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APPROXIMATION AND SHAPE PRESERVING PROPERTIES OF THE NONLINEAR BASKAKOV OPERATOR OF MAX-PRODUCT KIND

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Abstract. Starting from the study of the *Shepard nonlinear operator of max-prod type* in [2], [3], in the recent monograph [5], Open Problem 5.5.4, pp. 324-326, the *Baskakov max-prod type operator* is introduced and the question of the approximation order by this operator is raised. The aim of this note is to obtain for the discussed operator an upper pointwise estimate of the approximation error of the form $C\omega_1(f;\sqrt{\frac{x(1+x)}{n}})$ (with the explicit constant $C = 12$) and to prove by a counterexample that in some sense, for arbitrary f this type of order of approximation with respect to $\omega_1(f;)$ cannot be improved. However, for some subclasses of functions including for example the nondecreasing concave functions, the essentially better order of approximation $\omega_1(f;\frac{x+1}{n})$ is obtained. Finally, some shape preserving properties are proved.

1. Introduction

Starting from the study of the *Shepard nonlinear operator of max-prod type* in [2], [3], by the Open Problem 5.5.4, pp. 324-326 in the recent monograph [5], the following *nonlinear Baskakov operator of max-prod type* is introduced (here $\bigvee$ means maximum)

$$V_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{\infty} b_{n,k}(x)},$$

where $b_{n,k}(x) = \binom{n+k-1}{k} x^k/(1+x)^{n+k}$.

The aim of this note is to obtain for the discussed operator an upper pointwise estimate of the approximation error of the form $C\omega_1(f;\sqrt{\frac{x(1+x)}{n}})$ (with the explicit

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constant $C = 12$) Also, one proves by a counterexample that in some sense, in general this type of order of approximation with respect to $\omega_1(f;\cdot)$ cannot be improved. However, for some subclasses of functions, including for example the bounded, nondecreasing concave functions, the essentially better order $\omega_1(f;(x+1)/n)$ is obtained. This allows us to put in evidence large classes of functions (e.g. bounded, nondecreasing concave polygonal lines on $[0,\infty)$) for which the order of approximation given by the max-product Baskakov operator, is essentially better than the order given by the linear Baskakov operator. Finally, some shape preserving properties are presented.

Section 2 presents some general results on nonlinear operators, in Section 3 we prove several auxiliary lemmas, Section 4 contains the approximation results, while in Section 5 we present some shape preserving properties.

2. Preliminaries

For the proof of the main result we need some general considerations on the so-called nonlinear operators of max-prod kind. Over the set of positive reals, $\mathbb{R}_+$, we consider the operations $\vee$ (maximum) and $\cdot$, product. Then $(\mathbb{R}_+,\vee,\cdot)$ has a semiring structure and we call it as Max-Product algebra.

Let $I\subset\mathbb{R}$ be a bounded or unbounded interval, and

$$CB_+(I)=\{f:I\to\mathbb{R}_+; f \text{ continuous and bounded on } I\}.$$

The general form of $L_n : CB_+(I)\to CB_+(I)$, (called here a discrete max-product type approximation operator) studied in the paper will be

$$L_n(f)(x)=\bigvee_{i=0}^{n}K_{n,i}(x)\cdot f(x_{n,i}),$$

or

$$L_n(f)(x)=\bigvee_{i=0}^{\infty}K_{n,i}(x)\cdot f(x_{n,i}),$$

where $n\in\mathbb{N}$, $f\in CB_+(I)$, $K_{n,i}\in CB_+(I)$ and $x_i\in I$, for all i. These operators are nonlinear, positive operators and moreover they satisfy a pseudo-linearity condition of the form

$$L_n(\alpha\cdot f\vee\beta\cdot g)(x)=\alpha\cdot L_n(f)(x)\vee\beta\cdot L_n(g)(x), \forall\alpha,\beta\in\mathbb{R}_+, f,g\in CB_+(I).$$

In this section we present some general results on these kinds of operators which will be useful later in the study of the Baskakov max-product kind operator considered in Introduction.

Lemma 2.1. ([1]) *Let $I\subset\mathbb{R}$ be a bounded or unbounded interval,*

$$CB_+(I)=\{f:I\to\mathbb{R}_+; f \text{ continuous and bounded on } I\},$$

and $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ *be a sequence of operators satisfying the following properties :*

(i) If $f, g \in CB_+(I)$ *satisfy* $f \le g$ *then* $L_n(f) \le L_n(g)$ *for all* $n \in N$ *;*

(ii) $L_n(f+g) \le L_n(f) + L_n(g)$ *for all* $f, g \in CB_+(I)$.

Then for all $f, g \in CB_+(I)$, $n \in N$ *and* $x \in I$ *we have*

$$|L_n(f)(x) - L_n(g)(x)| \le L_n(|f-g|)(x).$$

Proof. Since is very simple, we reproduce here the proof in [1]. Let $f, g \in CB_+(I)$. We have $f = f - g + g \le |f-g| + g$, which by the conditions $(i)-(ii)$ successively implies $L_n(f)(x) \le L_n(|f-g|)(x) + L_n(g)(x)$, that is $L_n(f)(x) - L_n(g)(x) \le L_n(|f-g|)(x)$.

Writing now $g = g - f + f \le |f-g| + f$ and applying the above reasonings, it follows $L_n(g)(x) - L_n(f)(x) \le L_n(|f-g|)(x)$, which combined with the above inequality gives $|L_n(f)(x) - L_n(g)(x)| \le L_n(|f-g|)(x)$. □

Remarks. 1) It is easy to see that the Baskakov max-product operator satisfy the conditions (i) and (ii) in Lemma 2.1. In fact, instead of (i) it satisfies the stronger condition

$$L_n(f \vee g)(x) = L_n(f)(x) \vee L_n(g)(x),\ \ f, g \in CB_+(I).$$

Indeed, taking in the above equality $f \le g$, $f, g \in CB_+(I)$, it easily follows $L_n(f)(x) \le L_n(g)(x)$.

2) In addition, it is immediate that the Baskakov max-product operator is positive homogeneous, that is $L_n(\lambda f) = \lambda L_n(f)$ for all $\lambda \ge 0$.

Corollary 2.2. ([1]) *Let* $L_n : CB_+(I) \to CB_+(I)$, $n \in N$ *be a sequence of operators satisfying the conditions (i)-(ii) in Lemma 2.1 and in addition being positive homogenous. Then for all* $f \in CB_+(I)$, $n \in N$ *and* $x \in I$ *we have*

$$|f(x) - L_n(f)(x)| \le \left[\frac{1}{\delta} L_n(\varphi_x)(x) + L_n(e_0)(x)\right] \omega_1(f;\delta)_I + f(x) \cdot |L_n(e_0)(x) - 1|,$$

where $\delta > 0$, $e_0(t) = 1$ *for all* $t \in I$, $\varphi_x(t) = |t - x|$ *for all* $t \in I$, $x \in I$, $\omega_1(f;\delta)_I = \max\{|f(x) - f(y)|; x, y \in I, |x-y| \le \delta\}$ *and if* I *is unbounded then we suppose that there exists* $L_n(\varphi_x)(x) \in \mathbb{R}_+ \bigcup \{+\infty\}$, *for any* $x \in I$, $n \in \mathbb{N}$.

Proof. The proof is identical with that for positive linear operators and because of its simplicity we reproduce it below. Indeed, from the identity

$$L_n(f)(x) - f(x) = [L_n(f)(x) - f(x) \cdot L_n(e_0)(x)] + f(x)[L_n(e_0)(x) - 1],$$

it follows (by the positive homogeneity and by Lemma 2.1)

$$|f(x) - L_n(f)(x)| \le |L_n(f(x))(x) - L_n(f(t))(x)| + |f(x)| \cdot |L_n(e_0)(x) - 1| \le$$
$$L_n(|f(t) - f(x)|)(x) + |f(x)| \cdot |L_n(e_0)(x) - 1|.$$

Now, since for all $t, x \in I$ we have

$$|f(t) - f(x)| \leq \omega_1(f; |t-x|)_I \leq \left[\frac{1}{\delta}|t-x| + 1\right]\omega_1(f;\delta)_I,$$

replacing above we immediately obtain the estimate in the statement. □

An immediate consequence of Corollary 2.2 is the following.

Corollary 2.3. ([1]) *Suppose that in addition to the conditions in Corollary 2.2, the sequence $(L_n)_n$ satisfies $L_n(e_0) = e_0$, for all $n \in N$. Then for all $f \in CB_+(I)$, $n \in N$ and $x \in I$ we have*

$$|f(x) - L_n(f)(x)| \leq \left[1 + \frac{1}{\delta}L_n(\varphi_x)(x)\right]\omega_1(f;\delta)_I.$$

The nonlinear max-product Baskakov operator satisfies for all $n \in \mathbb{N}$, $n \geq 2$ all the hypothesis in the Lemma 2.1, Corollaries 2.2 and 2.3 as can be seen from the following considerations.

Lemma 2.4. *Let $n \in \mathbb{N}$, $n \geq 2$. We have*

$$\bigvee_{k=0}^{\infty} b_{n,k}(x) = b_{n,j}(x), \text{ for all } x \in \left[\frac{j}{n-1}, \frac{j+1}{n-1}\right], j = 0, 1, 2, ...$$

Proof. First we show that for fixed $n \in \mathbb{N}, n \geq 2$ and $0 \leq k < k+1$ we have

$$0 \leq b_{n,k+1}(x) \leq b_{n,k}(x), \text{ if and only if } x \in [0, (k+1)/(n-1)].$$

Indeed, the inequality one reduces to

$$0 \leq \binom{n+k}{k+1}\frac{x^{k+1}}{(1+x)^{n+k+1}} \leq \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}},$$

which after simple calculus is obviously equivalent to

$$0 \leq x \leq \frac{k+1}{n-1}.$$

By taking $k = 0, 1, 2, ..$ in the inequality just proved above, we get

$$b_{n,1}(x) \leq b_{n,0}(x), \text{ if and only if } x \in [0, 1/(n-1)],$$

$$b_{n,2}(x) \leq b_{n,1}(x), \text{ if and only if } x \in [0, 2/(n-1)],$$

$$b_{n,3}(x) \leq b_{n,2}(x), \text{ if and only if } x \in [0, 3/(n-1)],$$

so on,

$$b_{n,k+1}(x) \leq b_{n,k}(x), \text{ if and only if } x \in [0, (k+1)/(n-1)],$$

and so on.

From all these inequalities, reasoning by recurrence we easily obtain :

$$\text{if } x \in [0, 1/(n-1)] \text{ then } b_{n,k}(x) \leq b_{n,0}(x), \text{ for all } k = 0, 1, 2, ...$$

$$\text{if } x \in [1/(n-1), 2/(n-1)] \text{ then } b_{n,k}(x) \leq b_{n,1}(x), \text{ for all } k = 0, 1, 2, ...$$

$$\text{if } x \in [2/(n-1), 3/(n-1)] \text{ then } b_{n,k}(x) \leq b_{n,2}(x), \text{ for all } k = 0, 1, 2, ...$$

and so on, in general

$$\text{if } x \in [j/(n-1), (j+1)/(n-1)] \text{ then } b_{n,k}(x) \leq b_{n,j}(x), \text{ for all } k = 0, 1, 2, ...$$

which proves the lemma. □

In what follows we need some notations.

For each $n \in \mathbb{N}$, $n \geq 2$, $k, j \in \{0, 1, 2, ..., \}$, and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$, $x > 0$ let us denote

$$m_{k,n,j}(x) = \frac{b_{n,k}(x)}{b_{n,j}(x)} = \frac{\binom{n+k-1}{k}}{\binom{n+j-1}{j}} \left(\frac{x}{1+x}\right)^{k-j}.$$

and for $x = 0$ let us denote $m_{0,n,0}(x) = 1$ and $m_{k,n,0}(x) = 0$ for all $k \in \{1, 2, ..., \}$.

Also, for any $n \in \mathbb{N}$, $n \geq 2$, $k \in \{0, 1, .., \}$ and $j \in \{0, 1, .., \}$, let us define the functions $f_{k,n,j} : [\frac{j}{n-1}, \frac{j+1}{n-1}] \to \mathbb{R}$,

$$f_{k,n,j}(x) = m_{k,n,j}(x) f\left(\frac{k}{n}\right) = \frac{\binom{n+k-1}{k}}{\binom{n+j-1}{j}} \left(\frac{x}{1+x}\right)^{k-j} f\left(\frac{k}{n}\right).$$

From Lemma 2.4, it follows that for each $j \in \{0, 1, ..., \}$ and for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ we can write

$$V_n^{(M)}(f)(x) = \bigvee_{k=0}^{\infty} f_{k,n,j}(x).$$

Lemma 2.5. *Let $n \in \mathbb{N}$, $n \geq 2$. For all $k, j \in \{0, 1, 2, ...\}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ we have*

$$m_{k,n,j}(x) \leq 1.$$

Proof. Let $j \in \{0, 1, ..., \}$ and let $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. By Lemma 2.4, it immediately follows that

$$m_{k,n,j}(x) \leq m_{j,n,j}(x).$$

Since $m_{j,n,j}(x) = 1$, the conclusion of the lemma is immediate. □

Lemma 2.6. *For any arbitrary bounded function $f : [0, \infty) \to \mathbb{R}_+$, $V_n^{(M)}(f)$ is positive, bounded, continuous and satisfies $V_n^{(M)}(f)(0) = f(0)$, for all $n \in \mathbb{N}$, $n \geq 3$.*

Proof. The positivity of $V_n^{(M)}(f)$ is immediate. Also, taking into account that $b_{n,0}(0) = 1$ and $b_{n,k}(0) = 0$ for all $k \in \{1, 2, ..., \}$ we immediately obtain that $V_n^{(M)}(f)(0) = f(0)$.

If f is bounded then let $M \in \mathbb{R}_+$ be such that $f(x) \leq M$ for all $x \in [0, \infty)$. Let $x \in [0, \infty)$ and let $j \in \{0, 1, ...\}$ be such that $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Then

$$V_n^{(M)}(f)(x) = \bigvee_{k=0}^{\infty} f_{k,n,j}(x) = \bigvee_{k=0}^{\infty} m_{k,n,j}(x) f\left(\frac{k}{n}\right).$$

Since by Lemma 2.5. we have $m_{k,n,j}(x) \leq 1$ for all $k \in \{0, 1, ..., \}$ and since $f\left(\frac{k}{n}\right) \leq M$ for all $k \in \{0, 1, ..., \}$, it is immediate that $V_n^{(M)}(f)(x) \leq M$.

With respect to continuity, it suffices to prove that on each subinterval of the form $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, with $j \in \{0, 1, ...\}$, $V_n^{(M)}(f)$ is continuous. For this purpose, for $j \in \{0, 1, ...\}$ fixed and for any $l \in \mathbb{N}$ let us define the function $g_{l,j} : [\frac{j}{n-1}, \frac{j+1}{n-1}] \to \mathbb{R}_+$, $g_{l,j}(x) = \bigvee_{k=0}^{l} f_{k,n,j}(x)$. It is clear that for each $l \in \mathbb{N}$ the function $g_{l,j}$ is continuous on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, as a maximum of finite number of continuous functions. Since, for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ we have

$$0 \leq V_n^{(M)}(f)(x) = \max\left\{\bigvee_{k=0}^{l} f_{k,n,j}(x), \bigvee_{k=l+1}^{\infty} f_{k,n,j}(x)\right\}$$

$$\leq \bigvee_{k=0}^{l} f_{k,n,j}(x) + \bigvee_{k=l+1}^{\infty} f_{k,n,j}(x),$$

it follows that for all $l \in \mathbb{N}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ we have

$$0 \leq V_n^{(M)}(f)(x) - g_{l,j}(x) \leq \bigvee_{k=l+1}^{\infty} f_{k,n,j}(x) = \bigvee_{k=l+1}^{\infty} m_{k,n,j}(x) f\left(\frac{k}{n}\right)$$

$$\leq M \bigvee_{k=l+1}^{\infty} m_{k,n,j}(x).$$

For $l \geq j$, by the proof Lemma 2.4 it follows that

$$m_{l,n,j}(x) \geq m_{l+1,n,j}(x) \geq m_{l+2,n,j}(x) \geq ....$$

Also, for $l \geq j$ it is easy to prove that $m_{l,n,j}(x) \leq m_{l,n,j}(\frac{j+1}{n-1})$ for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. From all these reasonings it follows that

$$0 \leq V_n^{(M)}(f)(x) - g_{l,j}(x) \leq M m_{l+1,n,j}(\frac{j+1}{n-1})$$

for all $l \geq j$. Let us consider the sequence $(a_l)_{l \geq j}$, $a_l = M m_{l+1,n,j}(\frac{j+1}{n})$. By simple calculus we get $\lim_{l \to \infty} \frac{a_{l+1}}{a_l} = \lim_{l \to \infty} \left(\frac{(n+l+1)(j+1)}{(l+2)(n+j)}\right) = \frac{j+1}{n+j} < 1$, which immediately implies that $\lim_{l \to \infty} a_l = 0$. This implies that $V_n^{(M)}(f)$ is the uniform limit of a sequence of continuous functions on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, $g_{l,j}$, $l \in \mathbb{N}$, which implies the continuity of $V_n^{(M)}(f)$ on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$. □

Remark. From Lemmas 2.4-2.6, it is clear that $V_n^{(M)}(f)$ satisfies all the conditions in Lemma 2.1, Corollary 2.2 and Corollary 2.3 for $I = [0, \infty)$.

3. **Auxiliary Results**

Remark. Note since by Lemma 2.6 we have $V_n^{(M)}(f)(0) = f(0)$ for all $n \geq 3$, notice that in the notations, proofs and statements of the all approximation results, that is in Lemmas 3.1, 3.2, Theorem 4.1, Lemma 4.2, Corollaries 4.4, 4.5, in fact we always may suppose that $x > 0$.

For each $n \in \mathbb{N}$, $n \geq 3$, $k, j \in \{0, 1, 2, ...,\}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$, let us denote

$$M_{k,n,j}(x) = m_{k,n,j}(x) \left| \frac{k}{n} - x \right|.$$

It is clear that if $k \geq \frac{n}{n-1}(j+1)$ then

$$M_{k,n,j}(x) = m_{k,n,j}(x)(\frac{k}{n} - x)$$

and if $k \leq \frac{n}{n-1} j$ then

$$M_{k,n,j}(x) = m_{k,n,j}(x)(x - \frac{k}{n}).$$

Also, for each $n \in \mathbb{N}$, $n \geq 3$, $k, j \in \mathbb{N}$, $k \geq \frac{n}{n-1}(j+1)$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ let us denote

$$\overline{M}_{k,n,j}(x) = m_{k,n,j}(x)(\frac{k}{n-1} - x)$$

and for each $n \in \mathbb{N}$, $n \geq 3$, $k, j \in \mathbb{N}$, $k \leq \frac{n}{n+1} j$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ let us denote

$$\underline{M}_{k,n,j}(x) = m_{k,n,j}(x)(x - \frac{k}{n-1}).$$

Lemma 3.1. *Let $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ and $n \in \mathbb{N}$, $n \geq 3$.*

(i) For all $k, j \in \{0, 1, 2, ...,\}$ with $k \geq \frac{n}{n-1}(j+1)$ we have

$$M_{k,n,j}(x) \leq \overline{M}_{k,n,j}(x).$$

(ii) For all $k, j \in \mathbb{N}$, $k \geq \frac{n}{n-2}(j+1)$ we have

$$\overline{M}_{k,n,j}(x) \leq 2M_{k,n,j}(x).$$

(iii) For all $k, j \in \mathbb{N}$, $k \leq \frac{n}{n+1} j$ we have

$$\underline{M}_{k,n,j}(x) \leq M_{k,n,j}(x) \leq 2\underline{M}_{k,n,j}(x).$$

Proof. (i) The inequality $M_{k,n,j}(x) \leq \overline{M}_{k,n,j}(x)$ is immediate.

(ii) Since the function $h(x) = \frac{\frac{k}{n-1}-x}{\frac{k}{n}-x}$ is nondecreasing on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$ we get

$$\frac{\overline{M}_{k,n,j}(x)}{M_{k,n,j}(x)} = \frac{\frac{k}{n-1} - x}{\frac{k}{n} - x} \leq \frac{\frac{k}{n-1} - \frac{j+1}{n-1}}{\frac{k}{n} - \frac{j+1}{n-1}} = \frac{n(k-j-1)}{n(k-j-1)-k}.$$

We have

$$\begin{aligned} \frac{n(k-j-1)}{n(k-j-1)-k} &\leq 2 \Leftrightarrow n(k-j-1) \leq 2n(k-j-1) - 2k \\ &\Leftrightarrow 2k \leq n(k-j-1) \Leftrightarrow n(j+1) \leq k(n-2) \\ &\Leftrightarrow k \geq \frac{n}{n-2}(j+1). \end{aligned}$$

which proves (ii).

(iii) The inequality $\underline{M}_{k,n,j}(x) \leq M_{k,n,j}(x)$ is immediate.

On the other hand, tacking account of the fact that the function $h(x) = \frac{x-\frac{k}{n}}{x-\frac{k}{n-1}}$ is nonincreasing on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$ we get

$$\frac{M_{k,n,j}(x)}{\underline{M}_{k,n,j}(x)} = \frac{x - \frac{k}{n}}{x - \frac{k}{n-1}} \leq \frac{\frac{j}{n-1} - \frac{k}{n}}{\frac{j}{n-1} - \frac{k}{n-1}} = \frac{n(j-k)+k}{n(j-k)}.$$

We have

$$\begin{aligned} \frac{n(j-k)+k}{n(j-k)} &\leq 2 \Leftrightarrow n(j-k)+k \leq 2n(j-k) \\ &\Leftrightarrow k \leq n(j-k) \Leftrightarrow k(n+1) \leq nj \Leftrightarrow k \leq \frac{n}{n+1}j. \end{aligned}$$

which proves (iii). □

Lemma 3.2. *Let $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ and $n \in \mathbb{N}$, $n \geq 3$.*

(i) If $j \in \{0, 1, 2, ...\}$ is such that $k \geq \frac{n}{n-1}(j+1)$ and

$$n[(k-j)^2 - (k+1)] + kj - j^2 - k^2 - j \geq 0,$$

then $\overline{M}_{k,n,j}(x) \geq \overline{M}_{k+1,n,j}(x)$.

(ii) If $k \in \{1, 2, ...j\}$ is such that $k \leq \frac{n}{n+1}j$ and

$$n[(k-j)^2 - k] + kj - j^2 - k^2 + k \geq 0,$$

then $\underline{M}_{k,n,j}(x) \geq \underline{M}_{k-1,n,j}(x)$.

Proof. (i) We observe that

$$\frac{\overline{M}_{k,n,j}(x)}{\overline{M}_{k+1,n,j}(x)} = \frac{k+1}{n+k} \cdot \frac{x+1}{x} \cdot \frac{\frac{k}{n-1} - x}{\frac{k+1}{n-1} - x}.$$

Since the function $g(x) = \frac{x+1}{x} \cdot \frac{\frac{k}{n-1} - x}{\frac{k+1}{n-1} - x}$ clearly is nonincreasing, it follows that $g(x) \geq g(\frac{j+1}{n-1}) = \frac{n+j}{j+1} \cdot \frac{k-j-1}{k-j}$ for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$.

Then

$$\frac{\overline{M}_{k,n,j}(x)}{\overline{M}_{k+1,n,j}(x)} \geq \frac{k+1}{n+k} \cdot \frac{n+j}{j+1} \cdot \frac{k-j-1}{k-j}.$$

Through simple calculus we obtain

$$\begin{aligned} & (k+1)(n+j)(k-j-1) - (n+k)(j+1)(k-j) \\ = \quad & n[(k-j)^2 - (k+1)] + kj - j^2 - k^2 - j \end{aligned}$$

which proves (i).

(ii) We observe that

$$\frac{\underline{M}_{k,n,j}(x)}{\underline{M}_{k-1,n,j}(x)} = \frac{n+k-1}{k} \cdot \frac{x}{1+x} \cdot \frac{x - \frac{k}{n-1}}{x - \frac{k-1}{n}}.$$

Since the function $h(x) = \frac{x}{1+x} \cdot \frac{x - \frac{k}{n-1}}{x - \frac{k-1}{n-1}}$ is nondecreasing, it follows that $h(x) \geq h(\frac{j}{n-1}) = \frac{j}{n+j-1} \cdot \frac{j-k}{j-k+1}$ for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$.

Then

$$\frac{\underline{M}_{k,n,j}(x)}{\underline{M}_{k-1,n,j}(x)} \geq \frac{n+k-1}{k} \cdot \frac{j}{n+j-1} \cdot \frac{j-k}{j-k+1}.$$

Through simple calculus we obtain

$$\begin{aligned} & j(n+k-1)(j-k) - k(n+j-1)(j-k+1) \\ = \quad & n[(j-k)^2 - k] + kj - j^2 - k^2 + k \end{aligned}$$

which proves (ii) and the lemma. □

4. **Approximation Results**

If $V_n^{(M)}(f)(x)$ represents the *Baskakov operator of max-product kind* defined in Introduction, then the first main result of this section is the following.

Theorem 4.1. *Let $f : [0, \infty) \to \mathbb{R}_+$ be bounded and continuous on $[0, \infty)$. Then we have the estimate*

$$|V_n^{(M)}(f)(x) - f(x)| \leq 12\omega_1 \left(f, \sqrt{\frac{x(x+1)}{n-1}}\right), n \in \mathbb{N}, n \geq 4, x \in [0, \infty),$$

where

$$\omega_1(f, \delta) = \sup\{|f(x) - f(y)|; x, y \in [0, \infty), |x - y| \leq \delta\}.$$

Proof. It is easy to check that the max-product Baskakov operator fulfills the conditions in Corollary 2.3 and we have

$$|V_n^{(M)}(f)(x) - f(x)| \leq \left(1 + \frac{1}{\delta_n} V_n^{(M)}(\varphi_x)(x)\right) \omega_1(f, \delta_n), \tag{4.1}$$

where $\varphi_x(t) = |t - x|$. So, it is enough to estimate

$$E_n(x) := V_n^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k=0}^{\infty} b_{n,k}(x) \left|\frac{k}{n} - x\right|}{\bigvee_{k=0}^{\infty} b_{n,k}(x)}, x \in [0, \infty).$$

Let $x \in [j/n - 1, (j+1)/n - 1]$ where $j \in \{0, 1, ..., \}$ is fixed, arbitrary. By Lemma 2.4 we easily obtain

$$E_n(x) = \max_{k=0,1,...,} \{M_{k,n,j}(x)\}, x \in [j/n - 1, (j+1)/n - 1].$$

In all what follows we may suppose that $j \in \{1, 2, ..., \}$, because for $j = 0$ we get $E_n(x) < 5\sqrt{\frac{x(x+1)}{n-1}}$, for all $x \in [0, 1/n - 1]$. Indeed, in this case we have

$$M_{k,n,0}(x) = \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k \left|\frac{k}{n} - x\right|,$$

which for $k = 0$ gives

$$M_{k,n,0}(x) = x = \sqrt{x} \cdot \sqrt{x} \leq \sqrt{x} \cdot \frac{1}{\sqrt{n-1}} \leq \sqrt{\frac{x(x+1)}{n-1}}.$$

Also, for $k = 1$ we have $x \leq \frac{2}{n}$ which implies $\left|\frac{1}{n} - x\right| \leq \frac{1}{n}$ and further one

$$M_{1,n,0}(x) \leq \binom{n}{1} \left(\frac{x}{1+x}\right) \cdot \frac{1}{n} = \frac{x}{1+x} \leq x \leq \sqrt{\frac{x(x+1)}{n-1}}.$$

Suppose now that $k \geq 2$. We observe that in this case all the hypothesis of the Lemma 3.1 (i) are fulfilled, therefore in this case we have $M_{k,n,0}(x) \leq \overline{M}_{k,n,0}(x)$. Also by Lemma 3.2 (i), for $j = 0$ it follows that $\overline{M}_{k,n,0}(x) \geq \overline{M}_{k+1,n,0}(x)$ for every $k \geq 2$ such that $(n-1)k^2 - nk - n \geq 0$. Because the function $f(x) = (n-1)x^2 - nx - n$, $x \geq 1$ is nondecreasing and because $f(\sqrt{n}) \geq 0$, it follows that $\overline{M}_{k,n,0}(x) \geq \overline{M}_{k+1,n,0}(x)$ for every $k \in \mathbb{N}$, $k \geq \sqrt{n}$. Let us denote $A = \{k \in \mathbb{N}, 2 \leq k \leq \sqrt{n} + 1\}$ and let $k \in A$.

We have by Lemma 2.5

$$
\begin{aligned}
& \overline{M}_{k,n,0}(x) = \\
= \quad & \binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k\left(\frac{k}{n-1}-x\right) \\
\leq \quad & \binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k\frac{k}{n-1} \leq \binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k\cdot\frac{3k}{2n} \\
= \quad & \frac{(n+k-1)!}{k!(n-1)!}\cdot\left(\frac{x}{1+x}\right)^k\cdot\frac{3k}{2n} = \binom{n+k-1}{k-1}\left(\frac{x}{1+x}\right)^{k-1}\left(\frac{3x}{2(1+x)}\right) \\
= \quad & \binom{n+k-1}{k-1}\left(\frac{1/n}{1+1/n}\right)^{k-1}\left(\frac{x}{1+x}\cdot\frac{1+1/n}{1/n}\right)^{k-1}\cdot\left(\frac{3x}{2(1+x)}\right) \\
= \quad & \binom{n+k-1}{k-1}\left(\frac{1/n}{1+1/n}\right)^{k-1}\left(\frac{(n+1)x}{1+x}\right)^{k-1}\cdot\left(\frac{3x}{2(1+x)}\right) \\
= \quad & m_{k-1,n+1,0}\left(\frac{1}{n}\right)\left(\frac{(n+1)x}{1+x}\right)^{k-1}\cdot\left(\frac{3x}{2(1+x)}\right) \\
\leq \quad & \left(\frac{(n+1)x}{1+x}\right)^{k-1}\cdot\left(\frac{3x}{2(1+x)}\right).
\end{aligned}
$$

Since the function $g(x) = \left(\frac{(n+1)x}{1+x}\right)^{k-1}$ is nondecreasing on the interval $[0, \frac{1}{n-1}]$, it follows that

$$
g(x) \leq g\left(\frac{1}{n-1}\right) = \left(\frac{n+1}{n}\right)^{k-1}
$$

for all $x \in [0, \frac{1}{n-1}]$. Then

$$
\begin{aligned}
\overline{M}_{k,n,0}(x) \quad \leq \quad & \frac{3}{2}\cdot\left(\frac{n+1}{n}\right)^{k-1}\cdot\frac{x}{1+x} \leq \frac{3}{2}\cdot\left(\frac{n+1}{n}\right)^{\sqrt{n}}\cdot\frac{x}{1+x} \\
< \quad & \frac{3}{2}\cdot\left(\frac{n+1}{n}\right)^{n}\cdot\frac{x}{1+x} < \frac{3e}{2}\cdot x \leq \frac{3e}{2}\cdot\sqrt{\frac{x(x+1)}{n-1}} < 5\sqrt{\frac{x(x+1)}{n-1}}.
\end{aligned}
$$

Based on the above results, we obtain

$$
\begin{aligned}
& E_n(x) = \\
= \quad & \max_{k=0,1,\ldots,}\{M_{k,n,0}(x)\} \leq \max\{M_{0,,n,0}(x), M_{1,n,0}(x), \max_{k=2,3,\ldots,}\{\overline{M}_{k,n,0}(x)\}\} \\
= \quad & \max\{M_{0,,n,0}(x), M_{1,n,0}(x), \max_{k\in A}\{\overline{M}_{k,n,0}(x)\}\} < 5\sqrt{\frac{x(x+1)}{n-1}}.
\end{aligned}
$$

So it remains to obtain an upper estimate for each $M_{k,n,j}(x)$ when $j = 1, 2, ...,$ is fixed, $x \in [j/(n-1), (j+1)/(n-1)]$ and $k = 0, 1, ...,$. In fact we will prove that

$$M_{k,n,j}(x) < 6\sqrt{\frac{x(x+1)}{n}}, \text{ for all } x \in [j/(n-1), (j+1)/(n-1)], k = 0, 1, ..., \quad (4.2)$$

which immediately will imply that

$$E_n(x) \leq 6\sqrt{\frac{x(x+1)}{n}}, \text{ for all } x \in [0, \infty), n \in \mathbb{N},$$

and taking $\delta_n = 6\sqrt{\frac{x(x+1)}{n}}$ in (4.1), we immediately obtain the estimate in the statement.

In order to prove (4.2) we distinguish the following cases:
1) $\frac{n}{n+1} \cdot j \leq k \leq \frac{n}{n-1} \cdot (j+1)$; 2) $k > \frac{n}{n-1} \cdot (j+1)$ and 3) $k < \frac{n}{n+1} \cdot j$.
Case 1) We have

$$\frac{k}{n} - x \leq \frac{\frac{n}{n-1} \cdot (j+1)}{n} - \frac{j}{n-1} = \frac{j+1}{n-1} - \frac{j}{n-1} = \frac{1}{n-1}.$$

On the other hand

$$\begin{aligned} \frac{k}{n} - x &\geq \frac{\frac{n}{n+1} \cdot j}{n} - \frac{j+1}{n-1} = \frac{j}{n+1} - \frac{j+1}{n-1} = \frac{-2j}{(n-1)(n+1)} - \frac{1}{n-1} \\ &\geq \frac{-2x}{n+1} - \frac{1}{n-1}. \end{aligned}$$

Therefore $\left|\frac{k}{n} - x\right| \leq \frac{2x}{n-1} + \frac{1}{n-1}$. It is immediate that $\frac{x}{n-1} \leq \sqrt{\frac{x(x+1)}{n-1}}$ for all $x \geq 0$. On the other hand, $\frac{1}{n-1} = \sqrt{\frac{1}{n-1}} \cdot \sqrt{\frac{1}{n-1}} \leq \sqrt{\frac{1}{n-1}} \cdot \sqrt{\frac{j}{n-1}} \leq \sqrt{\frac{1}{n-1}} \cdot \sqrt{x} \leq \sqrt{\frac{x(x+1)}{n-1}}$. It follows that

$$M_{k,n,j}(x) = m_{k,n,j}(x) \left|\frac{k}{n} - x\right| \leq \left|\frac{k}{n} - x\right| \leq 3\sqrt{\frac{x(x+1)}{n-1}}.$$

Case 2). Subcase a). Suppose first that $n[(k-j)^2 - (k+1)] + kj - j^2 - k^2 - j < 0$. Denoting $k = j + \alpha$, the previous inequality becomes $\alpha^2(n-1) - \alpha(n+j) - (j+1)(n+j) < 0$ where evidently $\alpha \geq 1$.Let us define the function $f(t) = t^2(n-1) - t(n+j) - (j+1)(n+j)$, $t \in \mathbb{R}$. We claim that $f\left(\sqrt{\frac{3(j+1)(n+j)}{n-1}}\right) > 0$ which will imply $\alpha < \sqrt{\frac{3(j+1)(n+j)}{n-1}}$ and further one $k - j < \sqrt{\frac{3(j+1)(n+j)}{n-1}}$. Indeed, after simple calculation we get

$$
\begin{aligned}
f\left(\sqrt{\frac{3(j+1)(n+j)}{n-1}}\right) &= (n+j)\sqrt{j+1}\left(2\sqrt{j+1}-\sqrt{\frac{3(n+j)}{n-1}}\right)\\
&= (n+j)\sqrt{j+1}\left(\sqrt{4j+4}-\sqrt{3+\frac{3j+3}{n-1}}\right)\\
&\geq (n+j)\sqrt{j+1}\left(\sqrt{4j+4}-\sqrt{3+\frac{3j+3}{2}}\right)\\
&= (n+j)\sqrt{j+1}\left(\sqrt{4j+4}-\sqrt{4+\frac{3j+1}{2}}\right)>0
\end{aligned}
$$

where we used the obvious inequality $4j > \frac{3j+1}{2}$ for all $j \geq 1$.

Based on the above results we have

$$
\begin{aligned}
&\overline{M}_{k,n,j}(x) =\\
=\ & m_{k,n,j}(x)\left(\frac{k}{n-1}-x\right) \leq \frac{k}{n-1}-x \leq \frac{k}{n-1}-\frac{j}{n-1}\\
=\ & \frac{k-j}{n-1} < \frac{\sqrt{\frac{3(j+1)(n+j)}{n-1}}}{n-1} = \frac{1}{\sqrt{n-1}}\cdot\sqrt{\frac{3(j+1)(n+j)}{(n-1)^2}}\\
\leq\ & \frac{1}{\sqrt{n-1}}\cdot\sqrt{\frac{6j(n+j)}{(n-1)^2}}\\
=\ & \frac{1}{\sqrt{n-1}}\cdot\sqrt{\frac{6j}{n-1}}\cdot\sqrt{\frac{n+j}{n-1}}\\
=\ & \frac{1}{\sqrt{n-1}}\cdot\sqrt{\frac{6j}{n-1}}\cdot\sqrt{\frac{n+j-1}{n-1}}\cdot\sqrt{\frac{n+j}{n+j-1}}\\
=\ & \frac{1}{\sqrt{n-1}}\cdot\sqrt{\frac{6j}{n-1}}\cdot\sqrt{1+\frac{j}{n-1}}\cdot\sqrt{\frac{n+j}{n+j-1}} \leq \sqrt{\frac{6x(x+1)}{n-1}}\cdot\frac{2}{\sqrt{3}}\\
=\ & 2\sqrt{2}\sqrt{\frac{x(x+1)}{n-1}}.
\end{aligned}
$$

Subcase b). Suppose now that $n[(k-j)^2-(k+1)]+kj-j^2-k^2-j \geq 0$. Because n and j are fixed, we can define the real function

$$
\begin{aligned}
g(x) &= n[(x-j)^2-(x+1)]+xj-j^2-x^2-j\\
&= (n-1)x^2-x(2nj-j+n)+nj^2-n-j^2-j,
\end{aligned}
$$

for all $x \in \mathbb{R}$. For $x \geq \frac{n}{n-1}(j+1)$ we get

$$\begin{aligned} g'(x) &= 2(n-1)x - 2nj + j - n \geq 2(n-1) \cdot \frac{n(j+1)}{n-1} - 2nj + j - n \\ &= n + j > 0. \end{aligned}$$

Therefore, g is nondecreasing on the interval $[\frac{n}{n-1}(j+1), \infty)$. Since

$$g\left(\frac{n}{n-1}(j+1)\right) = -nj - n - j^2 - j < 0$$

and because $\lim_{x\to\infty} g(x) = \infty$, by the monotonicity of g too, it follows that there exists $\overline{k} \in \mathbb{N}$, $\overline{k} > \frac{n}{n-1}(j+1)$ of minimum value, such that $g(\overline{k}) = n[(\overline{k}-j)^2 - (\overline{k}+1)] + \overline{k}j - j^2 - \overline{k}^2 - j \geq 0$. Denote $k_1 = \overline{k} - 1$ where evidently $k_1 \geq j+1$. If $k_1 \geq \frac{n}{n-1}(j+1)$, then from the properties of g and by the way we choose $\overline{k}$ it results that $g(k_1) < 0$. If $k_1 < \frac{n}{n-1}(j+1)$, then $j < k_1 < \frac{n}{n-1}(j+1)$. Since g is a quadratic function and because $g(j) < 0$ and $g\left(\frac{n}{n-1}(j+1)\right) < 0$, it is immediate that we get to the same conclusion as in the other case, that is $g(k_1) < 0$ or equivalently $\alpha^2(n-1) - \alpha(n+j) - (j+1)(n+j) < 0$, where $k_1 = j + \alpha$. Using the same technique as in subcase a) we get $k_1 - j < \sqrt{\frac{3(j+1)(n+j)}{n-1}}$. Then

$$\begin{aligned} \overline{M}_{\overline{k},n,j}(x) &= m_{\overline{k},n,j}(x)(\frac{\overline{k}}{n-1} - x) \leq \frac{\overline{k}}{n-1} - x \\ &\leq \frac{\overline{k}}{n-1} - \frac{j}{n-1} = \frac{k_1 - j}{n-1} + \frac{1}{n-1} \\ &< 2\sqrt{2}\sqrt{\frac{x(x+1)}{n-1}} + \frac{1}{n-1} \\ &\leq 2\sqrt{2}\sqrt{\frac{x(x+1)}{n-1}} + \sqrt{\frac{x(x+1)}{n-1}} < 4\sqrt{\frac{x(x+1)}{n-1}}. \end{aligned}$$

By Lemma 3.2., (i) it follows that $\overline{M}_{\overline{k},n,j}(x) \geq \overline{M}_{\overline{k}+1,n,j}(x) \geq ....$ We thus obtain $\overline{M}_{k,n,j}(x) < 4\sqrt{\frac{x(x+1)}{n-1}}$ for any $k \in \{\overline{k}, \overline{k}+1, ..., \}$.

Therefore, in both subcases, by Lemma 3.1, (i) too, we get

$$M_{k,n,j}(x) < 4\sqrt{\frac{x(x+1)}{n-1}}.$$

Case 3). Subcase a). Suppose first that $n[(k-j)^2 - k] + kj - j^2 - k^2 + k < 0$. Denoting $k = j - \alpha$ the previous inequality becomes $\alpha^2(n-1) + \alpha(n+j-1) - nj - j^2 + j < 0$ where evidently $\alpha \geq 1$. Let us define the function $f(t) = t^2(n-1) + t(n+j-1) - nj - j^2 + j$, $t \in \mathbb{R}$. Because $f\left(\sqrt{\frac{j(n+j-1)}{n-1}}\right) = (n+j-1) \cdot \sqrt{\frac{j(n+j-1)}{n-1}} > 0$ it follows that $\alpha < \sqrt{\frac{j(n+j-1)}{n-1}}$ and further one $j-k < \sqrt{\frac{j(n+j-1)}{n-1}}$. Then we obtain

$$\begin{aligned}
&\underline{M}_{k,n,j}(x) = \\
= \quad & m_{k,n,j}(x)(x-\frac{k}{n-1}) \leq \frac{j+1}{n-1} - \frac{k}{n-1} = \frac{j-k}{n-1} + \frac{1}{n-1} \\
< \quad & \frac{\sqrt{\frac{j(n+j-1)}{n-1}}}{n-1} + \sqrt{\frac{x(x+1)}{n-1}} = \sqrt{\frac{1}{n-1}} \cdot \sqrt{\frac{j(n+j-1)}{(n-1)^2}} + \sqrt{\frac{x(x+1)}{n-1}} \\
= \quad & \sqrt{\frac{1}{n-1}} \cdot \sqrt{\frac{j}{(n-1)}} \cdot \sqrt{1+\frac{j}{(n-1)}} + \sqrt{\frac{x(x+1)}{n-1}} \leq 2\sqrt{\frac{x(x+1)}{n-1}}.
\end{aligned}$$

Subcase b). Suppose now that $n[(k-j)^2-k]+kj-j^2-k^2+k \geq 0$. Because n and j are fixed we can define the real function

$$\begin{aligned}
g(x) &= n[(x-j)^2-x]+xj-j^2-x^2+x \\
&= (n-1)x^2 - x(2nj+j+n+1)+nj^2-j^2,
\end{aligned}$$

for all $x \in \mathbb{R}$. For $x \leq \frac{n}{n+1} \cdot j$ we get

$$\begin{aligned}
g'(x) &= 2(n-1)x-(2nj+j+n+1) \leq \frac{2(n-1)nj}{n+1} - (2nj+j+n+1) \\
&\leq 2nj-(2nj+j+n+1) = -j-n-1 < 0.
\end{aligned}$$

Therefore, g in nonincreasing on the interval $[0, \frac{nj}{n+1}]$. We have

$$\begin{aligned}
g(\frac{nj}{n+1}) &= \frac{(n-1)n^2j^2}{(n+1)^2} - \frac{nj}{n+1} \cdot (2nj+j+n+1)+nj^2-j^2 \\
&\leq \frac{n^2j^2}{n+1} - \frac{nj}{n+1} \cdot (2nj+j+n+1)+nj^2-j^2 \\
&= \frac{-n^2j^2-nj^2-n^2j-nj}{n+1} + nj^2-j^2 = \frac{-n^2j-nj}{n+1} - j^2 \\
&= -nj-j^2 < 0.
\end{aligned}$$

Based on the above result and because $g(0) > 0$, by the monotonicity of g too, it follows that there exists $\widetilde{k} \in \mathbb{N}$, $\widetilde{k} < \frac{nj}{n+1}$ of maximum value, such that $g(\widetilde{k}) = n[(\widetilde{k}-j)^2-\widetilde{k}]+\widetilde{k}j-j^2-\widetilde{k}^2+\widetilde{k} \geq 0$. Denoting $k_2 = \widetilde{k}+1$ and reasoning as in case (ii), subcase b) we obtain $g(k_2) < 0$. Further, reasoning as in case (iii), subcase a) we obtain $j-k_2 < \sqrt{\frac{j(n+j-1)}{n-1}}$. It follows

$$\begin{aligned}
\underline{M}_{\widetilde{k},n,j}(x) &= m_{\widetilde{k},n,j}(x)(x-\frac{\widetilde{k}}{n-1}) \leq \frac{j+1}{n-1} - \frac{\widetilde{k}}{n-1} \\
&= \frac{j-k_2}{n-1} + \frac{2}{n-1} < 3\sqrt{\frac{x(x+1)}{n-1}}.
\end{aligned}$$

By Lemma 3.2, (ii) it follows that $\underline{M}_{\widetilde{k},n,j}(x) \geq \underline{M}_{\widetilde{k}-1,n,j}(x) \geq ... \geq \underline{M}_{0,n,j}(x)$. We thus obtain $\underline{M}_{k,n,j}(x) < 3\sqrt{\frac{x(x+1)}{n-1}}$ for any $k \in \{0, 1, ..., \widetilde{k}\}$

In both subcases, by Lemma 3.1, (iii) too, we get $M_{k,n,j}(x) < 6\sqrt{\frac{x(x+1)}{n-1}}$.

In conclusion, collecting all the estimates in the above cases and subcases we easily get the relationship (4.2), which completes the proof. □

Remark. In what follows we prove that the order of approximation in Theorem 4.1 cannot be improved. Indeed, for $n \in \mathbb{N}$, $n \geq 3$ let us take $j_n = (n-1)^2 - 1$, $k_n = j_n + [\sqrt{\frac{(j_n+1)(n+j_n)}{n-1}}] + 1 = j_n + [(n-1)\sqrt{n}] + 1$, $x_n = \frac{j_n+1}{n-1} = n-1$. Because $\lim_{n\to\infty}\left(k_n - \frac{n}{n-2}\cdot(j+1)\right) = \infty$, it follows that there exists $n_0 \in \mathbb{N}$, $n_0 \geq 3$ such that $k_n \geq \frac{n}{n-2}\cdot(j_n+1)$ for each $n \in \mathbb{N}$, $n \geq n_0$.Then, according to Lemma 3.1 (ii) for each $n \in \mathbb{N}$, $n \geq n_0$ we can write

$$\overline{M}_{k_n,n,j_n}(x_n) = \frac{\binom{n+k_n-1}{k_n} x_n^{k_n}/(1+x_n)^{n+k_n}}{\binom{n+j_n-1}{j_n} x_n^{j_n}/(1+x_n)^{n+j_n}} \left(\frac{k_n}{n-1} - x_n\right)$$

$$= \frac{(n+k_n-1)!}{(n+j_n-1)!}\cdot\frac{j_n!}{k_n!}\left(\frac{x_n}{1+x_n}\right)^{k_n-j_n}\left(\frac{k_n}{n-1} - x_n\right)$$

$$\geq \frac{(n+j_n)(n+j_n+1)...(n+k_n-1)}{(j_n+1)(j_n+2)...k_n}\left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1}\cdot$$

$$\cdot \left(\frac{k_n}{n-1} - \frac{j_n+1}{n-1}\right)$$

$$= \frac{(n+j_n)(n+j_n+1)...(n+k_n-1)}{(j_n+1)(j_n+2)...k_n}\left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1}\cdot$$

$$\cdot \frac{k_n - j_n}{n-1}\cdot\frac{k_n-j_n-1}{k_n-j_n}$$

$$\geq \frac{(n+j_n)(n+j_n+1)...(n+k_n-1)}{(j_n+1)(j_n+2)...k_n}\left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1}\cdot\frac{k_n-j_n}{2(n-1)}.$$

Since

$$\frac{k_n-j_n}{n-1} = \frac{[\sqrt{\frac{(j_n+1)(n+j_n)}{n-1}}]+1}{n-1} \geq \frac{\sqrt{\frac{(j_n+1)(n+j_n)}{n-1}}}{n-1} = \frac{\sqrt{\frac{(j_n+1)}{n-1}(1+\frac{j_n+1}{n-1})}}{\sqrt{n-1}}$$

$$= \frac{\sqrt{x_n(x_n+1)}}{\sqrt{n-1}},$$

it follows that

$$\overline{M}_{k_n,n,j_n}(x_n) \geq$$

$$\frac{(n+j_n)(n+j_n+1)...(n+k_n-1)}{(j_n+1)(j_n+2)...k_n}\left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1}\cdot\frac{\sqrt{x_n(x_n+1)}}{2\sqrt{n-1}}.$$

It is easy to prove that if $0 < a \leq b$ then $\frac{b}{a} \geq \frac{b+1}{a+1}$. Using this result, we get

$$\frac{n+j_n}{j_n+1} \geq \frac{n+j_n+1}{j_n+2} \geq \dots \geq \frac{n+k_n-1}{k_n},$$

which implies

$$\overline{M}_{k_n,n,j_n}(x_n)$$
$$\geq \left(\frac{n+k_n-1}{k_n}\right)^{[(n-1)\sqrt{n}]+1} \left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1} \cdot \frac{\sqrt{x_n(x_n+1)}}{2\sqrt{n-1}}.$$

We have

$$\begin{aligned}
&\lim_{n\to\infty} \left(\frac{n+k_n-1}{k_n}\right)^{[(n-1)\sqrt{n}]+1} \left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1} \\
= & \lim_{n\to\infty} \left(\frac{(n-1)^2+(n-1)+[(n-1)\sqrt{n}]}{(n-1)^2+[(n-1)\sqrt{n}]}\right)^{[(n-1)\sqrt{n}]+1} \cdot \\
\cdot & \left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1} \\
= & \lim_{n\to\infty} \left(\frac{(n-1)^2+(n-1)+(n-1)\sqrt{n}}{(n-1)^2+(n-1)\sqrt{n}}\right)^{(n-1)\sqrt{n}+1} \cdot \\
\cdot & \left(\frac{n-1}{n}\right)^{(n-1)\sqrt{n}+1} \\
= & \lim_{n\to\infty} \left(\frac{n+\sqrt{n}}{(n-1)+\sqrt{n}}\right)^{(n-1)\sqrt{n}+1} \left(\frac{n-1}{n}\right)^{(n-1)\sqrt{n}+1} \\
= & \lim_{n\to\infty} \left(1+\frac{1}{n-1+\sqrt{n}}\right)^{(n-1)\sqrt{n}+1} \left(1-\frac{1}{n}\right)^{(n-1)\sqrt{n}+1}
\end{aligned}$$

$$= \lim_{n\to\infty} \left[1+\left(\frac{1-\sqrt{n}}{n(n-1+\sqrt{n})} - \frac{1}{n(n-1+\sqrt{n})}\right)\right]^{(n-1)\sqrt{n}+1} = e^{-1}.$$

It follows that there exists $n_1 \in \mathbb{N}$, such that

$$\left(\frac{n+k_n-1}{k_n}\right)^{[(n-1)\sqrt{n}]+1} \left(\frac{n-1}{n}\right)^{[(n-1)\sqrt{n}]+1} \geq e^{-2}$$

for any $n \geq n_1$. Then we get

$$\overline{M}_{k_n,n,j_n}(x_n) \geq \frac{\sqrt{x_n(x_n+1)}}{2e^2\sqrt{n-1}}$$

for all $n \geq \max\{n_0, n_1\}$. Taking into account Lemma 3.1, (ii) too, it follows that for all $n \geq \max\{n_0, n_1\}$ we have $M_{k_n,n,j_n}(x_n) \geq \frac{\sqrt{x_n(x_n+1)}}{4e^2\sqrt{n-1}}$ for all $n \geq \max\{n_0, n_1\}$, which combined with the fact $\lim\limits_{n\to\infty} x_n = \infty$ will imply the desired conclusion.

In what follows we will prove that for large subclasses of functions f, the order of approximation $\omega_1(f;\sqrt{x(x+1)/(n-1)})$ in Theorem 4.1 can essentially be improved to $\omega_1(f;(x+1)/(n-1))$.

For this purpose, for any $n \in \mathbb{N}$, $n \geq 3$, $k \in \{0,1,..,\}$ and $j \in \{0,1,..,\}$, let us denote $A_j = \{k \in \mathbb{N} : j \leq k \leq \frac{n}{n-1}(j+1)+1\}$.

We need the following auxiliary lemmas.

Lemma 4.2. *Let $f : [0,\infty) \to [0,\infty)$ be bounded and suppose that there exists $j \in \{0,1,...,\}$* and *$x \in [j/(n-1),(j+1)/(n-1)]$ such that*

$$V_n^{(M)}(f)(x) = \bigvee_{k\in A_j} f_{k,n,j}(x),$$

Then

$$\left|V_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f;\frac{x+1}{n-1}\right), n \geq 3.$$

Proof. We distinguish two cases:

Case (i) Suppose that $V_n^{(M)}(f)(x) \leq f(x)$. Because $V_n^{(M)}(f)(x) \geq f_{j,n,j}(x) = f(\frac{j}{n})$ it follows that $f(\frac{j}{n}) \leq V_n^{(M)}(f)(x) \leq f(x)$, which implies

$$\left|V_n^{(M)}(f)(x) - f(x)\right| = f(x) - V_n^{(M)}(f)(x) \leq f(x) - f(\frac{j}{n}).$$

By simple calculation we have $0 \leq x - \frac{j}{n} \leq \frac{j+1}{n-1} - \frac{j}{n} = \frac{j}{(n-1)n} + \frac{1}{n-1} \leq \frac{x+1}{n-1}$. Therefore, in this case we obtain

$$\left|V_n^{(M)}(f)(x) - f(x)\right| \leq \omega_1\left(f;\frac{x+1}{n-1}\right).$$

Case (ii) Suppose that $V_n^{(M)}(f)(x) > f(x)$. From the hypothesis we get that there exists $\overline{k} \in A_j$ such that $V_n^{(M)}(f)(x) = f_{\overline{k},n,j}(x)$, which implies

$$\begin{aligned}\left|V_n^{(M)}(f)(x) - f(x)\right| &= V_n^{(M)}(f)(x) - f(x) = f_{\overline{k},n,j}(x) - f(x)\\ &= m_{\overline{k},n,j}(x)f(\frac{\overline{k}}{n}) - f(x) \leq f(\frac{\overline{k}}{n}) - f(x).\end{aligned}$$

We have $\frac{\overline{k}}{n} - x \leq \frac{\frac{n}{n-1}(j+1)+1}{n} - \frac{j}{n-1} = \frac{1}{n-1} + \frac{1}{n} \leq \frac{2}{n-1} \leq \frac{2(x+1)}{n-1}$. On the other hand we have

$$\begin{aligned}\frac{\overline{k}}{n} - x &\geq \frac{j}{n} - \frac{j+1}{n-1} = \frac{-j}{n(n-1)} - \frac{1}{n-1} \geq \frac{-x}{n} - \frac{1}{n-1}\\ &\geq \frac{-x}{n-1} - \frac{1}{n-1} = \frac{-(x+1)}{n-1}.\end{aligned}$$

Therefore, we obtain $\left|\frac{\overline{k}}{n} - x\right| \leq \frac{2(x+1)}{n-1}$ and it follows that $\left|V_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f;\frac{x+1}{n-1}\right)$ which proves the lemma. □

Lemma 4.3. *If the function* $f:[0,\infty)\to[0,\infty)$ *is concave, then the function* $g:(0,\infty)\to[0,\infty), g(x)=\frac{f(x)}{x}$ *is nonincreasing.*

Proof. Let $x,y\in(0,\infty)$ be with $x\le y$. Then

$$f(x)=f\left(\frac{x}{y}y+\frac{y-x}{y}0\right)\ge\frac{x}{y}f(y)+\frac{y-x}{y}f(0)\ge\frac{x}{y}f(y),$$

which implies $\frac{f(x)}{x}\ge\frac{f(y)}{y}$. $\square$

Corollary 4.4. *If* $f:[0,\infty)\to[0,\infty)$ *is bounded, nondecreasing and such that the function* $g:(0,\infty)\to[0,\infty), g(x)=\frac{f(x)}{x}$ *is nonincreasing, then*

$$\left|V_n^{(M)}(f)(x)-f(x)\right|\le 2\omega_1\left(f;\frac{x+1}{n-1}\right),\ \textit{for all } x\in[0,\infty), n\ge 3.$$

Proof. Since f is nondecreasing it follows (see the proof of Theorem 5.3 in the next section)

$$V_n^{(M)}(f)(x)=\bigvee_{k\ge j}^{\infty}f_{k,n,j}(x),\ \text{for all } x\in[j/(n-1),(j+1)/(n-1)].$$

Let $x\in[0,\infty)$ and $j\in\{0,1,...,\}$ such that $x\in[\frac{j}{n-1},\frac{j+1}{n-1}]$. Let $k\in\{1,2...,\}$ be with $k\ge j$. Then

$$\begin{aligned}f_{k+1,n,j}(x)&=\frac{\binom{n+k}{k+1}}{\binom{n+j-1}{j}}(\frac{x}{1+x})^{k+1-j}f(\frac{k+1}{n})\\&=\frac{\binom{n+k-1}{k}}{\binom{n+j-1}{j}}\cdot\frac{n+k}{k+1}(\frac{x}{1+x})^{k-j}\frac{x}{1+x}f(\frac{k+1}{n}).\end{aligned}$$

Since $g(x)$ is nonincreasing we get $\frac{f(\frac{k+1}{n})}{\frac{k+1}{n}}\le\frac{f(\frac{k}{n})}{\frac{k}{n}}$ that is $f(\frac{k+1}{n})\le\frac{k+1}{k}f(\frac{k}{n})$. From $x\le\frac{j+1}{n-1}$ it follows

$$\begin{aligned}f_{k+1,n,j}(x)&\le\frac{\binom{n+k-1}{k}}{\binom{n+j-1}{j}}(\frac{x}{1+x})^{k-j}\frac{j+1}{n+j}\cdot\frac{n+k}{k+1}\cdot\frac{k+1}{k}f(\frac{k}{n})\\&=f_{k,n,j}(x)\frac{j+1}{n+j}\cdot\frac{n+k}{k}=\frac{(n+j)k+n(j+1-k)+k}{(n+j)k}\cdot f_{k,n,j}(x).\end{aligned}$$

Since for each $k\ge\frac{n}{n-1}(j+1)$ we get $n(j+1-k)+k\le 0$, it follows that $f_{k,n,j}(x)\ge f_{k+1,n,j}(x)$ for any $k\ge\frac{n}{n-1}(j+1)$ which will immediately imply that $V_n^{(M)}(f)(x)=\bigvee_{k\in A_j}f_{k,n,j}(x)$. By Lemma 4.2 we immediately obtain the desired conclusion. $\square$

Corollary 4.5. *Let $f : [0, \infty) \to [0, \infty)$ be a bounded, nondecreasing, concave function. Then*

$$\left|V_n^{(M)}(f)(x) - f(x)\right| \le 2\omega_1\left(f; \frac{x+1}{n-1}\right), \text{ for all } x \in [0, \infty), n \ge 3.$$

Proof. The proof is immediate by Lemma 4.3 and Corollary 4.4. □

Remarks. 1) If we suppose, for example, that in addition to the hypothesis in Corollary 4.4, $f : [0, \infty) \to [0, \infty)$ is a Lipschitz function, that is there exists $M > 0$ such that $|f(x) - f(y)| \le M|x - y|$, for all $x, y \in [0, \infty)$, then it follows that the order of pointwise approximation on $[0, \infty)$ by $V_n^{(M)}(f)(x)$ is $\frac{x+1}{n-1}$, which is essentially better than the order $\frac{a}{\sqrt{n}}$ obtained from Theorem 4.1 on each compact subinterval $[0, a]$ for f Lipschitz function on $[0, \infty)$.

2) It is known that for the linear Baskakov operator given by

$$V_n(f)(x) = (1+x)^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k f(k/n),$$

the following pointwise approximation result is known (see [4])

$$|V_n(f)(x) - f(x)| \le C\omega_2^{\varphi}(f; \sqrt{x(1+x)/n}), x \in [0, \infty), n \in \mathbb{N}, \tag{4.3}$$

where $\varphi(x) = \sqrt{x(1+x)}$ and $\omega_2^{\varphi}(f; \delta)$ is the Ditzian-Totik second order modulus of smoothness on $[0, \infty)$ defined by

$$\omega_2^{\varphi}(f; \delta)$$
$$= \sup\{\sup\{|f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x))|; x \ge h^2/(1 - h^2)\}, h \in [0, \delta]\},$$

with $\delta < 1$.

Now, for example, if f has the second derivative bounded by the constant K on $[0, \infty)$, because in this case we have $\omega_2^{\varphi}(f; \delta) \le K\delta^2$, then by (4.3) we obtain the estimate

$$|V_n(f)(x) - f(x)| \le CK\frac{x(1+x)}{n}, x \in [0, \infty), n \in \mathbb{N},$$

while by Corollary 4.5 it follows the much better estimate (on large subintervals of $[0, \infty)$)

$$|V_n^{(M)}(f)(x) - f(x)| \le \frac{4\|f'\|(1+x)}{n}, x \in [0, \infty), n \in \mathbb{N}, n \ge 3.$$

Also, if f is, for example a nondecreasing concave polygonal line on $[0, \infty)$, constant on an interval $[a, \infty)$, then by simple reasonings we get that $\omega_2^{\varphi}(f; \delta) \sim \delta$ for $\delta \le 1$ and by (4.3) it easily follows the estimate

$$|V_n(f)(x) - f(x)| \le C\frac{\sqrt{x(1+x)}}{\sqrt{n}}, x \in [0, \infty), n \in \mathbb{N}, \tag{4.4}$$

while because such of function f obviously is a Lipschitz function on $[0, \infty)$ (as having bounded all the derivative numbers) by Corollary 4.5 we get the essentially better estimate than in (4.4)

$$|V_n^{(M)}(f)(x) - f(x)| \leq \frac{C(1+x)}{n}, x \in [0, \infty), n \in \mathbb{N}, n \geq 3.$$

In a similar manner, by Corollary 4.4 we can produce many subclasses of functions for which the order of approximation given by the max-product Baskakov operator is essentially better than the order of approximation given by the linear Baskakov operator. Intuitively, the max-product Baskakov operator has better approximation properties than its linear counterpart, for non-differentiable functions in a finite number of points (with the graphs having some "corners"), as for example for functions defined as a maximum of a finite number of continuous functions on $[0, \infty)$.

3) Since it is clear that a bounded nonincreasing concave function on $[0, \infty)$ necessarily one reduces to a constant function, the approximation of such functions is not of interest.

5. **Shape Preserving Properties**

In this section we will present some shape preserving properties.

Remark. Note that because of the continuity of $V_n^{(M)}(f)(x)$ on $[0, \infty)$ in Lemma 2.6, it will suffice to prove the shape properties of $V_n^{(M)}(f)(x)$ on $(0, \infty)$ only. As a consequence, in the notations and proofs below we always may suppose that $x > 0$.

Lemma 5.1. *Let $n \in \mathbb{N}$, $n \geq 3$. If $f : [0, \infty) \to \mathbb{R}_+$ is a nondecreasing function then for any $k \in \{0, 1, ..., \}$, $j \in \{0, 1, ..., \}$ with $k \leq j$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$ we have $f_{k,n,j}(x) \geq f_{k-1,n,j}(x)$.*

Proof. Because $k \leq j$, by direct computation it follows that $m_{k,n,j}(x) \geq m_{k-1,n,j}(x)$. From the monotonicity of f we get $f\left(\frac{k}{n}\right) \geq f\left(\frac{k-1}{n}\right)$. Thus we obtain

$$m_{k,n,j}(x) f\left(\frac{k}{n}\right) \geq m_{k-1,n,j}(x) f\left(\frac{k-1}{n}\right),$$

which proves the lemma. □

Corollary 5.2. *Let $n \in \mathbb{N}$, $n \geq 3$. If $f : [0, \infty) \to \mathbb{R}_+$ is nonincreasing then $f_{k,n,j}(x) \geq f_{k+1,n,j}(x)$ for any $k \in \{0, 1, ..., \}$, $j \in \{0, 1, .., \}$ with $k \geq j$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$.*

Proof. Because $k \geq j$, by direct computation it follows that $m_{k,n,j}(x) \geq m_{k+1,n,j}(x)$. From the monotonicity of f we get $f\left(\frac{k}{n}\right) \geq f\left(\frac{k+1}{n}\right)$. Thus we obtain

$$m_{k,n,j}(x) f\left(\frac{k}{n}\right) \geq m_{k+1,n,j}(x) f\left(\frac{k+1}{n}\right),$$

which proves the corollary. □

Theorem 5.3. *If $f : [0, \infty) \to \mathbb{R}_+$ is nondecreasing and bounded (on $[0, \infty)$), then $V_n^{(M)}(f)$ is nondecreasing and bounded, for any $n \in \mathbb{N}$ with $n \geq 3$.*

Proof. Because $V_n^{(M)}(f)$ is continuous on $[0, \infty)$, it suffice to prove that on each subinterval of the form $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, with $j \in \{0, 1, ..., \}$, $V_n^{(M)}(f)$ is nondecreasing.

So let $j \in \{0, 1, ..., \}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Because f is nondecreasing, from Lemma 5.1 it follows that

$$f_{j,n,j}(x) \geq f_{j-1,n,j}(x) \geq f_{j-2,n,j}(x) \geq ... \geq f_{0,n,j}(x).$$

But then it is immediate that

$$V_n^{(M)}(f)(x) = \bigvee_{k=j}^{\infty} f_{k,n,j}(x),$$

for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Clearly that for $k \geq j$ the function $f_{k,n,j}$ is nondecreasing and since $V_n^{(M)}(f)$ is defined as the supremum of nondecreasing functions, it follows that it is nondecreasing. □

Corollary 5.4. *If $f : [0, \infty) \to \mathbb{R}_+$ is nonincreasing then $V_n^{(M)}(f)$ is nonincreasing, for any $n \in \mathbb{N}$ with $n \geq 3$.*

Proof. Because $V_n^{(M)}(f)$ is continuous on $[0, \infty)$, it suffice to prove that on each subinterval of the form $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, with $j \in \{0, 1, ..., \}$, $V_n^{(M)}(f)$ is nonincreasing.

So let $j \in \{0, 1, ..., \}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Because f is nonincreasing, from Corollary 5.2 it follows that

$$f_{j,n,j}(x) \geq f_{j+1,n,j}(x) \geq f_{j+2,n,j}(x) \geq ... \geq f_{n,n,j}(x).$$

But then it is immediate that

$$V_n^{(M)}(f)(x) = \bigvee_{k=0}^{j} f_{k,n,j}(x),$$

for all $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Clearly that for $k \leq j$ the function $f_{k,n,j}$ is nonincreasing and since $V_n^{(M)}(f)$ is defined as the maximum of nonincreasing functions, it follows that it is nonincreasing. □

In what follows, let us consider the following concept generalizing the monotonicity and convexity.

Definition 5.5. Let $f : [0, \infty) \to \mathbb{R}$ be continuous on $[0, \infty)$. One says that f is quasi-convex on $[0, \infty)$ if it satisfies the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}, \text{ for all } x, y \in [0, \infty) \text{ and } \lambda \in [0, 1].$$

(see e.g. the book [5], p. 4, (iv)).

Remark. By [6], the continuous function f is quasi-convex on the bounded interval $[0, a]$, equivalently means that there exists a point $c \in [0, a]$ such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, a]$. But this property easily can be extended to continuous quasiconvex functions on $[0, \infty)$, in the sense that there exists $c \in [0, \infty]$ ($c = \infty$ by convention for nonincreasing functions on $[0, \infty)$) such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, \infty)$. This easily follows from the fact that the quasiconvexity of f on $[0, \infty)$ means the quasiconvexity of f on any bounded interval $[0, a]$, with arbitrary large $a > 0$.

The class of quasi-convex functions includes the both classes of nondecreasing functions and of nonincreasing functions (obtained from the class of quasi-convex functions by taking $c = 0$ and $c = \infty$, respectively). Also, it obviously includes the class of convex functions on $[0, \infty)$.

Corollary 5.6. *If $f : [0, \infty) \to \mathbb{R}_+$ is continuous, bounded and quasi-convex on $[0, \infty)$ then $V_n^{(M)}(f)$ is quasi-convex on $[0, \infty)$ for any $n \in \mathbb{N}$ with $n \geq 3$.*

Proof. If f is nonincreasing (or nondecreasing) on $[0, \infty)$ (that is the point $c = \infty$ (or $c = 0$) in the above Remark) then by the Corollary 5.4 (or Theorem 5.3, respectively) it follows that for all $n \in \mathbb{N}$, $V_n^{(M)}(f)$ is nonincreasing (or nondecreasing) on $[0, \infty)$.

Suppose now that there exists $c \in (0, \infty)$, such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, \infty)$. Define the functions $F, G : [0, \infty) \to \mathbb{R}_+$ by $F(x) = f(x)$ for all $x \in [0, c]$, $F(x) = f(c)$ for all $x \in [c, \infty)$ and $G(x) = f(c)$ for all $x \in [0, c]$, $G(x) = f(x)$ for all $x \in [c, \infty)$.

It is clear that F is nonincreasing and continuous on $[0, \infty)$, G is nondecreasing and continuous on $[0, \infty)$ and that $f(x) = \max\{F(x), G(x)\}$, for all $x \in [0, \infty)$.

But it is easy to show (see also Remark 1 after the proof of Lemma 2.1) that

$$V_n^{(M)}(f)(x) = \max\{V_n^{(M)}(F)(x), V_n^{(M)}(G)(x)\}, \text{ for all } x \in [0, \infty),$$

where by the Corollary 5.4 and Theorem 5.3 , $V_n^{(M)}(F)(x)$ is nonincreasing and continuous on $[0, \infty)$ and $V_n^{(M)}(G)(x)$ is nondecreasing and continuous on $[0, \infty)$. We have two cases : 1) $V_n^{(M)}(F)(x)$ and $V_n^{(M)}(G)(x)$ do not intersect each other ; 2) $V_n^{(M)}(F)(x)$ and $V_n^{(M)}(G)(x)$ intersect each other.

Case 1). We have $\max\{V_n^{(M)}(F)(x), V_n^{(M)}(G)(x)\} = V_n^{(M)}(F)(x)$ for all $x \in [0, \infty)$ or $\max\{V_n^{(M)}(F)(x), V_n^{(M)}(G)(x)\} = V_n^{(M)}(G)(x)$ for all $x \in [0, \infty)$, which obviously proves that $V_n^{(M)}(f)(x)$ is quasi-convex on $[0, \infty)$.

Case 2). In this case it is clear that there exists a point $c' \in [0, \infty)$ such that $V_n^{(M)}(f)(x)$ is nonincreasing on $[0, c']$ and nondecreasing on $[c', \infty)$, which by the considerations in the above Remark implies that $V_n^{(M)}(f)(x)$ is quasiconvex on $[0, \infty)$ and proves the corollary. $\square$

It is of interest to exactly calculate $V_n^{(M)}(f)$ for $f(x) = e_0(x) = 1$ and for $f(x) = e_1(x) = x$. In this sense we can state the following.

Lemma 5.7. *For all* $x \in [0, \infty)$ *and* $n \in \mathbb{N}$, $n \geq 3$ *we have* $V_n^{(M)}(e_0)(x) = 1$ *and*

$$V_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,0}(x)}{b_{n,0}(x)} = \frac{x}{1+x}, \text{ if } x \in [0, 1/n],$$

$$V_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,1}(x)}{b_{n,0}(x)} = \frac{(n+1)x^2}{(1+x)^2}, \text{ if } x \in [1/n, 1/(n-1)],$$

$$V_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,1}(x)}{b_{n,1}(x)} = \frac{x}{1+x} \cdot \frac{n+1}{n}, \text{ if } x \in [1/(n-1), 2/n],$$

$$V_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,2}(x)}{b_{n,1}(x)} = \frac{x^2}{(1+x)^2} \cdot \frac{(n+1)(n+2)}{2n}, \text{ if } x \in [2/n, 2/(n-1)],$$

$$V_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,2}(x)}{b_{n,2}(x)} = \frac{x}{1+x} \cdot \frac{n+2}{n}, \text{ if } x \in [2/(n-1), 3/n],$$

$$V_n^{(M)}(e_1)(x) = x \cdot \frac{b_{n+1,3}(x)}{b_{n,2}(x)} = \frac{x^2}{(1+x)^2} \cdot \frac{(n+2)(n+3)}{3n}, \text{ if } x \in [3/n, 3/(n-1)],$$

and so on, in general we have

$$V_n^{(M)}(e_1)(x) = \frac{x}{1+x} \cdot \frac{n+j}{n}, \text{ if } x \in [j/(n-1), (j+1)/n],$$

$$V_n^{(M)}(e_1)(x) = \frac{x^2}{(1+x)^2} \cdot \frac{(n+j)(n+j+1)}{n(j+1)}, \text{ if } x \in [(j+1)/n, (j+1)/(n-1)],$$

for $j \in \{0, 1, ..., \}$.

Proof. The formula $V_n^{(M)}(e_0)(x) = 1$ is immediate by the definition of $V_n^{(M)}(f)(x)$.

To find the formula for $V_n^{(M)}(e_1)(x)$ we will use the explicit formula in Lemma 2.4 which says that

$$\bigvee_{k=0}^{\infty} b_{n,k}(x) = b_{n,j}(x), \text{ for all } x \in \left[\frac{j}{n-1}, \frac{j+1}{n-1}\right], j = 0, 1, ...,$$

where $b_{n,k}(x) = \binom{n+k-1}{k} x^k/(1+x)^{n+k}$.

Since

$$\max_{k=0,1...,} \left\{ b_{n,k}(x) \frac{k}{n} \right\} = \max_{k=1,...,n} \left\{ b_{n,k}(x) \frac{k}{n} \right\} = x \cdot \max_{k=0,1...,} \{ b_{n+1,k}(x) \},$$

we obtain

$$V_n^{(M)}(e_1)(x) = x \cdot \frac{\bigvee_{k=0}^{\infty} b_{n+1,k}(x)}{\bigvee_{k=0}^{\infty} b_{n,k}(x)}$$

Now the conclusion of the lemma is immediate by applying Lemma 2.4 to both expressions $\bigvee_{k=0}^{\infty} b_{n+1,k}(x)$, $\bigvee_{k=0}^{\infty} b_{n,k}(x)$, taking into account that we get the following division of the interval $[0,\infty)$

$$0 < \frac{1}{n} \leq \frac{1}{n-1} \leq \frac{2}{n} \leq \frac{2}{n-1} \leq \frac{3}{n} \leq \frac{3}{n-1} \leq \frac{4}{n} \leq \frac{4}{n-1} ...., .$$

□

Remarks. 1) The convexity of f on $[0,\infty)$ is not preserved by $V_n^{(M)}(f)$ as can be seen from Lemma 5.8. Indeed, while $f(x) = e_1(x) = x$ is obviously convex on $[0,\infty)$, it is easy to see that $V_n^{(M)}(e_1)$ is not convex on $[0,1]$.

2) Also, if f is supposed to be starshaped on $[0,\infty)$ (that is $f(\lambda x) \leq \lambda f(x)$ for all $x, \lambda \in [0,\infty)$), then again by Lemma 5.8 it follows that $V_n^{(M)}(f)$ for $f(x) = e_1(x)$ is not starshaped on $[0,\infty)$, although $e_1(x)$ obviously is starshaped on $[0,\infty)$.

Despite of the absence of the preservation of the convexity, we can prove the interesting property that for any arbitrary nonincreasing function f, the max-product Baskakov operator $V_n^{(M)}(f)$ is piecewise convex on $[0,\infty)$. We present the following.

Theorem 5.8. *Let $n \in \mathbb{N}$ be with $n \geq 3$. For any nonincreasing function $f : [0,\infty) \to [0,\infty)$, $V_n^{(M)}(f)$ is convex on any interval of the form $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, $j = 0, 1, ...,$.*

Proof. From the proof of Corollary 5.4 we have

$$V_n^{(M)}(f)(x) = \bigvee_{k=0}^{j} f_{k,n,j}(x),$$

for any $j \in \{0, 1, ..., \}$ and $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$.

We will prove that for any fixed j and $k \leq j$, each function $f_{k,n,j}(x)$ is convex on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, which will imply that $V_n^{(M)}(f)$ can be written as a maximum of some convex functions on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$.

Since $f \geq 0$ it suffices to prove that the functions $g_{k,j} : [0,1] \to \mathbb{R}_+$, $g_{k,j}(x) = \left(\frac{x}{1+x}\right)^{k-j}$ are convex on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$.

For $k = j$, $g_{j,j}$ is constant so is convex.

For $k = j-1$ it follows $g_{j-1,j}(x) = \frac{x+1}{x}$ for any $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$. Then $g''_{j-1,j}(x) = \frac{2}{x^3} > 0$ for any $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$.

If $k \leq j-2$ then $g''_{k,j}(x) = (k-j)\left(\frac{x}{1+x}\right)^{k-j-2} \cdot \frac{1}{(x+1)^4} \cdot (k-j-1-2x) > 0$, for any $x \in [\frac{j}{n-1}, \frac{j+1}{n-1}]$.

Since all the functions $g_{k,j}$ are convex on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$, we get that $V_n^{(M)}(f)$ is convex on $[\frac{j}{n-1}, \frac{j+1}{n-1}]$ as maximum of these functions, which proves the theorem. $\square$

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