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Fractional order partial hyperbolic differential equations involving Caputo's derivative

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Abstract. In the present paper we investigate the existence and uniqueness of solutions of the Darboux problem for the initial value problems (IVP for short), for some classes of hyperbolic fractional order differential equations by using some fixed point theorems.

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1. Introduction

The idea of fractional calculus and fractional order differential equations and inclusions has been a subject of interest not only among mathematicians, but also among physicists and engineers. Indeed, we can find numerous applications in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [14, 15, 19, 20, 22, 27]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [3], Kilbas *et al.* [17], Miller and Ross [21], Samko *et al.* [26], the papers of Abbas and Benchohra [1, 2], Abbas *et al.* [4, 5], Belarbi *et al.* [8], Benchohra *et al.* [9, 10, 11], Diethelm [13], Kaufmann and Mboumi [16], Kilbas and Marzan [18], Mainardi [19], Podlubny *et al* [25], Vityuk [28], Vityuk and Golushkov [29], Vityuk and Mykhailenko [30, 31], Zhang [32] and the references therein.

Applied problems require definitions of fractional derivative allowing the utilization of physically interpretable initial conditions. Caputo's fractional derivative, originally introduced by Caputo [12] and afterwards adopted in the theory of linear viscoelasticity, satisfies this demand. For a consistent bibliography on this topic, historical remarks and examples we refer to [6, 7, 23, 24].

In [33], Zhang considered the existence and uniqueness of positive solutions for the following fractional order system

$$\begin{cases} D_\theta^r u(x,y) = f(x,y,u(x,y), D_\theta^{\rho_1}u(x,y),\dots, \\ D_\theta^{\rho_n}u(x,y)); \text{ if } (x,y)\in(0,a]\times(0,b], \\ u(x,0)=u(0,y)=0, \end{cases} \tag{1.1}$$

where $r=(\alpha,\beta)\in(0,1]\times(0,1],\ \rho_i=(\delta_i,\gamma_i);\ i=1,\dots,n$, and $0\le\gamma_i<\alpha,\ 0\le\delta_i<\beta$, and D_θ^r is Riemann-Liouville fractional derivative.

In the present paper we investigate the existence and uniqueness of solutions to fractional order system

$$^cD_\theta^r u(x,y) = f(x,y,u(x,y),^c D_\theta^\rho u(x,y)); \text{ if } (x,y)\in J:=[0,a]\times[0,b], \tag{1.2}$$

$$\begin{cases} u(x,0)=\varphi(x);\ x\in[0,a], \\ u(0,y)=\psi(y);\ y\in[0,b], \\ \varphi(0)=\psi(0), \end{cases} \tag{1.3}$$

where $a,b>0,\ \theta=(0,0),\ r=(r_1,r_2),\ \rho=(\rho_1,\rho_2),\ 0<\rho_i<r_i\le 1;\ i=1,2,\ ^cD_\theta^r$ is the standard Caputo's fractional derivative of order r, $f: J\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is a given function, $\varphi:[0,a]\to\mathbb{R}^n$, and $\psi:[0,b]\to\mathbb{R}^n$ are given absolutely continuous functions. We present three results for the problem (1.2)-(1.3), the two first results are based on Schauder's Fixed Point Theorem (Theorems 3.3 and 3.4) and the third one on Banach's contraction principle (Theorem 3.5). As an extension to the problem (4.1)-(4.2), we present two similar results (Theorems 4.1 and 4.2).

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J)$ we denote the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty = \sup_{(x,y)\in J}\|w(x,y)\|,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$.

As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into $\mathbb{R}^n$ and $L^1(J)$ is the space of Lebesgue-integrable functions $w: J\to\mathbb{R}^n$ with the norm

$$\|w\|_{L^1} = \int_0^a\int_0^b\|w(x,y)\|dydx.$$

Definition 2.1. [29] *Let $r=(r_1,r_2)\in(0,\infty)\times(0,\infty),\ \theta=(0,0)$ and $u\in L^1(J)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by*

$$(I_\theta^r u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}u(s,t)dtds.$$

In particular,

$$(I_\theta^\theta u)(x,y) = u(x,y),\ (I_\theta^\sigma u)(x,y) = \int_0^x \int_0^y u(s,t)dtds;\ \text{for a.a } (x,y) \in J,$$

where $\sigma = (1,1)$.

For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $(I_\theta^r u) \in C(J)$, moreover

$$(I_\theta^r u)(x,0) = (I_\theta^r u)(0,y) = 0;\ x \in [0,a],\ y \in [0,b].$$

Example 2.2. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, then

$$I_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2},\ \text{for almost all } (x,y) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$, the mixed second order partial derivative.

Definition 2.3. [29] *Let $r \in (0,1] \times (0,1]$ and $u \in L^1(J)$. The Caputo fractional-order derivative of order r of u is defined by the expression ${}^cD_\theta^r u(x,y) = (I_\theta^{1-r} D_{xy}^2 u)(x,y)$.*

The case $\sigma = (1,1)$ is included and we have

$$({}^cD_\theta^\sigma u)(x,y) = (D_{xy}^2 u)(x,y),\ \text{for almost all } (x,y) \in J.$$

Example 2.4. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$, then

$$D_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2},\ \text{for almost all } (x,y) \in J.$$

For $w, {}^c D_\theta^\rho w \in C(J)$, denote

$$\|w(x,y)\|_1 = \|w(x,y)\| + \|{}^cD_\theta^\rho w(x,y)\|.$$

We define the space X as the following

$$X = \{w \in C(J) \text{ having the Caputo fractional derivative of order } \rho,$$
$$\text{and } {}^cD_\theta^\rho w \in C(J)\}.$$

In the space X we define the norm

$$\|w\|_X = \sup_{(x,y)\in J} \|w(x,y)\|_1.$$

It is easy to see that $(X, \|.\|_X)$ is a Banach space.

3. Existence of solutions

Let us start by defining what we mean by a solution of the problem (1.2)-(1.3).

Definition 3.1. *A function $u \in X$ is said to be a solution of (1.2)-(1.3) if u satisfies equation (1.2) and conditions (1.3) on J.*

For the existence of solutions for the problem (1.2)-(1.3) we need the following lemma. Its proof is easily and left to the reader.

Lemma 3.2. *A function $u \in X$ is a solution of problem (1.2)-(1.3) if and only if u satisfies*

$$u(x,y) = \mu(x,y) + I_\theta^r f(x,y,u(x,y),{}^c D_\theta^\rho u(x,y));\ (x,y) \in J,$$

where

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

Further, we present conditions for the existence of a solution of problem (1.2)-(1.3) by using Schauder's Fixed Point Theorem. In the following result we assume a sublinear growth condition on the right hand side, namely the function f.

Theorem 3.3. *Assume*

(H_1) *The function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous,*

(H_2) *There exist constants $c, c_i > 0;\ i = 0, 1$ and $0 < \tau_j < 1;\ j = 0, 1$ such that*

$$\|f(x,y,u(x,y),{}^c D_\theta^\rho u)\| \le c + c_0\|u\|^{\tau_0} + c_1\|{}^c D_\theta^\rho u\|^{\tau_1},$$

for any $u \in \mathbb{R}^n$ and all $(x,y) \in J$.

Then there exists at least a solution for IVP (1.2)-(1.3) on J.

Proof. Transform the problem (1.2)-(1.3) into a fixed point problem. Consider the operator $N : X \to X$ defined by,

$$N(u)(x,y) = \mu(x,y) + I_\theta^r f(x,y,u(x,y),{}^c D_\theta^\rho u(x,y)). \tag{3.1}$$

By Lemma 3.2, the problem of finding the solutions of the IVP (1.2)-(1.3) is reduced to finding the solutions of the operator equation $N(u) = u$. Differentiating both sides of (3.1) by applying the Caputo fractional derivative, we get

$${}^c D_\theta^\rho (Nu)(x,y) = {}^c D_\theta^\rho \mu(x,y) + I_\theta^{r-\rho} f(x,y,u(x,y),{}^c D_\theta^\rho u(x,y)). \tag{3.2}$$

Since $N(u)$ and ${}^c D_\theta^\rho (Nu)$ are continuous on J, then N maps X into itself.

From (H_1) and the Arzela-Ascoli Theorem, the operator N is completely continuous.

Let $\tau = \max\{\tau_0, \tau_1\}$ and $B_R = \{u \in X : \|u\|_X \le R\}$ be a closed bounded and convex subset of X, where

$$R > \max\{1, A, B, C, D\},$$

where

$$A = 4\|\mu\|_\infty + \frac{4ca^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)},$$

$$B = 4\|{}^c D_\theta^\rho \mu\|_\infty + \frac{4ca^{r_1-\rho_1}b^{r_2-\rho_2}}{\Gamma(1+r_1-\rho_1)\Gamma(1+r_2-\rho_2)},$$

$$C = \left(\frac{\Gamma(1+r_1)\Gamma(1+r_2)}{4(c_0+c_1+2)a^{r_1}b^{r_2}}\right)^{\frac{1}{1-\tau}},$$

$$D = \left(\frac{\Gamma(1+r_1-\rho_1)\Gamma(1+r_2-\rho_2)}{4(c_0+c_1+2)a^{r_1-\rho_1}b^{r_2-\rho_2}}\right)^{\frac{1}{1-\tau}}.$$

By (H_2), for every $u \in B_R$ and $(x,y) \in J$ we have

$$\begin{aligned}
& \|N(u)(x,y)\| \\
\leq\ & \|\mu(x,y)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\
\times\ & \|f(s,t,u(s,t),{}^cD_\theta^\rho u(s,t)))\|dtds \\
\leq\ & \|\mu(x,y)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\
\times\ & (c + c_0\|u(s,t)\|^{\tau_0} + c_1\|{}^cD_\theta^\rho u(s,t)\|^{\tau_1})dtds \\
\leq\ & \|\mu\|_\infty + \frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}(c + c_0\|u\|_X^{\tau_0} + c_1\|{}^cD_\theta^\rho u\|_X^{\tau_1}) \\
\leq\ & \|\mu\|_\infty + \frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\Big(c + (c_0+1)R^{\tau_0} + (c_1+1)R^{\tau_1}\Big) \\
\leq\ & \|\mu\|_\infty + \frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\Big(c + (c_0+c_1+2)R^{\tau}\Big) \\
=\ & \|\mu\|_\infty + \frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\Big(c + R^{\tau-1}R(c_0+c_1+2)\Big) \\
\leq\ & \frac{R}{4} + \frac{R}{4} = \frac{R}{2},
\end{aligned}$$

and

$$\begin{aligned}
& \|{}^cD_\theta^\rho N(u)(x,y)\| \\
\leq\ & \|{}^cD_\theta^\rho \mu(x,y)\| + \frac{1}{\Gamma(r_1-\rho_1)\Gamma(r_2-\rho_2)} \int_0^x \int_0^y (x-s)^{r_1-\rho_1-1} \\
\times\ & (y-t)^{r_2-\rho_2-1} f(s,t,u(s,t),{}^cD_\theta^\rho u(s,t)))\|dtds \\
\leq\ & \|{}^cD_\theta^\rho \mu(x,y)\| + \frac{1}{\Gamma(r_1-\rho_1)\Gamma(r_2-\rho_2)} \int_0^x \int_0^y (x-s)^{r_1-\rho_1-1} \\
\times\ & (y-t)^{r_2-\rho_2-1}(c + c_0\|u(s,t)\|^{\tau_0} + c_1\|{}^cD_\theta^\rho u(s,t)\|^{\tau_1})dtds \\
\leq\ & \|{}^cD_\theta^\rho \mu\|_\infty + \frac{a^{r_1-\rho_1}b^{r_2-\rho_2}}{\Gamma(1+r_1-\rho_1)\Gamma(1+r_2-\rho_2)}(c + c_0\|u\|_X^{\tau_0} + c_1\|{}^cD_\theta^\rho u\|_X^{\tau_1}) \\
\leq\ & \|{}^cD_\theta^\rho \mu\|_\infty + \frac{a^{r_1-\rho_1}b^{r_2-\rho_2}}{\Gamma(1+r_1-\rho_1)\Gamma(1+r_2-\rho_2)} \\
\times\ & \Big(c + (c_0+1)R^{\tau_0} + (c_1+1)R^{\tau_1}\Big) \\
\leq\ & \|{}^cD_\theta^\rho \mu\|_\infty + \frac{a^{r_1-\rho_1}b^{r_2-\rho_2}}{\Gamma(1+r_1-\rho_1)\Gamma(1+r_2-\rho_2)}\Big(c + R^{\tau-1}R(c_0+c_1+2)\Big) \\
\leq\ & \frac{R}{4} + \frac{R}{4} = \frac{R}{2}.
\end{aligned}$$

Thus, for every $u \in B_R$ and $(x,y) \in J$ we have

$$\begin{aligned} \|N(u)(x,y)\|_1 &= \|N(u)(x,y)\| + \|^c D_\theta^\rho N(u)(x,y)\| \\ &\leq \frac{R}{2} + \frac{R}{2} = R. \end{aligned}$$

Hence $\|N(u)\|_X \leq R$ for $u \in B_R$, that is, $N(B_R) \subseteq B_R$. Schauder's fixed point theorem implies that the operator N has at least a fixed point $u^* \in B_R$. By Lemma 3.2, the problem (1.2)-(1.3) has a solution $u^* \in B_R$.
In the following result we assume a superlinear growth condition on the function f.

Theorem 3.4. *Assume* (H_1) *and the following hypothesis holds*

(H_2') *There exist constants* $d_i > 0;\ i = 0,1$ *and* $\nu_j > 1;\ j = 0,1$ *such that*

$$\|f(x,y,u(x,y),^c D_\theta^\rho u)\| \leq d_0\|u\|^{\nu_0} + d_1\|^c D_\theta^\rho u\|^{\nu_1},$$

for any $u \in \mathbb{R}^n$ *and all* $(x,y) \in J$.

Then the IVP (1.2)*-*(1.3) *has at least a solution on* J.

Proof. Consider the operator N defined by (3.1). In a similar way as in Theorem 3.3, we can complete this proof, provided if we take the closed, bounded and convex subset $B_R = \{u \in X : \|u\|_X \leq R\}$ of the space X, where

$$R < \min\left\{1, \frac{\|\mu\|_\infty}{3}, \mathcal{A}, \mathcal{B}\right\},$$

where

$$\mathcal{A} = \left(\frac{\Gamma(1+r_1)\Gamma(1+r_2)}{3(c_0+c_1+2)a^{r_1}b^{r_2}}\right)^{\frac{1}{1-\nu}},$$

$$\mathcal{B} = \left(\frac{\Gamma(1+r_1-\rho_1)\Gamma(1+r_2-\rho_2)}{3(c_0+c_1+2)a^{r_1-\rho_1}b^{r_2-\rho_2}}\right)^{\frac{1}{1-\nu}},$$

and

$$\nu = \min\{\nu_0, \nu_1\}.$$

Now, we present a uniqueness result for the problem (1.2)-(1.3) based on Banach's contraction principle.

Theorem 3.5. *Assume* (H_1) *and the following hypothesis holds*

(H_3) *There exist positive functions* $g, h \in C(J)$ *satisfying*

$$(I_\theta^r g)(x,y) + (I_\theta^{r-\rho})g(x,y) < \frac{1}{2},\ (I_\theta^r h)(x,y) + (I_\theta^{r-\rho}h)(x,y) < \frac{1}{2},$$

such that

$$\|f(x,y,u,^c D_\theta^\rho u) - f(x,y,v,^c D_\theta^\rho v)\| \leq g(x,y)\|u-v\| + h(x,y)\|^c D_\theta^\rho u -^c D_\theta^\rho v\|,$$

for all $(x,y) \in J$ *and* $u, v \in \mathbb{R}^n$.

Then the IVP (1.2)*-*(1.3) *has a unique solution on* J.

Proof. Consider the operator N defined in (3.1). Let $u, v \in X$. By assumption (H_3), for $(x,y) \in J$, we have

$$\begin{aligned}
& \|N(u)(x,y) - N(v)(x,y)\|_1 \\
= & \|I_\theta^r\Big(f(x,y,u(x,y),^c D_\theta^\rho u(x,y)) - f(x,y,v(x,y),^c D_\theta v^\rho(x,y))\Big)\| \\
+ & \|^c D_\theta^r I_\theta^\rho\Big(f(x,y,u(x,y),^c D_\theta^\rho u(x,y)) - f(x,y,v(x,y),^c D_\theta v^\rho(x,y))\Big)\| \\
\leq & I_\theta^r\Big(g(x,y)\|u(x,y)-v(x,y)\| + h(x,y)\|^c D_\theta^\rho u(x,y) -^c D_\theta^\rho v(x,y)\|\Big) \\
+ & I_\theta^{r-\rho}\Big(g(x,y)\|u(x,y)-v(x,y)\| + h(x,y)\|^c D_\theta^\rho u(x,y) -^c D_\theta^\rho v(x,y)\|\Big) \\
\leq & \Big(I_\theta^r g(x,y) + I_\theta^{r-\rho} g(x,y)\Big)\|u(x,y)-v(x,y)\| \\
+ & \Big(I_\theta^r h(x,y) + I_\theta^{r-\rho} h(x,y)\Big)\|^c D_\theta^\rho u(x,y) -^c D_\theta^\rho v(x,y)\| \\
\leq & \frac{1}{2}\|u(x,y)-v(x,y)\| + \frac{1}{2}\|^c D_\theta^\rho u(x,y) -^c D_\theta^\rho v(x,y)\| \\
\leq & \frac{1}{2}\|u(x,y)-v(x,y)\|_1.
\end{aligned}$$

Hence

$$\|N(u) - N(v)\|_X \leq \frac{1}{2}\|u - v\|_X,$$

which implies that N is a contraction operator. Then Banach's Contraction Principle assures that the operator N has a unique fixed point $u^* \in X$.

4. More general existence results

In this section, we present (without proof) two existence results to the more general class of fractional order IVP for the system

$$\begin{aligned}
^c D_\theta^r u(x,y) = f(x,y,u(x,y),^c D_\theta^{\rho_1} u(x,y),^c D_\theta^{\rho_2} u(x,y), ..., \\
^c D_\theta^{\rho_m} u(x,y)); \text{ if } (x,y) \in J,
\end{aligned} \tag{4.1}$$

$$\begin{cases} u(x,0) = \varphi(x); \ x \in [0,a], \\ u(0,y) = \psi(y); \ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{4.2}$$

where $J := [0,a] \times [0,b],\ a,b > 0,\ \theta = (0,0),\ r = (r_1, r_2),\ \rho_i = (\rho_{i,1}, \rho_{i,2}),\ 0 < \rho_{i,j} < r_j \leq 1;\ i = 1, \ldots, m,\ j = 1,2$ and f is a given continuous function.

For $w,^c D_\theta^{\rho_i} w \in C(J);\ i = 1, \ldots, m$, denote

$$\|w(x,y)\|_1 = \|w(x,y)\| + \sum_{i=1}^m \|^c D_\theta^{\rho_i} w(x,y)\|.$$

We define the following space

$$\begin{aligned}
\widetilde{X} = \{w \in C(J) \text{ having the Caputo fractional derivative of order } \rho_i, \\
\text{and } ^c D_\theta^{\rho_i} w \in C(J);\ i = 1, \ldots, m\}.
\end{aligned}$$

The space $\widetilde{X}$ is a Banach space with the norm

$$\|w\|_{\widetilde{X}} = \sup_{(x,y)\in J} \|w(x,y)\|_1.$$

The following result for the problem (4.1)-(4.2) is based on Schauder's fixed point theorem.

Theorem 4.1. *Assume that the function f satisfying one of the following conditions:*

(H_4) *There exist constants $c, c_i > 0;\ i = 0, \ldots, m$ and $0 < \tau_j < 1;\ j = 1, \ldots, m$ such that*

$$\|f(x,y,u(x,y),{}^c D_\theta^{\rho_1} u, {}^c D_\theta^{\rho_2} u, ..., {}^c D_\theta^{\rho_m} u)\| \leq c + c_0\|u\|^{\tau_0} + \sum_{i=1}^{m} c_i \|{}^c D_\theta^{\rho_i} u\|^{\tau_i},$$

for any $u \in \mathbb{R}^n$ and all $(x,y) \in J$.

(H_4') *There exist constants $d_i > 0;\ i = 0, 1, \ldots, m$ and $\nu_j > 1;\ j = 0, 1, \ldots, m$ such that*

$$\|f(x,y,u(x,y),{}^c D_\theta^{\rho_1} u, {}^c D_\theta^{\rho_2} u, ..., {}^c D_\theta^{\rho_m} u)\| \leq d_0\|u\|^{\nu_0} + \sum_{i=1}^{m} d_i \|{}^c D_\theta^{\rho_i} u\|^{\nu_i},$$

for any $u \in \mathbb{R}^n$ and all $(x,y) \in J$.

Then there exists at least a solution for IVP (4.1)-(4.2) on J.

By means of the Banach contraction principle, we have the following result for problem (4.1)-(4.2).

Theorem 4.2. *Assume*

(H_5) *There exist positive functions $g, h_i \in C(J);\ i = 1, \ldots, m$ satisfying*

$$(I_\theta^r g)(x,y) + \sum_{i=1}^{m} (I_\theta^{r-\rho_i} g)(x,y) < \frac{1}{2},$$

$$\sum_{i=1}^{m} (I_\theta^r h_i)(x,y) + \sum_{j=1}^{m}\sum_{i=1}^{m} (I_\theta^{r-\rho_j} h_i)(x,y) < \frac{1}{2},$$

such that

$$\|f(x,y,u,{}^c D_\theta^{\rho_1} u, \ldots, {}^c D_\theta^{\rho_m} u) - f(x,y,v,{}^c D_\theta^{\rho_1} v, \ldots, {}^c D_\theta^{\rho_m} v)\| \leq g(x,y)\|u-v\|$$

$$+ \sum_{i=1}^{m} h_i(x,y) \|{}^c D_\theta^{\rho_i} u - {}^c D_\theta^{\rho_i} v\|,$$

for all $(x,y) \in J$ and $u, v \in \mathbb{R}^n$.

Then the IVP (4.1)-(4.2) has a unique solution on J.

5. An example

As an application of our results we consider the following partial hyperbolic differential equations of the form

$$^{c}D^{r}_{\theta}u(x,y) = \frac{72}{72+9xy^2|u(x,y)|+8x^2y|^{c}D^{\rho}_{\theta}u(x,y)|}; \text{ if } (x,y)\in[0,1]\times[0,1], \quad (5.1)$$

$$u(x,0)=x,\; u(0,y)=y^2;\; x,y\in[0,1]. \quad (5.2)$$

Set for $(x,y)\in[0,1]\times[0,1]$

$$f(x,y,u(x,y),^{c}D^{r}_{\theta}u(x,y)) = \frac{72}{72+9xy^2|u(x,y)|+8x^2y|^{c}D^{\rho}_{\theta}u(x,y)|}.$$

Clearly, the function f is continuous. For each $u,\overline{u},\; v,\overline{v}\in\mathbb{R}$ and $(x,y)\in[0,1]\times[0,1]$ we have

$$\begin{aligned} & |f(x,y,u(x,y),v(x,y)) - f(x,y,\overline{u}(x,y),\overline{v}(x,y))| \\ \leq\; & \frac{1}{8}xy^2\|u-\overline{u}\| + \frac{1}{9}x^2y\|v-\overline{v}\|. \end{aligned}$$

Hence condition (H_3) is satisfied with

$$g(x,y) = \frac{1}{8}xy^2 \text{ and } h(x,y) = \frac{1}{9}x^2y.$$

For each $(x,y)\in[0,1]\times[0,1]$ and $0<r_i<\rho_i\leq 1;\; i=1,2$ we have

$$(I^{r}_{\theta}g)(x,y) + (I^{r-\rho}_{\theta}g)(x,y) \leq \frac{2\Gamma(2)\Gamma(3)}{8\Gamma(2)\Gamma(3)} = \frac{1}{4} < \frac{1}{2},$$

and

$$(I^{r}_{\theta}h)(x,y) + (I^{r-\rho}_{\theta}h)(x,y) \leq \frac{2\Gamma(2)\Gamma(3)}{9\Gamma(2)\Gamma(3)} = \frac{2}{9} < \frac{1}{2}.$$

Consequently, Theorem 3.5 implies that problem (5.1)-(5.2) has a unique solution on $[0,1]\times[0,1]$.

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