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Approximation methods for second order nonlinear polylocal problems

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Abstract. Consider the problem:

$$\begin{aligned} y''(x) + f(x,y) &= 0, \qquad x \in [0,1] \\ y(a) &= \alpha \\ y(b) &= \beta, \qquad a, b \in (0,1). \end{aligned}$$

This is not a two-point boundary value problem since $a, b \in (0,1)$. It is possible to solve this problem by dividing it into the three problems: a two-point boundary value problem (BVP) on $[a,b]$ and two initial-value problems (IVP), on $[0,a]$ and $[b,1]$. The aim of this work is to present two solution procedures: one based on B-splines of order $k+2$ and the other based on a combination of B-splines (order $k+2$) with a $(k+1)$-order Runge-Kutta method. Then, we give two numerical examples and compare the methods experimentally.

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1. Introduction

Consider the problem (PVP - **P**olylocal **V**alue **P**roblem):

$$y''(x) + f(x,y) = 0, \qquad x \in [0,1] \tag{1.1}$$

$$y(a) = \alpha \tag{1.2}$$

$$y(b) = \beta, \qquad a, b \in (0,1), \ a < b. \tag{1.3}$$

where a, b, α, $\beta \in \mathbb{R}$. *This is not a two-point boundary value problem*, since $a, b \in (0,1)$.

We try to solve the problem using two methods:

- a collocation method based on B-splines of order $k+2$;
- a combined method based on B-splines (order $k+2$) and a Runge-Kutta method(order $k+1$).

The methods are new in this context: the conditions are stated at interior points. Also it is shown that the Runge-Kutta method does not degrade the accuracy provided by the collocation method for the BVP.

Our choice to use these methods is based on the following reasons :

1. We write the code using the function `spcol` in MATLAB Spline Toolbox.
2. It is the most suitable method, for a general purpose code, among the finite element ones. See [2, 17, 21], where complexity comparisons which support the above claim are made and collocation, when efficiently implemented, is shown to be competitive with finite differences using extrapolation.
3. Theoretical results on the convergence of collocation method are given in [6, 16].
4. Several representative test problems demonstrate the stability and flexibility [7].
5. For each Newton iteration, the resulting linear algebraic system of equations (after using Newton method with quasilinearization) is solved using methods given in [8].

We also consider the BVP :

$$y''(x) + f(x,y) = 0, \qquad x \in [a,b] \tag{1.4}$$

$$y(a) = \alpha \tag{1.5}$$

$$y(b) = \beta, \tag{1.6}$$

To apply the collocation theory, we need to have an isolated solution $y(x)$ of the problem (1.4)+(1.5)+(1.6), and this occurs if the above linearized problem for $y(x)$ is uniquely solvable. R.D Russel and L.F.Shampine [22] study the existence and the uniqueness of the isolated solution.

Theorem 1.1. [22] *Suppose that $y(x)$ is a solution of the boundary value problem (1.4)+(1.5)+(1.6), that the functions*

$$f(x,z) \text{ and } \frac{\partial f(x,z)}{\partial y}$$

are defined and continuous for $a \le x \le b$, and $|z-y| \le \delta$, $\delta > 0$, and the homogeneous equation $y''(x) = 0$ subject to the homogeneous boundary conditions (1.5)+(1.6) has only the trivial solution. If the linear homogeneous equation

$$z''(x) + \frac{\partial f(x,y)}{\partial y} z(x) = 0$$

has only trivial solution, then this is sufficient to guarantee that there exists a $\sigma > 0$ such $y(x)$ is the unique solution of problem BVP in the sphere:

$$\{w : \|w - y''\| \le \sigma\}.$$

For the existence and uniqueness of an IVP, we recall the following result.

Theorem 1.2. [15, pp. 112-113]*Suppose that* $D = \{a \le x \le b, -\infty < y < \infty\}$ *and* $f(x, y)$ *is continuous on* D. *If* f *satisfies a Lipschitz condition on* D *in the variable* y, *then the initial value problem (IVP)*

$$\begin{cases} y' & = & z, \\ z' & = & -f(x,y), \ a \le x \le b, \\ y(a) & = & \alpha, \\ y'(a) & = & \varsigma, \end{cases} \tag{1.7}$$

has a unique solution $y(x)$ *for* $a \le x \le b$.

If the problem BVP has the unique solution, the requirement $y(x) \in C^2[0,1]$ ensure the existence and the uniqueness of the solution of PVP.

2. The collocation method for solving the polylocal problem using B-splines

2.1. B-splines bases of degree k (order $k+1$)

For reason of efficiency, stability, flexibility in order, and continuity, we choose B-splines as the basis functions. Efficient algorithms for calculating with B-splines are given by deBoor [9, 10] and Risler[20].

Consider a sequence of knots $t_0, \ldots, t_m$, such that $t_i \le t_{i+1}$ for all i.

Definition 2.1. Let $t = (t_0, \ldots, t_m)$. For $x \in \mathbb{R}$, $0 \le i \le m-k-1$, we define B-splines of degree k as follows:

$$\begin{cases} B_{i,0} = \begin{cases} 1, \text{ if } t_i \le x < t_{i+1} \\ 0, \text{ otherwise} \end{cases} \\ B_{i,k}(x) = w_{i,k}(x) B_{i,k-1}(x) + (1 - w_{i+1,k}(x)) B_{i+1,k-1}(x), \end{cases} \tag{2.1}$$

where

$$w_{i,k}(x) = \begin{cases} \frac{x - t_i}{t_{i+k} - t_i}, & \text{if } t_i < t_{i+k} \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

If $s(x) = \sum_{r=0}^{m-k-1} c_r B_{r,k}(x)$, then its derivatives can be found for $x \in (t_j, t_{j+k})$ from (see for more details [4, pp. 62]):

$$s^{(i)}(x) = \sum_{l=j-k+i-1}^{j} c_{l,i+1} B_{l,k-i}(x), \tag{2.3}$$

where

$$c_{l,i+1} := \begin{cases} c_l, \text{ if } i = 0 \\ (k-i) \frac{c_{l,i} - c_{l-1,i}}{t_{l+k-i} - t_l}, \text{ if } i > 0. \end{cases} \tag{2.4}$$

To evaluate $B_{j,k}^{(i)}(x)$, we take $c_r = \delta_{rj}$, for $r = 0, \ldots, m-k-1$, in (2.3) and (2.4).

2.2. Principles of the method

First we are interested to a global approach for the solution of problem (1.1) + (1.2) + (1.3). Let Δ be a partition of $[0,1]$ like

$$\Delta : 0 = x_0 < x_1 < \cdots < x_{N-1} = 1. \tag{2.5}$$

We insert the points a and b into the partition. Suppose $x_l = a$ and $x_{l+p} = b$, $0 < l < N+1$, $1 < l+p < N+1$. The multiplicity of each point inner point is k, and the multiplicity of endpoints is $k+2$. Let

$$H_i := x_{i+1} - x_i, \qquad i = 0, \ldots, N \tag{2.6}$$

be the step sizes.

We construct the following collocation points

$$\xi_{ij} := x_i + H_i \rho_j;\ i = 0, 1, \ldots, N-1,\ j = 1, 2, \ldots, k, \tag{2.7}$$

on each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, N-1$, where

$$0 < \rho_1 < \rho_2 < \cdots < \rho_k < 1 \tag{2.8}$$

are the roots of k-th Legendre polynomial (see [5] for more details). We insert the points a and b into the set of collocation point, so we obtain $n = Nk + 2$ points.

Remark 2.2. If a or b coincide with one of the previously computed collocation point, then we increment N.

One renumbers the collocation points, such that the first is $\xi_0 := x_0 + H_0 \rho_0$, and the last is $\xi_{n-1} := x_N + H_N \rho_k$, where $n = Nk + 2$. The dimension of our spline space must be $n = Nk + 2$. Using notations in section 2.1, we have $m = (N+1)k + 4$. Therefore, the partition of $[0,1]$ becomes: [23, pp. 65]:

$$\overline{\Delta} : 0 = x_0 \leq x_1 \leq \cdots \leq x_m = 1. \tag{2.9}$$

Definition 2.3. A function $v(x)$ is in the family $L(\overline{\Delta}, k, p)$ if $v(x)$ is a polynomial of degree k on each subinterval of $\overline{\Delta}$ and $v \in C^p[0,1]$. The subfamily $L'(\overline{\Delta}, k, p)$ consists of all functions in $L(\overline{\Delta}, k, p)$ which satisfy the boundary conditions (1.5) + (1.6).

Suppose a partition (2.9) of $[0,1]$ and a sequence of partitions $\overline{\Delta}_n (n = 1, 2, 3, \ldots)$ satisfying

$$\lim_{n \to \infty} h(\overline{\Delta}_n) = 0$$

are given. If we form a set of points $S_n (n = 1, 2, \ldots)$ like in (2.9), then, for a large n (see [25] for more details), there is a unique element $u_{\overline{\Delta}_n}(x)$ of $L'(\overline{\Delta}_n, k+1, 1)$ satisfying (1.4) at each point of S_n and

$$\left\| u_{\overline{\Delta}_n}(x) - y(x) \right\| \leq \delta. \tag{2.10}$$

The approximate solution $y_n(x)$ and its derivatives up to order two converge uniformly to $y(x)$ and to its derivatives of corresponding orders. Moreover the rate of convergence is bounded by

$$\left\| u_{\overline{\Delta}_n}(x)^{(k)} - y^{(k)}(x) \right\| \le \theta F_n(u''),\ k = 0, 1, \tag{2.11}$$

where θ is a constant independent of n and $F_n(u'')$ is the error of the best uniform approximation to $y''(x)$ in $L(\overline{\Delta}_n, k-1, 0)$.

We wish to find an approximate solution of the problem (1.1)+(1.2)+(1.3) in $L(\overline{\Delta}, k+1, 1)$, having the following form:

$$u_{\overline{\Delta}}(x) = \sum_{i=0}^{n-1} c_i B_{i,k+1}(x), \tag{2.12}$$

where $B_{i,k+1}(x)$ is a B-spline of order $(k+2)$ with knots $\{x_i\}_{i=0}^{m}$.

Remark 2.4. Our approximation method is inspired from ([11], chap. 2,5)

Let

$$J = \{0, \dots, n-1\} \backslash \{l, l+p\}.$$

We impose the conditions:

(c1) The approximate solution (2.12) satisfies the differential equation (1.1) at ξ_j, $j \in J$, where ξ_j are the collocation points.

(c2) The solution satisfies $u_{\overline{\Delta}}(\xi_l) = \alpha$, $u_{\overline{\Delta}}(\xi_{l+p}) = \beta$ (we recall that $a = \xi_l$, $b = \xi_{l+p}$).

The conditions **(c1)** and **(c2)** yield a nonlinear system with n equations:

$$\begin{cases} \sum\limits_{i=0}^{n-1} c_i B_{i,k+1}(a) = \alpha, & j = l, \\ \sum\limits_{i=0}^{n-1} c_i B''_{i,k+1}(\xi_j) + f\left(\xi_j, \sum\limits_{i=0}^{n-1} c_i B_{i,k+1}(\xi_j)\right) = 0, & j \in J, \\ \sum\limits_{i=0}^{n-1} c_i B_{i,k+1}(b) = \beta, & j = l+p, \end{cases} \tag{2.13}$$

with unknowns $(c_i)_{i=0}^{n-1}$. If $F = [F_0, F_1, \dots, F_{n-1}]^T$ are the functions defined by the equations of the nonlinear systems, using the quasilinearization of Newton method [4, pp. 52-55], we find the next approximation by means of

$$c^{(k+1)} = c^{(k)} - w^{(k)}, \tag{2.14}$$

where $c^{(k)}$ is the vector of unknowns obtained at the k-th step, and $w^{(k)}$ is the solution of the linear system:

$$F'(c^{(k)})w = F(c^{(k)}). \tag{2.15}$$

The Jacobian matrix $F' = (J_{ij})$ is banded and it is given by

$$J_{ij} = \begin{cases} B_{j,k+1}(a), & \text{for } i = l \\ B_{j,k+1}(b), & \text{for } i = l + p \\ B''_{j,k+1}(\xi_i) + \frac{\partial f}{\partial y}\left(\xi_i, \sum\limits_{i=1}^{n-1} c_i B_{j,k+1}(\xi_i)\right) B_{j,k+1}(\xi_i), & \text{for } i \in J. \end{cases} \tag{2.16}$$

To solve (1.1)+(1.2)+(1.3) we use the method presented in [7, pp. 670-674] and [24, pp. 771-795]. An initial approximation $u^{(0)} \in C^1[0,1]$ is required.

The successful **stopping criterion** [1] is

$$\left\|u^{(k+1)} - u^{(k)}\right\| \leq abstol + \left\|u^{(k+1)}\right\| reltol,$$

where, *abstol* and *reltol* is the absolute and the relative error tolerance, respectively, and the norm is the usual uniform convergence norm. The reliability of the error-estimation procedure being used for stopping criterion was verified in [3]. Papers on this topics exploit the almost block diagonal structure of collocation matrix and recommend an LU factorization (see [8, 3]).

3. A combined method using B-splines and Runge-Kutta methods

Our second method consists of the decomposition of the problem (1.1) + (1.2) + (1.3) into three problems:

1. A BVP on $[a, b]$ (problem (1.4)+(1.5)+(1.6));
2. Two IVPs on $[0, a]$ and $[b, 1]$.

Also we suppose that the problem (1.4)+(1.5)+(1.6) satisfies hypothesis of the Theorem 1.1, which ensures a sufficient condition to guarantee that there exists a $\sigma > 0$ such that $y(x)$ is the unique solution of problem BVP in the sphere

$$\{w : \|w - y''\| \leq \sigma\}.$$

Due to conditions in Theorems 1.1 and 1.2, the problem (1.1)+(1.2)+(1.3) has a unique solution. To solve the problem (1.4)+(1.5)+(1.6), we use the collocation method presented in Section 2. This time, we consider a partition of $[a, b]$ as follows

$$\overline{\Delta} : a = x_0 < x_1 < \cdots < x_N = b.$$

The multiplicity of a and b is $k + 2$ and the multiplicity of inner points is k. The dimension of spline space is again $Nk + 2$, and the nonlinear system is analogous to (2.13).

For the solution of the two initial value problems, we use a Runge-Kutta method of appropriate order. This needs good approximations of $y'(a)$ and $y'(b)$, which could be obtained with no additional effort during the collocation. Let $u_{\overline{\Delta}}(x)$ be the approximation computed by the combined method.

Theorem 3.1. *If u is an isolated solution of* (1.1)+ (1.2)+(1.3), *f has continuous second order partial derivatives and the initial guess is sufficiently close to u, then the combined method is convergent to u and its accuracy is $O(h^{k+1})$, where h is the norm of the partition $\overline{\Delta}$ given by (*2.9*).*

Proof. For the problem (1.4)+(1.5)+(1.6) we apply Theorem 5.147, page 257 in [4]. We conclude that Newton method, applied to Δ, converges quadratically to the restriction of u to Δ, and the accuracy for the approximation and its derivative is $O(h^{k+1})$, that is

$$|u_{\Delta}^{(j)}(x) - y^{(j)}(x)| = O(h^{k+1}), \quad x \in [a,b],\ j = 0,1.$$

We extend convergence and the accuracy to the whole interval $[0,1]$ by using the stability and the convergence of Runge-Kutta methods. A $(k+1)$-order explicit Runge-Kutta method is consistent and stable, so it is convergent, and its accuracy is $O(h^{k+1})$. Thus the final solution has the same accuracy. The stability and convergence of Runge-Kutta method are guaranteed by Theorems 5.3.1, page 285 and 5.3.2, page 288 in [13]. □

4. Some considerations on complexity

We will give a rough estimation of the complexity of our methods. We start with the first method. In the sequel, B will be the cost for B-spline evaluation and f the time for a function evaluation.

The time required to construct the collocation matrix is $C_0 = 2(Nk + 1)(k+2)B$.

To construct the Jacobian we need $Nk(k+2)(B+f)$. The construction of the right-hand side requires $(Nk+2)B + NkB + Nkf$. So, for the linear system construction, we obtain

$$W_1 = ((B+f)k^2 + (4B+3f)k)N + B$$

For a banded linear system with bandwidth w the total cost for solution, using LU with pivoting is $n(\frac{w^2}{2} + w)$ (see [12, pp. 79-80]). In our case, $n = Nk + 2$, and $w = \frac{3}{2}(k+2)$, and the cost for the solution of the linear system will be

$$W_2 = \left(\frac{21}{2}k + \frac{9}{8}k^3 + \frac{15}{2}k^2\right) N + 15k + 21 + \frac{9k^2}{4}.$$

The cost of Newton step is $W_s = W_1 + W_2$, that is,

$$W_s = \left[\frac{9k^3}{8} + \left(f + B + \frac{15}{2}\right) k^2 + \left(3f + 4B + \frac{21}{2}\right) k\right] N + 2B + 15k + 21 + \frac{9k^2}{4}.$$

The total cost is $IW_s + C_0$, where I is the number of steps required in Newton methods. Since the convergence is quadratic, if the final tolerance is ε, assuming $\delta_{i+1} = c\delta_i^2$, where δ_i is the error at the ith step, we obtain [18, pp. 295-297]

$$I = \frac{1}{\log 2} \log \frac{\log |c| + \log \varepsilon}{\log |c| + \log |\delta_0|}.$$

For the second method, the same analysis works for BVP solution part. We have an additional amount of work for Runge-Kutta method. If the number of stages is s and the number of points is p, the cost is $O(psf)$.

5. Implementation and numerical examples

We implemented the ideas from previous sections in MATLAB 2010a. Our code uses MATLAB Spline Toolbox and sparse matrices (see [26]). The function `spcol` allows us to compute easily the collocation matrix. For IVPs the solver `ode45` works fine. To avoid the error propagation, we chose for (BVP) B-splines of order 4 (degree 3) or order 5 (degree 4).

We implemented two functions: `polycollocnelin`, global B-spline collocation, and `polycalnlinRK`, the combined method (B-spline collocation + Runge-Kutta).

Consider the following examples:

1. [14] Consider the PVP

$$\begin{aligned}
&y''(x) + y^3(x) + \frac{4-(x-x^2)^3}{(x+1)^3} = 0;\ x \in (0,1)\\
&y(1/4) = 3/20;\ y(1/2) = 1/6
\end{aligned} \tag{5.1}$$

with exact solution

$$y(x) = \frac{x-x^2}{x+1}.$$

2.

$$\begin{aligned}
&y''(x) + e^{-y(x)} = 0;\ x \in [0,1]\\
&y(\pi/6) = \ln(3/2),\ y(\pi/4) = \ln((2+\sqrt{2})/2)
\end{aligned} \tag{5.2}$$

with the exact solution

$$y = \ln(\sin(x)+1).$$

We applied both methods to each example.

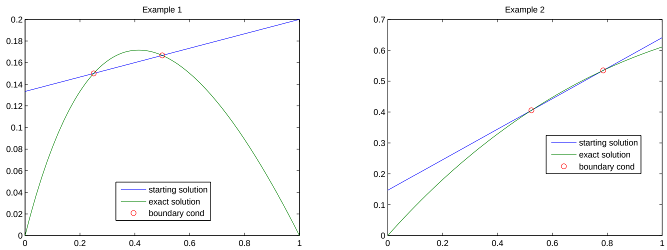
Figure 1 shows the exact solutions and the starting functions. The error plots for both methods, in semi-logarithmic scale, are given in Figure 2 for the first example and in Figure 3 for the second example, respectively.

We chose as starting function the Lagrange interpolation polynomial that takes the values α and β at a and b.

Table 1 gives the residuals $e_\Delta^{(j)}\|y^{(j)} - y_\Delta^{(j)}\|$, for $j = 0,1,2$, for the global method based on B-splines. For the residuals it holds $\|y^{(j)} - y_\Delta^{(j)}\| = O(|\Delta\|)^{k+2-j}$, for $j = 0,1,2$. To check this experimentally, we plot the residuals versus $1/\Delta$, for various values of N in a log-log scale (see Figure 4, the left column, for Example (5.1) and Figure 4, the right column, for Example (5.2)).

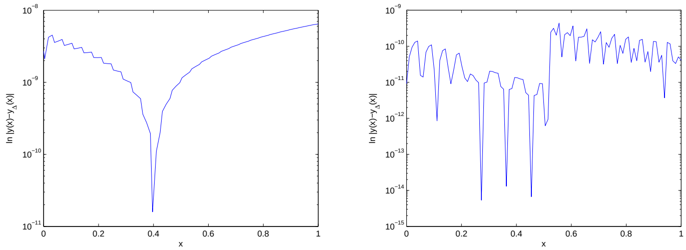
In order to compare the costs (run-times) experimentally we used MATLAB functions `tic` and `toc`. The results are given in Table 2.

The time for combined method is a bit larger.



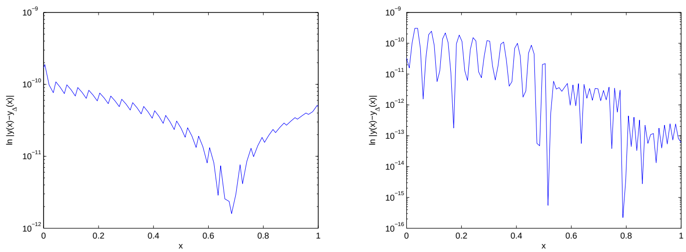
(a) Example 1 (b) Example 2

FIGURE 1. Exact solution and starting approximation



(a) Global collocation (b) Combined method

FIGURE 2. Error plot for example (5.1)



(a) Global collocation (b) Combined method

FIGURE 3. Error plot for example (5.2)

The next numerical experiment compares the running time of our methods to the running time of a pseudospectral method (see [19] for implementation details of the latter). As example, we consider a variant of Bratu's

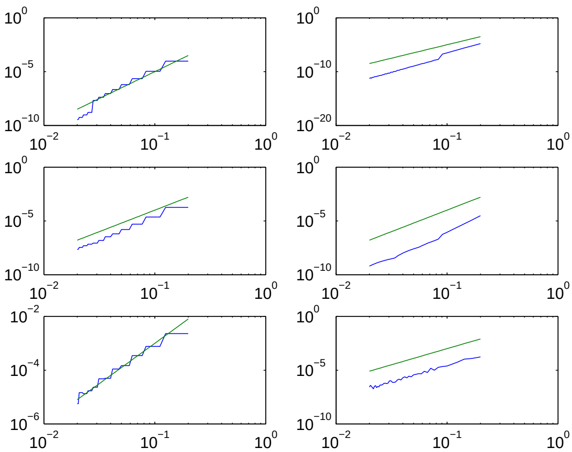


FIGURE 4. Order estimation for Example (5.1) (left) and Example (5.2) (right): $e_\Delta^{(0)}$ - up, $e_\Delta^{(1)}$ - middle, and $e_\Delta^{(2)}$ - bottom

	Example (5.1)			Example (5.2)		
N	$\|y-y_\Delta\|$	$\|y'-y'_\Delta\|$	$\|y''-y''_\Delta\|$	$\|y-y_\Delta\|$	$\|y'-y'_\Delta\|$	$\|y''-y''_\Delta\|$
5	6e-05	0.000123	0.00229	1.04e-05	2.15e-05	0.000172
6	6e-05	0.000123	0.00229	3.72e-06	8.47e-06	0.000124
7	6e-05	0.000123	0.00229	1.59e-06	3.96e-06	0.000106
8	6e-05	0.000123	0.00229	7.37e-07	2.02e-06	5.65e-05
9	6.9e-06	1.63e-05	0.000769	3.84e-07	1.16e-06	3.61e-05
10	6.9e-06	1.63e-05	0.000769	2.04e-07	6.81e-07	2.42e-05
11	6.9e-06	1.63e-05	0.000769	1.21e-07	4.46e-07	2.14e-05
12	6.9e-06	1.63e-05	0.000769	1.82e-08	2.02e-07	1.81e-05
13	1.4e-06	3.64e-06	0.000346	1.3e-08	1.44e-07	9.77e-06
14	1.4e-06	3.64e-06	0.000346	7.54e-09	1.09e-07	1.51e-05
15	1.4e-06	3.64e-06	0.000346	5.54e-09	8.32e-08	6.08e-06
16	1.4e-06	3.64e-06	0.000346	3.54e-09	6.22e-08	7.92e-06
17	3.99e-07	1.22e-06	0.000148	2.81e-09	4.65e-08	4.65e-06
18	3.99e-07	1.22e-06	0.000148	1.89e-09	3.51e-08	4.76e-06
19	3.99e-07	1.22e-06	0.000148	1.43e-09	2.98e-08	4.14e-06
20	3.99e-07	1.22e-06	0.000148	1.02e-09	2.5e-08	3.7e-06

TABLE 1. Error table for Examples (5.1) and (5.2)

problem [4, page 491] for $\lambda = 1$

$$
\begin{aligned}
&y'' + e^y = 0, \qquad x \in (0,1)\\
&y(0.2) = y(0.8) = 0.08918993462883.
\end{aligned}
$$

	Method 1	Method 2
First example	0.017501	0.022751
Second example	0.016387	0.021535

TABLE 2. Run times

ε	PseudoS	BS+RK	Global BS
10^{-5}	0.054	0.035	0.021
10^{-6}	0.077	0.043	0.023
10^{-7}	0.049	0.025	0.024
10^{-8}	0.055	0.031	0.031
10^{-9}	0.054	0.036	0.030
10^{-10}	0.058	0.026	0.028

TABLE 3. Running times for Bratu's problem

We chose 128 collocation points, and as starting function $y_0(t) = \frac{39}{70}x(x-1)$. The running times for various tolerances are given in Table 3.

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