathbb{E},$$
*and hence $f(z)$ is close-to-convex.*

**Remark 3.18.** When we select the dominant $q(z) = e^z$, then this dominant satisfies the condition (2.1) of Theorem 2.1 for real numbers $a, b$ $(\neq 0)$ with same sign and real numbers $\gamma,\ \beta(\neq 0)$ such that $0 < \dfrac{\gamma}{\beta} \leq 1$. Consequently, we obtain the following result:

**Theorem 3.19.** *Let $a, b$ $(\neq 0)$ be real numbers with same sign and $\gamma, \beta$ $(\neq 0)$ such that $0 < \dfrac{\gamma}{\beta} \leq 1$. Let $\phi$ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*
$$\left(\frac{zf'(z)}{\phi(g(z))}\right)^{\gamma} \left[a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right]^{\beta} \prec e^{\gamma z}\left[ae^z + bz\right]^{\beta}$$
*then*
$$\frac{zf'(z)}{\phi(g(z))} \prec e^z, \ z \in \mathbb{E}.$$

On choosing $g(z) = f(z)$ in above theorem, we obtain:

**Corollary 3.20.** *Let $a, b$ $(\neq 0)$ be real numbers with same sign and $\gamma, \beta$ $(\neq 0)$ be real numbers such that $0 < \dfrac{\gamma}{\beta} \leq 1$. Let $\phi$ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. If $f \in \mathcal{A}$,*
$$\frac{zf'(z)}{\phi(f(z))} \neq 0, \ z \in \mathbb{E},$$
*satisfy*
$$\left(\frac{zf'(z)}{\phi(f(z))}\right)^{\gamma} \left[a\frac{zf'(z)}{\phi(f(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right)\right]^{\beta} \prec e^{\gamma z}\left[ae^z + bz\right]^{\beta}$$

*then*

$$\frac{zf'(z)}{\phi(f(z))} \prec e^z, \ z \in \mathbb{E},$$

*i.e.* $f(z)$ *is* $\phi-$*like.*

On selecting $\phi(z) = z$ and $g(z) = f(z)$ in Theorem 3.19, we get:

**Corollary 3.21.** *Let* $a, b \ (\neq 0)$ *be real numbers with same sign and* $\gamma, \beta \ (\neq 0)$ *be real numbers such that* $0 < \dfrac{\gamma}{\beta} \leq 1$. *If* $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0, \ z \in \mathbb{E}$, *satisfy the differential subordination*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma} \left[(a-b)\frac{zf'(z)}{f(z)} + b\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\beta} \prec e^{\gamma z} \left[ae^z + bz\right]^{\beta}$$

*then*

$$\frac{zf'(z)}{f(z)} \prec e^z, \ z \in \mathbb{E},$$

*and hence* $f(z)$ *is starlike.*

On choosing $a = 1$ and $b = \alpha$ in above corollary, we obtain:

**Corollary 3.22.** *Let* $\alpha$ *be a non-zero real number and real numbers* $\gamma, \beta \ (\neq 0)$ *such that* $0 < \dfrac{\gamma}{\beta} \leq 1$. *If* $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0, \ z \in \mathbb{E}$, *satisfies*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma} \left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\beta} \prec e^{\gamma z} \left[e^z + \alpha z\right]^{\beta}$$

*then*

$$\frac{zf'(z)}{f(z)} \prec e^z, \ z \in \mathbb{E}.$$

*Therefore,* $f \in S^*$.

For $\phi(z) = g(z) = z$ in Theorem 3.19, we obtain the following result:

**Corollary 3.23.** *Let* $a, b \ (\neq 0)$ *be real numbers with same sign and* $\gamma, \beta \ (\neq 0)$ *be real numbers such that* $0 < \dfrac{\gamma}{\beta} \leq 1$. *If* $f \in \mathcal{A}$, $f'(z) \neq 0, \ z \in \mathbb{E}$, *satisfies*

$$(f'(z))^{\gamma} \left[af'(z) + b\frac{zf''(z)}{f'(z)}\right]^{\beta} \prec e^{\gamma z} \left[ae^z + bz\right]^{\beta}$$

*then*

$$f'(z) \prec e^z, \ z \in \mathbb{E},$$

*and hence* $f(z)$ *is close-to-convex.*

**Remark 3.24.** By selecting the dominant $q(z) = 1 + mz, \ 0 < m \leq 1$, we observed that the Condition (2.1) of Theorem 2.1 holds for all real numbers $a, b \ (\neq 0)$ such that $\dfrac{a}{b} > 0$, and $\gamma = 0$. Thus from Theorem 2.1, we have the following result:

**Theorem 3.25.** *Let $\phi$ be analytic function in the domain containing $g(\mathbb{E})$, where $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\backslash\{0\}$. Let real numbers $a, b\ (\neq 0)$ be such that $\dfrac{a}{b} > 0$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\left[a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\right)\right] \prec \left[a(1+mz) + \frac{bmz}{1+mz}\right]$$

*then*

$$\frac{zf'(z)}{\phi(g(z))} \prec 1 + mz,\ \textit{where}\ 0 < m \leq 1,\ z \in \mathbb{E}.$$

Taking $g(z) = f(z)$ in above theorem, we get the following result:

**Corollary 3.26.** *Let $\phi$ be analytic function in the domain containing $f(\mathbb{E})$, where $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E})\backslash\{0\}$. Let real numbers $a, b\ (\neq 0)$ be such that $\dfrac{a}{b} > 0$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\left[a\frac{zf'(z)}{\phi(f(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right)\right] \prec \left[a(1+mz) + \frac{bmz}{1+mz}\right]$$

*then*

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + mz,\ \textit{where}\ 0 < m \leq 1,\ z \in \mathbb{E},$$

*i.e. $f(z)$ is $\phi-$like.*

From Theorem 3.25, for $\phi(z) = z$ and $g(z) = f(z)$, we obtain:

**Corollary 3.27.** *Let $a, b\ (\neq 0)$ are real numbers having same sign. If $f\ \in \mathcal{A}$,*

$$\frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E},$$

*satisfies*

$$\left[(a-b)\frac{zf'(z)}{f(z)} + b\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \prec \left[a(1+mz) + \frac{bmz}{1+mz}\right]$$

*then*

$$\frac{zf'(z)}{f(z)} \prec 1 + mz,\ \textit{where}\ 0 < m \leq 1,\ z \in \mathbb{E},$$

*and hence $f(z)$ is starlike.*

On selecting $a = 1$ and $b = \alpha$ in above corollary, we get the following result:

**Corollary 3.28.** *Let $\alpha > 0$ be a real number. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies the differential subordination*

$$\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \prec \left[(1+mz) + \frac{\alpha mz}{1+mz}\right]$$

*then*

$$\frac{zf'(z)}{f(z)} \prec 1+mz,\ 0<m\leq 1,\ z\in\mathbb{E},$$

*and hence $f(z)$ is starlike.*

Selecting $\phi(z)=g(z)=z$, in Theorem 3.25, we have:

**Corollary 3.29.** *Let $a, b\ (\neq 0)$ be real numbers having same sign. If $f\in\mathcal{A}$, $f'(z)\neq 0$, $z\in\mathbb{E}$, satisfies*

$$\left[af'(z)+b\frac{zf''(z)}{f'(z)}\right] \prec \left[a(1+mz)+\frac{bmz}{1+mz}\right]$$

*then*

$$f'(z)\prec 1+mz,\ 0<m\leq 1,\ z\in\mathbb{E},$$

*and hence $f(z)$ is close-to-convex.*

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