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On some vertical cohomologies of complex Finsler manifolds

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Abstract. In this paper we study some vertical cohomologies of complex Finsler manifolds as vertical cohomology attached to a function and vertical Lichnerowicz cohomology. We also study a relative vertical cohomology attached to a function associated to a holomorphic Finsler subspace.

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Introduction

The study of vertical cohomology of complex Finsler manifolds was initiated by Pitiş and Munteanu in [13]. The main goal of this paper is to study some other vertical cohomologies for forms of type (p, q, r, s) on complex Finsler manifolds as cohomology attached to a function defined in [12] and Lichnerowicz cohomology studied by many authors, e.g. [3, 8, 16]. In this sense, in the first section following [1, 2, 9] and [13], we briefly recall some preliminaries notions about complex Finsler manifolds and $\overline{v}$-cohomology groups. In the second section, we define a vertical cohomology attached to a function for forms of type (p, q, r, s) on a compex Finsler manifold (M, F) and we explain how this cohomology depends on the function. In particular, we show that if the function does not vanish, then our cohomology is isomorphic with the vertical cohomology of (M, F). In the third section we define and we study a vertical Lichnerowicz cohomology for forms of type (p, q, r, s) on a complex Finsler manifold (M, F) and in the last section, we construct a relative vertical cohomology attached to a function associated to a holomorphic Finsler subspace. The methods used here are closely related to those used by [4], [12] and [16].

1. Preliminaries

1.1. Complex Finsler manifolds

Let $\pi : T^{1,0}M \to M$ be the holomorphic tangent bundle of a n-dimensional complex manifold M. Denote by $(\pi^{-1}(U), (z^k, \eta^k))$, $k = 1, \ldots, n$ the induced complex coordinates on $T^{1,0}M$, where $(U, (z^k))$ is a local chart domain of M. At local change charts on $T^{1,0}M$, the transformation rules of these coordinates are given by

$$z'^k = z'^k(z) \ , \ \eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j, \tag{1.1}$$

where z'^k are holomorphic functions and $\det(\frac{\partial z'^k}{\partial z^j}) \neq 0$.

It is well known that $T^{1,0}M$ has a natural structure of $2n$-dimensional complex manifold, because the transition functions $\frac{\partial z'^k}{\partial z^j}$ are holomorphic.

Denote by $\widetilde{M} = T^{1,0}M - \{o\}$, where o is the zero section of $T^{1,0}M$, and we consider $T_{\mathbb{C}}\widetilde{M} = T^{1,0}\widetilde{M} \oplus T^{0,1}\widetilde{M}$ the complexified tangent bundle of the real tangent bundle $T_{\mathbb{R}}\widetilde{M}$, where $T^{1,0}\widetilde{M}$ and $T^{0,1}\widetilde{M} = \overline{T^{1,0}\widetilde{M}}$ are the holomorphic and antiholomorphic tangent bundles of $\widetilde{M}$, respectively.

Let $V^{1,0}\widetilde{M} = \ker \pi_*$ be the holomorphic vertical bundle over $\widetilde{M}$ and $\mathcal{V}^{1,0}(\widetilde{M})$ the module of its sections, called *vector fields of v-type.*

A given supplementary subbundle $H^{1,0}\widetilde{M}$ of $V^{1,0}\widetilde{M}$ in $T^{1,0}\widetilde{M}$, i.e.

$$T^{1,0}\widetilde{M} = H^{1,0}\widetilde{M} \oplus V^{1,0}\widetilde{M} \tag{1.2}$$

defines a *complex nonlinear connection* on $\widetilde{M}$, briefly c.n.c. and we denote by $\mathcal{H}^{1,0}(\widetilde{M})$ the module of its sections, called *vector fields of h-type.*

By conjugation over all, we get a decomposition of the complexified tangent bundle, namely $T_{\mathbb{C}}\widetilde{M} = H^{1,0}\widetilde{M} \oplus V^{1,0}\widetilde{M} \oplus H^{0,1}\widetilde{M} \oplus V^{0,1}\widetilde{M}$.

The elements of the conjugates are called *vector fields of* $\overline{h}$*-type* and $\overline{v}$*-type*, respectively.

If $N_k^j(z, \eta)$ are the local coefficients of the c.n.c. then the following set of complex vector fields

$$\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\} \, , \, \{\frac{\partial}{\partial \eta^k}\} \, , \, \{\frac{\delta}{\delta \overline{z}^k} = \frac{\partial}{\partial \overline{z}^k} - N_{\overline{k}}^{\overline{j}} \frac{\partial}{\partial \overline{\eta}^j}\} \, , \, \{\frac{\partial}{\partial \overline{\eta}^k}\} \tag{1.3}$$

are called the local adapted bases of $\mathcal{H}^{1,0}(\widetilde{M})$, $\mathcal{V}^{1,0}(\widetilde{M})$, $\mathcal{H}^{0,1}(\widetilde{M})$ and $\mathcal{V}^{0,1}(\widetilde{M})$, respectively. The dual adapted bases are given by

$$\{dz^k\} \, , \, \{\delta\eta^k = d\eta^k + N_j^k dz^j\} \, , \, \{d\overline{z}^k\} \, , \, \{\delta\overline{\eta}^k = d\overline{\eta}^k + N_{\overline{j}}^{\overline{k}} d\overline{z}^j\}. \tag{1.4}$$

Throughout this paper, we consider the abreviate notations $\partial_k = \frac{\partial}{\partial z^k}$, $\dot{\partial}_k = \frac{\partial}{\partial \eta^k}$, $\delta_k = \frac{\delta}{\delta z^k}$ and its conjugates $\partial_{\overline{k}} = \frac{\partial}{\partial \overline{z}^k}$, $\dot{\partial}_{\overline{k}} = \frac{\partial}{\partial \overline{\eta}^k}$, $\delta_{\overline{k}} = \frac{\delta}{\delta \overline{z}^k}$.

Let us consider M be a strongly pseudoconvex complex Finsler manifold [1], i.e. M is endowed with a complex Finsler metric $F : T^{1,0}M \to \mathbb{R}_+ \cup \{0\}$ satisfying:

1. F^2 is smooth on $\widetilde{M}$;
2. $F(z, \eta) > 0$ for all $(z, \eta) \in \widetilde{M}$ and $F(z, \eta) = 0$ if and only if $\eta = 0$;

(3) $F(z,\lambda\eta) = |\lambda| F(z,\eta)$ for all $(z,\eta) \in T^{1,0}M$ and $\lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$;
(4) the complex hessian $(G_{j\overline{k}}) = (\dot{\partial}_j \dot{\partial}_{\overline{k}}\, (F^2))$ is positive definite on $\widetilde{M}$.

Let $(G^{\overline{m}j})$ be the inverse of $(G_{j\overline{m}})$. According to [1] and [9], a c.n.c. on (M,F) depending only on the complex Finsler metric F is the Chern-Finsler c.n.c., locally given by

$$\overset{CF}{N^j_k} = G^{\overline{m}j} \partial_k\, \dot{\partial}_{\overline{m}}\, (F^2) \tag{1.5}$$

and it has an important property, namely

$$\overset{CF}{R^i_{kj}} = \delta_k\, \overset{CF}{N^i_j} - \delta_j\, \overset{CF}{N^i_k} = 0. \tag{1.6}$$

In the sequel we will consider the adapted frames and coframes with respect to the Chern-Finsler c.n.c. and the hermitian metric structure G on $\widetilde{M}$ given by the Sasaki type lift of the fundamental tensor $G_{j\overline{k}}$, locally given by

$$G = G_{j\overline{k}} dz^j \otimes d\overline{z}^k + G_{j\overline{k}} \delta\eta^j \otimes \delta\overline{\eta}^k. \tag{1.7}$$

1.2. Vertical cohomology

According to [13], the set $\mathcal{A}(\widetilde{M})$ of complex valued differential forms on $\widetilde{M}$ is given by the direct sum

$$\mathcal{A}(\widetilde{M}) = \bigoplus_{p,q,r,s=0}^{n} \mathcal{A}^{p,q,r,s}(\widetilde{M}), \tag{1.8}$$

where $\mathcal{A}^{p,q,r,s}(\widetilde{M})$ or simply $\mathcal{A}^{p,q,r,s}$ is the set of all $(p+q+r+s)$-forms which can be non zero only when these act on p vector fields of h-type, on q vector fields of $\overline{h}$-type, on r vector fields of v-type, and on s vector fields of $\overline{v}$-type. The elements of $\mathcal{A}^{p,q,r,s}$ are called (p,q,r,s)*-forms* on $\widetilde{M}$.

With respect to the adapted coframes $\{dz^k, d\overline{z}^k, \delta\eta^k, \delta\overline{\eta}^k\}$ of $T^*_{\mathbb{C}}\widetilde{M}$ a form $\varphi \in \mathcal{A}^{p,q,r,s}$ is locally given by

$$\varphi = \frac{1}{p!q!r!s!} \varphi_{I_p \overline{J_q} K_r \overline{H_s}} dz^{I_p} \wedge d\overline{z}^{J_q} \wedge \delta\eta^{K_r} \wedge \delta\overline{\eta}^{H_s}, \tag{1.9}$$

where I_p denotes the ordered p-tuple $(i_1 \ldots i_p)$, J_q the ordered q-tuple $(j_1 \ldots j_q)$, K_r the ordered r-tuple $(k_1 \ldots k_r)$, H_s the ordered s-tuple $(h_1 \ldots h_s)$ and $dz^{I_p} = dz^{i_1} \wedge \ldots \wedge dz^{i_p}$, $d\overline{z}^{J_q} = d\overline{z}^{j_1} \wedge \ldots \wedge d\overline{z}^{j_q}$, $\delta\eta^{K_r} = \delta\eta^{k_1} \wedge \ldots \wedge \delta\eta^{k_r}$ and $\delta\overline{\eta}^{H_s} = \delta\overline{\eta}^{h_1} \wedge \ldots \wedge \delta\overline{\eta}^{h_s}$, respectively.

We notice that these forms are the $(p+r, q+s)$ *complex type* and according to [13] if (M,F) is a complex Finsler manifold endowed with the Chern-Finsler c.n.c., then by (1.6) the exterior differential d admits the decomposition

$$\begin{aligned} d\mathcal{A}^{p,q,r,s} \subset\ & \mathcal{A}^{p+1,q,r,s} \oplus \mathcal{A}^{p,q+1,r,s} \oplus \mathcal{A}^{p,q,r+1,s} \oplus \mathcal{A}^{p,q,r,s+1} \oplus \\ & \oplus \mathcal{A}^{p+1,q+1,r-1,s} \oplus \mathcal{A}^{p+1,q,r-1,s+1} \oplus \mathcal{A}^{p+1,q+1,r,s-1} \oplus \mathcal{A}^{p,q+1,r+1,s-1} \end{aligned} \tag{1.10}$$

which allows us to define eight morphisms of complex vector spaces if we consider the different components, namely

$$\begin{array}{lll} \partial_h & : & \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p+1,q,r,s} \quad ; \quad \partial_v : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q,r+1,s} \\ \partial_{\overline{h}} & : & \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q+1,r,s} \quad ; \quad \partial_{\overline{v}} : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q,r,s+1} \\ \partial_1 & : & \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p+1,q+1,r-1,s} ; \ \partial_2 : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p+1,q,r-1,s+1} \\ \partial_3 & : & \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p+1,q+1,r,s-1} ; \ \partial_4 : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q+1,r+1,s-1} \end{array}$$

We remark that these operators and the classical operators ∂ and $\overline{\partial}$ that appear in the decomposition $d = \partial + \overline{\partial}$ of the differential on a complex manifold are related by $\partial = \partial_h + \partial_v + \partial_3 + \partial_4$ and $\overline{\partial} = \partial_{\overline{h}} + \partial_{\overline{v}} + \partial_1 + \partial_2$.

The conjugated vertical differential operator $\partial_{\overline{v}}$ is locally given by

$$\partial_{\overline{v}}\varphi = \sum_k \dot{\partial}_{\overline{k}}\,(\varphi_{I_p\overline{J_q}K_r\overline{H_s}})\delta\overline{\eta}^k \wedge dz^{I_p} \wedge d\overline{z}^{J_q} \wedge \delta\eta^{K_r} \wedge \delta\overline{\eta}^{H_s}, \tag{1.11}$$

where the sum is after the indices $i_1 \leq \ldots \leq i_p$, $j_1 \leq \ldots \leq j_q$, $k_1 \leq \ldots \leq k_r$ and $h_1 \leq \ldots \leq h_s$, respectively. Also it satisfies

$$\partial_{\overline{v}}(\varphi \wedge \psi) = \partial_{\overline{v}}\varphi \wedge \psi + (-1)^{\deg\varphi}\varphi \wedge \partial_{\overline{v}}\psi$$

for any $\varphi \in \mathcal{A}^{p,q,r,s}$ and $\psi \in \mathcal{A}^{p',q',r',s'}$.

This operator has the property $\partial^2_{\overline{v}} = 0$ and in [13], a classical theory of de Rham cohomology is developed for the conjugated vertical differential $\partial_{\overline{v}}$, see also [9] pag. 89. Namely, the sequence

$$0 \longrightarrow \Phi^{p,q,r} \xrightarrow{i} \mathcal{F}^{p,q,r0} \xrightarrow{\partial_{\overline{v}}} \mathcal{F}^{p,q,r,1} \xrightarrow{\partial_{\overline{v}}} \mathcal{F}^{p,q,r,2} \xrightarrow{\partial_{\overline{v}}} \ldots \xrightarrow{\partial_{\overline{v}}} \ldots,$$

is a fine resolution for the sheaf $\Phi^{p,q,r}$ of germs of $\partial_{\overline{v}}$-closed $(p,q,r,0)$-forms on $\widetilde{M}$, where $\mathcal{F}^{p,q,r,s}$ are the sheaves of germs of (p,q,r,s)-forms. It is also given a de Rham type theorem for the $\overline{v}$-cohomology groups $H^{p,q,r,s}(\widetilde{M}) = Z^{p,q,r,s}(\widetilde{M})/B^{p,q,r,s}(\widetilde{M})$ of the complex Finsler manifold:

$$H^s(\widetilde{M}, \Phi^{p,q,r}) \approx H^{p,q,r,s}(\widetilde{M}),$$

where $Z^{p,q,r,s}(\widetilde{M})$ is the space of $\partial_{\overline{v}}$-closed (p,q,r,s)-forms and $B^{p,q,r,s}(\widetilde{M})$ is the space of $\partial_{\overline{v}}$-exact (p,q,r,s)-forms globally defined on $\widetilde{M}$.

2. Vertical cohomology attached to a function

In this section, we consider a new $\overline{v}$-cohomology attached to a function on the complex Finsler manifold (M,F). This new cohomology is also defined in terms of forms of type (p,q,r,s) on $\widetilde{M}$. More precisely, if (M,F) is a complex Finsler manifold and f is a function on $\widetilde{M}$, we define the coboundary operator

$$\partial_{\overline{v},f} : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q,r,s+1}\ ,\ \partial_{\overline{v},f}\varphi = f\partial_{\overline{v}}\varphi - (p+q+r+s)\partial_{\overline{v}}f \wedge \varphi. \tag{2.1}$$

It is easy to check that $\partial^2_{\overline{v},f} = 0$ and denote by $H^{p,q,r,s}_f(\widetilde{M})$ the cohomology groups of the differential complex $(\mathcal{A}^{p,q,r,\bullet}(\widetilde{M}), \partial_{\overline{v},f})$, called the *vertical de Rham cohomology groups attached to the function f* of complex Finsler manifold (M,F).

More generally, for any integer k, we define the coboundary operator

$$\partial_{\overline{v},f,k} : \mathcal{A}^{p,q,r,s} \to \mathcal{A}^{p,q,r,s+1} \ , \ \partial_{\overline{v},f,k}\varphi = f\partial_{\overline{v}}\varphi - (p+q+r+s-k)\partial_{\overline{v}}f \wedge \varphi. \quad (2.2)$$

We still have $\partial^2_{\overline{v},f,k} = 0$ and we denote by $H^{p,q,r,s}_{f,k}(\widetilde{M})$ the cohomology of this complex. We shall restrict our attention to the cohomology $H^{p,q,r,s}_{f}(\widetilde{M})$ but most results readily generalize to the cohomology $H^{p,q,r,s}_{f,k}(\widetilde{M})$.

Using (2.1), by direct calculus we obtain

Proposition 2.1. *If $f, g \in \mathcal{F}(\widetilde{M})$ then*

(i) $\partial_{\overline{v},f+g} = \partial_{\overline{v},f} + \partial_{\overline{v},g}$, $\partial_{\overline{v},0} = 0$, $\partial_{\overline{v},-f} = -\partial_{\overline{v},f}$;
(ii) $\partial_{\overline{v},fg} = f\partial_{\overline{v},g} + g\partial_{\overline{v},f} - fg\partial_{\overline{v}}$, $\partial_{\overline{v},1} = \partial_{\overline{v}}$, $\partial_{\overline{v}} = \frac{1}{2}(f\partial_{\overline{v},\frac{1}{f}} + \frac{1}{f}\partial_{\overline{v},f})$, *and*
(iii) $\partial_{\overline{v},f}(\varphi \wedge \psi) = \partial_{\overline{v},f}\varphi \wedge \psi + (-1)^{\deg\varphi}\varphi \wedge \partial_{\overline{v},f}\psi$.

Dependence on the function

A natural question to ask about the cohomology $H^{p,q,r,s}_{f}(\widetilde{M})$ is how it depends on the function f. Similar with the Proposition 3.2. from [12], we explain this fact for our vertical cohomology. We have

Proposition 2.2. *If $h \in \mathcal{F}(\widetilde{M})$ does not vanish, then the cohomology groups $H^{p,q,r,s}_{f}(\widetilde{M})$ and $H^{p,q,r,s}_{fh}(\widetilde{M})$ are isomorphic.*

Proof. For each $p, q, r, s \in \mathbb{N}$, consider the linear isomorphism

$$\phi^{p,q,r,s} : \mathcal{A}^{p,q,r,s}(\widetilde{M}) \to \mathcal{A}^{p,q,r,s}(\widetilde{M}) \ , \ \phi^{p,q,r,s}(\varphi) = \frac{\varphi}{h^{p+q+r+s}}. \quad (2.3)$$

If $\varphi \in \mathcal{A}^{p,q,r,s}(\widetilde{M})$, one checks easily that

$$\phi^{p,q,r,s+1}(\partial_{\overline{v},fh}\varphi) = \partial_{\overline{v},f}(\phi^{p,q,r,s}(\varphi)), \quad (2.4)$$

so $\phi^{p,q,r,s}$ induces an isomorphism between the cohomologies $H^{p,q,r,s}_{f}(\widetilde{M})$ and $H^{p,q,r,s}_{fh}(\widetilde{M})$. □

Corollary 2.3. *If the function f does not vanish, then $H^{p,q,r,s}_{f}(\widetilde{M})$ is isomorphic to the vertical de Rham cohomology $H^{p,q,r,s}(\widetilde{M})$.*

Proof. We take $h = \frac{1}{f}$ in the above proposition. □

3. Vertical Lichnerowicz cohomology

In this section we define a vertical Lichnerowicz cohomology for (p, q, r, s)-forms on a complex Finsler manifold (M, F) following the classical definition, e.g. [3, 8, 16].

Let $\omega \in \mathcal{A}^{0,0,0,1}(\widetilde{M})$ be a $\partial_{\overline{v}}$-closed $(0, 0, 0, 1)$-form on $\widetilde{M}$ and the map

$$\partial_{\overline{v},\omega} : \mathcal{A}^{p,q,r,s}(\widetilde{M}) \to \mathcal{A}^{p,q,r,s+1}(\widetilde{M}) \ , \ \partial_{\overline{v},\omega} = \partial_{\overline{v}} - \omega \wedge . \quad (3.1)$$

Since $\partial_{\overline{v}}\omega = 0$, we easily obtain that $\partial^2_{\overline{v},\omega} = 0$. The differential complex

$$0 \longrightarrow \mathcal{A}^{p,q,r,0}(\widetilde{M}) \xrightarrow{\partial_{\overline{v},\omega}} \mathcal{A}^{p,q,r,1}(\widetilde{M}) \xrightarrow{\partial_{\overline{v},\omega}} \dots \xrightarrow{\partial_{\overline{v},\omega}} \mathcal{A}^{p,q,r,n}(\widetilde{M}) \longrightarrow 0 \quad (3.2)$$

is called $\overline{v}$-*Lichnerowicz complex* of complex Finsler manifold (M, F); its cohomology groups $H^{p,q,r,s}_{\omega}(\widetilde{M})$ are called $\overline{v}$-*Lichnerowicz cohomology groups* of the complex Finsler manifold (M, F).

This is a version adapted to our study of the classical Lichnerowicz cohomology, motivated by Lichnerowicz's work [8] or Lichnerowicz-Jacobi cohomology on Jacobi and locally conformal symplectic geometry manifolds, see [3, 7]. We also notice that Vaisman in [16] studied it under the name of "adapted cohomology" on locally conformal Kähler (LCK) manifolds.

We notice that, locally, the $\overline{v}$-Lichnerowicz complex becames the $\overline{v}$-complex after a change $\varphi \mapsto e^f\varphi$ with f a function which satisfies $\partial_{\overline{v}}f = \omega$, namely $\partial_{\overline{v},\omega}$ is the unique differential in $\mathcal{A}^{p,q,r,s}(\widetilde{M})$ which makes the multiplication by the smooth function e^f an isomorphism of cochain $\overline{v}$-complexes $e^f : (\mathcal{A}^{p,q,r,\bullet}(\widetilde{M}), \partial_{\overline{v},\omega}) \to (\mathcal{A}^{p,q,r,\bullet}(\widetilde{M}), \partial_{\overline{v}})$.

Proposition 3.1. *The $\overline{v}$-Lichnerowicz cohomology depends only on the $\overline{v}$-class of ω. In fact, we have $H^{p,q,r,s}_{\omega-\partial_{\overline{v}}f}(\widetilde{M}) \approx H^{p,q,r,s}_{\omega}(\widetilde{M})$.*

Proof. Since $\partial_{\overline{v},\omega}(e^f\varphi) = e^f\partial_{\overline{v},\omega-\partial_{\overline{v}}f}\varphi$ it results that the map $[\varphi] \mapsto [e^f\varphi]$ is an isomorphism between $H^{p,q,r,s}_{\omega-\partial_{\overline{v}}f}(\widetilde{M})$ and $H^{p,q,r,s}_{\omega}(\widetilde{M})$. □

Example 3.2. Let us consider $\omega := \overline{\gamma}$ to be the conjugated vertical Liouville 1-form (the dual of the conjugated vertical Liouville vector field $\overline{\Gamma} = \overline{\eta}^k\frac{\partial}{\partial\overline{\eta}^k}$). Then, by the homogeneity conditions of complex Finsler metric, it is locally given by

$$\overline{\gamma} = \frac{G_{j\overline{k}}\eta^j}{F^2}\delta\overline{\eta}^k = \frac{1}{F^2}\frac{\partial F^2}{\partial\overline{\eta}^k}\delta\overline{\eta}^k = \partial_{\overline{v}}(\log F^2). \tag{3.3}$$

Then $\overline{\gamma}$ is a $\partial_{\overline{v}}$-closed $(0,0,0,1)$-form on $\widetilde{M}$ and we can consider the associated $\overline{v}$-Lichnerowicz cohomology groups $H^{p,q,r,s}_{\overline{\gamma}}(\widetilde{M})$.

As in the classical case, using the definition of $\partial_{\overline{v},\omega}$ we easily obtain

$$\partial_{\overline{v},\omega}(\varphi \wedge \psi) = \partial_{\overline{v}}\varphi \wedge \psi + (-1)^{\deg\varphi}\varphi \wedge \partial_{\overline{v},\omega}\psi.$$

Also, if ω_1 and ω_2 are two $\partial_{\overline{v}}$-closed $(0,0,0,1)$-forms on $\widetilde{M}$ then

$$\partial_{\overline{v},\omega_1+\omega_2}(\varphi \wedge \psi) = \partial_{\overline{v},\omega_1}\varphi \wedge \psi + (-1)^{\deg\varphi}\varphi \wedge \partial_{\overline{v},\omega_2}\psi,$$

which says that the wedge product induces the map

$$\wedge : H^{p,q,r,s_1}_{\omega_1}(\widetilde{M}) \times H^{p,q,r,s_2}_{\omega_2}(\widetilde{M}) \to H^{p,q,r,s_1+s_2}_{\omega_1+\omega_2}(\widetilde{M}).$$

Corollary 3.3. *The wedge product induces the following homomorphism*

$$\wedge : H^{p,q,r,s}_{\omega}(\widetilde{M}) \times H^{p,q,r,s}_{-\omega}(\widetilde{M}) \to H^{p,q,r,2s}(\widetilde{M}).$$

Now, using an argument inspired from [16], we prove that the $\overline{v}$-Lichnerowicz cohomology spaces $H^{p,q,r,s}_{\omega}(\widetilde{M})$ can also be obtained as the $\overline{v}$-cohomology spaces of $\widetilde{M}$ with the coefficients in the sheaf $\Phi^{p,q,r}_{\omega}$ of germs of $\partial_{\overline{v},\omega}$-closed $(p,q,r,0)$-forms.

Firstly, we notice that $\partial_{\overline{v},\omega}$ satisfies a Dolbeault type Lemma for (p,q,r,s)-forms on $\widetilde{M}$. Indeed, let φ be a local (p,q,r,s)-form such that $\partial_{\overline{v},\omega}\varphi = 0$. Since $\partial_{\overline{v}}\omega = 0$ and the lemma has to be local, we may suppose $\omega = -(\partial_{\overline{v}}\alpha)/\alpha$, where α is a nonzero smooth function on $\widetilde{M}$. Then, $\partial_{\overline{v},\omega}\varphi = 0$ means $\partial_{\overline{v}}(\alpha\varphi) = 0$, whence by Dolbeault type lemma for the operator $\partial_{\overline{v}}$, see [13], we have $\varphi = \partial_{\overline{v},\omega}(\psi/\alpha)$ for some local $(p, q, r, s-1)$-form ψ. This is exactly the requested result.

Then, we see that

$$0 \longrightarrow \Phi_{\omega}^{p,q,r} \xrightarrow{\ i\ } \mathcal{A}^{p,r,r,0}(\widetilde{M}) \xrightarrow{\partial_{\overline{v},\omega}} \mathcal{A}^{p,q,r,1}(\widetilde{M}) \xrightarrow{\partial_{\overline{v},\omega}} \dots \tag{3.4}$$

is a fine resolution of $\Phi_{\omega}^{p,q,r}$, which leads to

Proposition 3.4. *For every $\partial_{\overline{v}}$-closed $(0,0,0,1)$-form ω, one has the isomorphisms*

$$H^s(\widetilde{M}, \Phi_{\omega}^{p,q,r}) \approx H_{\omega}^{p,q,r,s}(\widetilde{M}).$$

For every ω as above, let us consider now the auxiliary vertical operator

$$\widetilde{\partial}_{\overline{v}} = \partial_{\overline{v}} - \frac{p+q+r+s}{2}\omega\wedge, \tag{3.5}$$

where (p, q, r, s) is the type of the form acted on. We notice that $\widetilde{\partial}_{\overline{v}}$ is an antiderivation of differential forms and it is easy to see that $\widetilde{\partial}_{\overline{v}}^2 = -\frac{1}{2}\omega\wedge\partial_{\overline{v}}$. Then $\widetilde{\partial}_{\overline{v}}$ defines a *twisted $\overline{v}$-cohomology*, [17], of (p, q, r, s)-forms on $\widetilde{M}$, which is given by

$$H_{\widetilde{\partial}_{\overline{v}}}^{p,q,r,\bullet}(\widetilde{M}) = \frac{\operatorname{Ker}\widetilde{\partial}_{\overline{v}}}{\operatorname{Im}\widetilde{\partial}_{\overline{v}} \cap \operatorname{Ker}\widetilde{\partial}_{\overline{v}}} \tag{3.6}$$

and is isomorphic to the cohomology of the $\overline{v}$-complex $(\widetilde{\mathcal{A}}^{p,q,r,\bullet}(\widetilde{M}), \widetilde{\partial}_{\overline{v}})$ consisting of the (p, q, r, s)-forms $\varphi \in \mathcal{A}^{p,q,r,s}(\widetilde{M})$ satisfying $\widetilde{\partial}_{\overline{v}}^2\varphi = -\omega\wedge\partial_{\overline{v}}\varphi = 0$.

The $\overline{v}$-complex $\widetilde{\mathcal{A}}^{p,q,r,\bullet}(\widetilde{M})$ admits a $\overline{v}$-subcomplex $\mathcal{A}_{\omega}^{p,q,r,\bullet}(\widetilde{M})$, namely, the ideal generated by ω. On this subcomplex, $\widetilde{\partial}_{\overline{v}} = \partial_{\overline{v}}$, which means that it is a $\overline{v}$-subcomplex of the usual $\overline{v}$-de Rham complex of $\widetilde{M}$. Hence, one has the homomorphisms

$$a : H^s(\mathcal{A}_{\omega}^{p,q,r,\bullet}(\widetilde{M})) \to H_{\widetilde{\partial}_{\overline{v}}}^{p,q,r,s}(\widetilde{M})\,,\ b : H^s(\mathcal{A}_{\omega}^{p,q,r,\bullet}(\widetilde{M})) \to H^{p,q,r,s}(\widetilde{M}, \mathbb{C}). \tag{3.7}$$

Now, we can easily construct a homomorphism

$$c : H_{\widetilde{\partial}_{\overline{v}}}^{p,q,r,s}(\widetilde{M}) \to H^{p,q,r,s+1}(\widetilde{M}, \mathbb{C}). \tag{3.8}$$

Indeed, if $[\varphi] \in H_{\widetilde{\partial}_{\overline{v}}}^{p,q,r,s}(\widetilde{M})$, where φ is $\widetilde{\partial}_{\overline{v}}$-closed (p, q, r, s)-form, then we put $c([\varphi]) = [\omega\wedge\varphi]$, and this produces the homomorphism from (3.8). We notice that the existence of c gives some relation between $\widetilde{\partial}_{\overline{v}}$ and the $\overline{v}$-cohomology of $\widetilde{M}$ with values in $\mathbb{C}$.

Remark 3.5. From (2.1) and (3.5) one gets

$$\frac{1}{f}\partial_{\overline{v},f} = \partial_{\overline{v}} - \frac{p+q+r+s}{2}\partial_{\overline{v}}(\log f^2)\wedge = \widetilde{\partial}_{\overline{v}},\ \text{with } \omega = \partial_{\overline{v}}(\log f^2). \tag{3.9}$$

Then, if f does not vanish, we have the homomorphisms

$$\widetilde{a} : H_f^{p,q,r,s}(\widetilde{M}) \to H_{\widetilde{\partial}_{\overline{v}}}^{p,q,r,s}(\widetilde{M}). \tag{3.10}$$

In particular, we can choose $f = F$ to be the complex Finsler function, and so $\frac{1}{F}\partial_{\overline{v},F} = \widetilde{\partial}_{\overline{v}}$ with $\omega = \overline{\gamma}$.

4. A relative vertical cohomology attached to a function

The relative de Rham cohomology was first defined in [4] p. 78. In this subsection we construct a similar version for our vertical cohomology of complex Finsler manifolds.

For the begining we need some basic notions about holomorphic Finsler subspaces. For more details see [9, 10, 11].

4.1. Holomorphic Finsler subspaces

Let (M,F) be a complex Finsler space, (z^k,η^k), $k=1,\ldots,n$ complex coordinates in a local chart, and $i:\mathcal{M}\hookrightarrow M$ a holomorphic immersion of an m-dimensional complex manifold $\mathcal{M}$ into M, locally given by $z^k=z^k(\xi^1,\ldots,\xi^m)$. Everywhere the indices $i,j,k,\ldots$ run from 1 to n and $\alpha,\beta,\gamma,\ldots$ run from 1 to $m\leq n$. Let $T^{1,0}\mathcal{M}$ and $T^{1,0}M$ be the corresponding holomorphic tangent bundles. By $i_{*,C}:T^{1,0}\mathcal{M}\to T^{1,0}M$ we denote the inclusion map between the manifolds $T^{1,0}\mathcal{M}$ and $T^{1,0}M$ (the complexified tangent inclusion map), that is $i_{*,C}(\xi,\theta)=(z(\xi),\eta(\xi,\theta))$, where $\xi=(\xi^\alpha)$, $\theta=\theta^\alpha\frac{\partial}{\partial\xi^\alpha}$, $\eta=\eta^k\frac{\partial}{\partial z^k}$. Then $i_{*,C}$ has the following local representation [9]:

$$z^k=z^k(\xi^1,\ldots,\xi^m),\ \eta^k=\theta^\alpha B^k_\alpha(\xi)\text{ where }B^k_\alpha(\xi)=\frac{\partial z^k}{\partial\xi^\alpha}. \tag{4.1}$$

The holomorphic immersion assumption implies that $B^k_{\overline{\alpha}}=\frac{\partial z^k}{\partial\overline{\xi}^\alpha}=0$ and $B^{\overline{k}}_\alpha=\frac{\partial\overline{z}^k}{\partial\xi^\alpha}=0$. In a point of the complexified tangent space $T_{\mathbb{C}}(T^{1,0}\mathcal{M})$, the local frame $\{\frac{\partial}{\partial\xi^\alpha},\frac{\partial}{\partial\theta^\alpha}\}$ is coupled to $\{\frac{\partial}{\partial z^k},\frac{\partial}{\partial\eta^k}\}$ as follows:

$$\frac{\partial}{\partial\xi^\alpha}=B^k_\alpha\frac{\partial}{\partial z^k}+B^k_{0\alpha}\frac{\partial}{\partial\eta^k},\ \frac{\partial}{\partial\theta^\alpha}=B^k_\alpha\frac{\partial}{\partial\eta^k}, \tag{4.2}$$

where $B^k_{0\alpha}=\frac{\partial B^k_\alpha}{\partial\xi^\beta}\theta^\beta$. Its dual basis satisfies the conditions

$$dz^k=B^k_\alpha d\xi^\alpha,\ d\eta^k=B^k_{0\alpha}d\xi^\alpha+B^k_\alpha d\theta^\alpha \tag{4.3}$$

and their conjugates.

In view of (4.1) the complex Finsler function F, with the metric tensor $G_{j\overline{k}}=\dot{\partial}_j\dot{\partial}_{\overline{k}}(F^2)$, induces a complex Finsler function $\mathcal{F}:T^{1,0}\mathcal{M}\to\mathbb{R}_+\cup\{0\}$ given by $\mathcal{F}(\xi,\theta)=F(z(\xi),\eta(\xi,\theta))=F(z^k(\xi),\theta^\alpha B^k_\alpha(\xi))$ with the metric tensor $\mathcal{G}_{\alpha\overline{\beta}}=B^j_\alpha B^{\overline{k}}_{\overline{\beta}}G_{j\overline{k}}$. Here $\mathcal{G}_{\alpha\overline{\beta}}=\dot{\partial}_\alpha\dot{\partial}_{\overline{\beta}}(\mathcal{F}^2)$ and $\dot{\partial}_\alpha=\frac{\partial}{\partial\theta^\alpha}$, $\dot{\partial}_{\overline{\beta}}=\frac{\partial}{\partial\overline{\theta}^\beta}$. By these considerations, the pair $(\mathcal{M},\mathcal{F})$ is said to be a holomorphic subspace of the complex Finsler space (M,F).

From (4.2) it is deduced that the distribution $V^{1,0}\widetilde{\mathcal{M}}$, spanned locally by $\{\frac{\partial}{\partial\theta^\alpha}\}$, $\alpha=1,\ldots,m$, is a subdistribution of the vertical distribution $V^{1,0}\widetilde{M}$ spanned in any point $(z(\xi),\eta(\xi,\theta))$ by $\{\frac{\partial}{\partial\eta^k}\}$, $k=1,\ldots,n$. We consider $V^{1,0\perp}\mathcal{M}$ an orthogonal complement, namely $V^{1,0}\widetilde{M}=V^{1,0}\mathcal{M}\oplus V^{1,0\perp}\mathcal{M}$, spanned in any point by the set of normal vectors $\{N_a=B^k_a\frac{\partial}{\partial\eta^k}\}$, $a=1,\ldots,n-m$, which we may assume orthonormal. Therefore, the functions $B^k_a(\xi,\theta)$ (and their conjugates) will satisfy the conditions $G_{j\overline{k}}(z(\xi),\eta(\xi,\theta))B^j_\alpha B^{\overline{k}}_{\overline{a}}=0$ and $G_{j\overline{k}}(z(\xi),\eta(\xi,\theta))B^j_aB^{\overline{k}}_{\overline{b}}=\delta_{a\overline{b}}$.

Let us now consider the moving frame $\mathcal{R} = \{B^k_\alpha(\xi)B^k_a(\xi,\theta)\}$ along the complex Finsler subspace $(\mathcal{M},\mathcal{F})$ and let $\mathcal{R}^{-1} = (B^\alpha_k B^a_k)^t$ be the inverse matrix associated to the moving frame $\mathcal{R}$. Evidently, B^α_k and B^a_k are functions of ξ,θ and

$$B^\alpha_k B^k_\beta = \delta^\alpha_\beta,\ B^\alpha_k B^k_a = 0,\ B^a_k B^k_b = \delta^a_b,\ B^k_\alpha B^\alpha_j + B^k_a B^a_j = \delta^k_j. \tag{4.4}$$

Let $\mathcal{N} = (\mathcal{N}^\alpha_\beta(\xi,\theta))$ be a c.n.c. on $T^{1,0}\mathcal{M}$ and consider its adapted basis $\{\delta_\alpha := \frac{\delta}{\delta\xi^\alpha} = \frac{\partial}{\partial\xi^\alpha} - \mathcal{N}^\beta_\alpha\frac{\partial}{\partial\theta^\beta}, \dot{\partial}_\alpha := \frac{\partial}{\partial\theta^\alpha}\}$ and $\{\delta_{\overline{\alpha}}, \dot{\partial}_{\overline{\alpha}}\}$ as well as its dual $\{d\xi^\alpha, \delta\theta^\alpha = d\theta^\alpha - \mathcal{N}^\alpha_\beta d\xi^\beta\}$ and $\{d\overline{\xi}^\alpha, \delta\overline{\theta}^\alpha = d\overline{\theta}^\alpha - \mathcal{N}^{\overline{\alpha}}_{\overline{\beta}} d\overline{\xi}^\beta\}$.

The c.n.c. $\mathcal{N}$ on $T^{1,0}\mathcal{M}$ is said to be *induced* by the c.n.c. N on $T^{1,0}M$ if $\delta\theta^\alpha = B^\alpha_k\delta\eta^k$. This condition implies [9], $\mathcal{N}^\alpha_\beta = B^\alpha_k(B^k_{0\beta} + N^k_j B^j_\beta)$.

Proposition 4.1. *(*[9, 10]*). The adapted bases are tied by*

$$\delta_\alpha = B^k_\alpha\delta_k + B^k_a H^a_\alpha\,\dot{\partial}_k,\ \dot{\partial}_\alpha = B^k_\alpha\,\dot{\partial}_k,$$
$$dz^k = B^k_\alpha d\xi^\alpha,\ \delta\eta^k = B^k_\alpha\delta\theta^\alpha + B^k_a H^a_\alpha d\xi^\alpha$$

with $H^a_\alpha = B^a_j(B^j_{0\alpha} + N^j_k B^k_\alpha)$.

By above proposition we easily deduce that $\delta_\alpha = B^k_\alpha\delta_k + H^a_\alpha N_a$ and $\dot{\partial}_k = B^\alpha_k\,\dot{\partial}_\alpha + B^a_k N_a$. A notable result in [9] asserts that the induced c.n.c. by the Chern-Finsler c.n.c. coincides with the intrinsic Chern-Finsler c.n.c. of the holomorphic subspace $(\mathcal{M},\mathcal{F})$.

4.2. Relative vertical cohomology

Now we return to the construction of a relative vertical cohomology attached to a function of complex Finsler manifolds. Let us denote by

$$J_C i = i_{*,C}.$$

By Proposition 4.1. we easily deduce that if $\varphi \in \mathcal{A}^{p,q,r,s}(\widetilde{M})$ is locally given by (1.9) then

$$(J_C i)^*\varphi \in \mathcal{A}^{p,q,r,s}(\widetilde{\mathcal{M}}) \oplus \bigoplus_{h=\overline{1,r};k=\overline{1,s}} \mathcal{A}^{p+h,q+k,r-h,s-k}(\widetilde{\mathcal{M}}). \tag{4.5}$$

Thus, $(J_C i)^*$ does not preserves the (p,q,r,s) type components of a form $\varphi \in \mathcal{A}^{p,q,r,s}(\widetilde{M})$, but we can eliminate this inconvenient if we take $p = q = m = \dim_{\mathbb{C}}\mathcal{M}$. Then, we easily obtain

Proposition 4.2. *If $\varphi \in \mathcal{A}^{m,m,r,s}(\widetilde{M})$ then $(J_C i)^*\varphi \in \mathcal{A}^{m,m,r,s}(\widetilde{\mathcal{M}})$.*

Proposition 4.3. *If $\varphi \in \mathcal{A}^{m,m,r,s}(\widetilde{M})$ then*

$$\partial_{\overline{v}}(J_C i)^*\varphi = (J_C i)^*\partial_{\overline{v}}\varphi. \tag{4.6}$$

Proof. Let $\varphi = \varphi_{I_m\overline{J_m}K_r\overline{H_s}} dz^{I_m}\wedge d\overline{z}^{J_m}\wedge\delta\eta^{K_r}\wedge\delta\overline{\eta}^{H_s} \in \mathcal{A}^{m,m,r,s}(\widetilde{M})$. By Proposition 4.1. we have

$$(J_C i)^*\varphi = \varphi_{A_m\overline{B_m}C_r\overline{D_s}}(\xi,\theta)d\xi^{A_m}\wedge d\overline{\xi}^{B_m}\wedge\delta\theta^{C_r}\wedge\delta\overline{\theta}^{D_s},$$

where

$$\varphi_{A_m\overline{B_m}C_r\overline{D_s}}(\xi,\theta) = B^{I_m}_{A_m}B^{\overline{J_m}}_{\overline{B_m}}B^{K_r}_{C_r}B^{\overline{H_s}}_{\overline{D_s}}\varphi_{I_m\overline{J_m}K_r\overline{H_s}}(z(\xi),\eta(\xi,\theta)) \tag{4.7}$$

and $B^{I_m}_{A_m} = B^{i_1}_{\alpha_1}(z(\xi)) \cdot \ldots \cdot B^{i_m}_{\alpha_m}(z(\xi))$ etc. Applying $\partial_{\overline{v}}$ from (1.11) it results

$$\partial_{\overline{v}}(J_C i)^*\varphi = \sum_{\alpha} \dot{\partial}_{\overline{\alpha}}\,(\varphi_{A_m\overline{B_m}C_r\overline{D_s}})\delta\overline{\theta}^{\alpha} \wedge d\xi^{A_m} \wedge d\overline{\xi}^{B_m} \wedge \delta\theta^{C_r} \wedge \delta\overline{\theta}^{D_s}. \tag{4.8}$$

Similarly, we have

$$(J_C i)^*\partial_{\overline{v}}\varphi = \\ \dot{\partial}_{\overline{k}}\,(\varphi_{I_m\overline{J_m}K_r\overline{H_s}})B^{\overline{k}}_{\overline{\alpha}}\delta\overline{\theta}^{\alpha} \wedge B^{I_m}_{A_m}d\xi^{A_m} \wedge B^{\overline{J_m}}_{\overline{B_m}}d\overline{\xi}^{B_m} \wedge B^{K_r}_{C_r}\delta\theta^{C_r} \wedge B^{\overline{H_s}}_{\overline{D_s}}\delta\overline{\theta}^{D_s}$$

and by (4.2) and (4.7) one gets

$$(J_C i)^*\partial_{\overline{v}}\varphi = \sum_{\alpha} \dot{\partial}_{\overline{\alpha}}\,(\varphi_{A_m\overline{B_m}C_r\overline{D_s}})\delta\overline{\theta}^{\alpha} \wedge d\xi^{A_m} \wedge d\overline{\xi}^{B_m} \wedge \delta\theta^{C_r} \wedge \delta\overline{\theta}^{D_s} \tag{4.9}$$

which completes the proof. $\square$

Now, if $f \in \mathcal{F}(M)$ then by (4.6) one gets

$$\partial_{\overline{v},(J_C i)^* f}(J_C i)^*\varphi = (J_C i)^*\partial_{\overline{v},f}\varphi, \text{ for any } \varphi \in \mathcal{A}^{m,m,r,s}(\widetilde{M}). \tag{4.10}$$

Indeed, for $\varphi \in \mathcal{A}^{m,m,r,s}(\widetilde{M})$ by direct calculus we have

$$\begin{aligned}
&\partial_{\overline{v},(J_C i)^* f}((J_C i)^*\varphi)\\
&= (J_C i)^* f\partial_{\overline{v}}((J_C i)^*\varphi) - (2m+r+s)\partial_{\overline{v}}((J_C i)^* f) \wedge (J_C i)^*\varphi\\
&= (J_C i)^* f(J_C i)^*(\partial_{\overline{v}}\varphi) - (2m+r+s)(J_C i)^*(\partial_{\overline{v}} f) \wedge (J_C i)^*\varphi\\
&= (J_C i)^*(f\partial_{\overline{v}}\varphi) - (J_C i)^*((2m+r+s)\partial_{\overline{v}} f \wedge \varphi)\\
&= (J_C i)^*(\partial_{\overline{v},f}\varphi).
\end{aligned}$$

We define the differential complex

$$\ldots \xrightarrow{\widetilde{\partial}_{\overline{v},f}} \mathcal{A}^{m,m,r,s}(J_C i) \xrightarrow{\widetilde{\partial}_{\overline{v},f}} \mathcal{A}^{m,m,r,s+1}(J_C i) \xrightarrow{\widetilde{\partial}_{\overline{v},f}} \ldots$$

where $\mathcal{A}^{m,m,r,s}(J_C i) = \mathcal{A}^{m,m,r,s}(\widetilde{M}) \oplus \mathcal{A}^{m,m,r,s-1}(\widetilde{\mathcal{M}})$ and

$$\widetilde{\partial}_{\overline{v},f}(\varphi,\psi) = (\partial_{\overline{v},f}\varphi, (J_C i)^*\varphi - \partial_{\overline{v},(J_C i)^* f}\psi). \tag{4.11}$$

Taking into account $\partial^2_{\overline{v},f} = \partial^2_{\overline{v},(J_C i)^* f} = 0$ and (4.10) we easily verify that $\widetilde{\partial}^2_{\overline{v},f} = 0$. Denote the cohomology groups of this complex by $H^{m,m,r,s}_f(J_C i)$.

Now, if we regraduate the complex $\mathcal{A}^{m,m,r,s}(\widetilde{\mathcal{M}})$ as

$$\widetilde{\mathcal{A}}^{m,m,r,s}(\widetilde{\mathcal{M}}) = \mathcal{A}^{m,m,r,s-1}(\widetilde{\mathcal{M}}),$$

then we obtain an exact sequence of differential complexes

$$0 \longrightarrow \widetilde{\mathcal{A}}^{m,m,r,s}(\widetilde{\mathcal{M}}) \xrightarrow{\alpha} \mathcal{A}^{m,m,r,s}(J_C i) \xrightarrow{\beta} \mathcal{A}^{m,m,r,s}(\widetilde{M}) \longrightarrow 0 \tag{4.12}$$

with the obvious mappings α and β given by $\alpha(\psi) = (0,\psi)$ and $\beta(\varphi,\psi) = \varphi$, respectively. From (4.12) we have an exact sequence in cohomologies

$$\ldots \longrightarrow H^{m,m,r,s-1}_{(J_C i)^* f}(\widetilde{\mathcal{M}}) \xrightarrow{\alpha^*} H^{m,m,r,s}_f((J_C i)^*) \xrightarrow{\beta^*} H^{m,m,r,s-1}_f(\widetilde{M}) \xrightarrow{\delta^*}$$

$$\xrightarrow{\delta^*} H^{m,m,r,s}_{(J_C i)^* f}(\widetilde{\mathcal{M}}) \xrightarrow{\alpha^*} \ldots.$$

It is easily seen that $\delta^* = (J_C i)^*$. Here μ^* denotes the corresponding map between cohomology groups. Let $\varphi \in \mathcal{A}^{m,m,r,s}(\widetilde{M})$ be a $\partial_{\overline{v},f}$-closed form, and $(\varphi, \psi) \in \mathcal{A}^{m,m,r,s}(J_C i)$. Then $\widetilde{\partial}_{\overline{v},f}(\varphi,\psi) = (0, (J_C i)^*\varphi - \partial_{\overline{v},(J_C i)^* f}\psi)$ and by the definition of the operator δ^* we have

$$\delta^*[\varphi] = [(J_C i)^*\varphi - \partial_{\overline{v},(J_C i)^* f}\psi] = [(J_C i)^*\varphi] = (J_C i)^*[\varphi].$$

Hence we finally get a long exact sequence

$$\ldots \longrightarrow H^{m,m,r,s-1}_{(J_C i)^* f}(\widetilde{\mathcal{M}}) \xrightarrow{\alpha^*} H^{m,m,r,s}_{f}((J_C i)^*) \xrightarrow{\beta^*} H^{m,m,r,s-1}_{f}(\widetilde{M}) \xrightarrow{(J_C i)^*}$$
$$\xrightarrow{(J_C i)^*} H^{m,m,r,s}_{(J_C i)^* f}(\widetilde{\mathcal{M}}) \xrightarrow{\alpha^*} \ldots .$$

Finally, similar to [14], we have

Corollary 4.4. *If $(\mathcal{M}, \mathcal{F})$ is an m-dimensional holomorphic Finsler subspace of an n-dimensional complex Finsler space (M, F), then*

(i) $\beta^* : H^{m,m,r,m+1}_{f}(J_C i) \to H^{m,m,r,m+1}_{f}(\widetilde{M})$ *is an epimorphism;*

(ii) $\alpha^* : H^{m,m,r,n}_{(J_C i)^* f}(\widetilde{\mathcal{M}}) \to H^{m,m,r,n+1}_{f}(J_C i)$ *is an epimorphism;*

(iii) $\beta^* : H^{m,m,r,s}_{f}(J_C i) \to H^{m,m,r,s}_{f}(\widetilde{M})$ *is an isomorphism for $s > m+1$;*

(iv) $\alpha^* : H^{m,m,r,s}_{(J_C i)^* f}(\widetilde{\mathcal{M}}) \to H^{m,m,r,s+1}_{f}(J_C i)$ *is an isomorphism for $s > n$;*

(v) $H^{m,m,r,s}_{f}(J_C i) = 0$ *for $s > \max\{m+1, n\}$.*

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