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SOME SUBCLASSES OF MEROMORPHICALLY UNIVALENT FUNCTIONS

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Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. Making use of certain linear operator, we introduce two novel subclasses $\sum_n(A, B, \lambda)$ and $\sum_{p,n}^*(A, B, \lambda)$ of meromorphically univalent functions in the punctured disc U^*. The main object of this paper is to investigate the various important properties and characteristics of these subclasses of meromorphically univalent functions. We extend the familiar concept of neighborhoods of analytic functions to these subclasses of meromorphically univalent functions. We also derive many result for the Hadamard products of functions belonging to the class $\sum_{p,n}^*(\alpha, \beta, \gamma, \lambda)$.

1. Introduction

Let $\sum$ denote the class of functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k. \tag{1.1}$$

which are analytic and univalent in the punctured disc

$$U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \backslash \{0\}$$

and which have a simple pole at the origin with residue one there. Define a linear operator as follows:

$$D^0 f(z) = f(z),$$

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$$D^1 f(z) = \frac{1}{z} + \sum_{k=0}^{\infty}(k+2)a_k z^k = \frac{(z^2 f(z))^{'}}{z},$$
$$D^2 f(z) = D(D^1 f(z)),$$

and (in general)

$$D^n f(z) = \frac{1}{z} + \sum_{k=0}^{\infty}(k+2)^n a_k z^k$$
$$= \frac{(z^2 D^{n-1} f(z))^{'}}{z} \quad (f \in \sum;\ n \in \mathbb{N} = \{1,2,...\}). \tag{1.2}$$

The linear operator D^n was considered by Uralegaddi and Somanath [15].

Let

$$F_{\lambda,n}(z) = (1-\lambda)D^n f(z) + \lambda z(D^n f(z))^{'}(f \in \sum; n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\};\ 0 \leq \lambda < \frac{1}{2}), \tag{1.3}$$

so that, obviously,

$$F_{\lambda,n}(z) = \frac{1-2\lambda}{z} + \sum_{k=0}^{\infty}(k+2)^n[1+\lambda(k-1)]a_k z^k (n \in \mathbb{N}_0;\ 0 \leq \lambda < \frac{1}{2}), \tag{1.4}$$

it is easily verified that

$$zF^{'}_{\lambda,n}(z) = F_{\lambda,n+1}(z) - 2F_{\lambda,n}(z). \tag{1.5}$$

For a function $f(z) \in \sum$, we say that f(z) is a member of the class $\sum_n(A,B,\lambda)$ if the function $F_{\lambda,n}(z)$ defined by (1.3) satisfies the inequality:

$$\left| \frac{z^2 F^{'}_{\lambda,n}(z) + (1-2\lambda)}{Bz^2 F^{'}_{\lambda,n}(z) + (1-2\lambda)A} \right| < 1 \quad (z \in U^*), \tag{1.6}$$

where (and throughout this paper) the parameters A, B, λ, p and n are constrained as follows:

$$-1 \leq A < B \leq 1,\ 0 < B \leq 1\,,\ 0 \leq \lambda < \frac{1}{2}; p \in \mathbb{N} \ \text{ and } n \in \mathbb{N}_0. \tag{1.7}$$

Furthermore, we say that a function $f(z) \in \sum^*_{p,n}(A,B,\lambda)$ whenever $f(z)$ is of the form [cf. Equation (1.1)]:

$$f(z) = \frac{1}{z} + \sum_{k=p}^{\infty} |a_k|\, z^k \quad (k \geq p\,;\ p \in \mathbb{N}). \tag{1.8}$$

We note that:

(i) $\sum_{p,0}^{*}((2\alpha\gamma-1)\beta,\ (2\gamma-1)\beta,0)=\sum_{p}(\alpha,\beta,\gamma)\,(0\leq\alpha<1\,;\,0<\beta\leq 1;\frac{1}{2}\leq\gamma\leq 1)$ (Cho et al. [6]);

(ii) $\sum_{1,0}^{*}((2\alpha\gamma-1)\beta,\ (2\gamma-1)\beta,0)=\sum_{1}(\alpha,\beta,\gamma)\,(0\leq\alpha<1\,;\,0<\beta\leq 1;\frac{1}{2}\leq\gamma\leq 1)$ (Cho et al. [5]);

(iii) $\sum_{1,0}^{*}(-A,-B,0)=\sum_{d}(A,B)(-1\leq B<A\leq 1\,;\,-1\leq B<0)$ (Cho [4]);

(iv) $\sum_{p,0}^{*}(B,A,\lambda)=\Omega^{+}(p;0;1,1,A,B,\lambda)=\Omega^{+}(p,A,B,\lambda)$ (Joshi et al. [9]).

Also we note that:

(v) $\sum_{p,n}^{*}((2\alpha\gamma-1)\beta,(2\gamma-1)\beta,\lambda)=\sum_{p,n}^{*}(\alpha,\beta,\gamma,\lambda)$

$$=\left\{f\in\sum\nolimits_{p}^{*}:\left|\frac{z^{2}F_{\lambda,n}^{'}(z)+(1-2\lambda)}{(2\gamma-1)z^{2}F_{\lambda,n}^{'}(z)+(1-2\lambda)(2\gamma\alpha-1)}\right|<\beta,\right.$$

$$\left.(z\in U^{*}\,;\,0\leq\alpha<1;\,0<\beta\leq 1;\frac{1}{2}\leq\gamma\leq 1;0\leq\lambda<\frac{1}{2}\,;\,n\in\mathbb{N}_{0})\right\};\tag{1.9}$$

(vi) $\sum_{p,n}^{*}((2\alpha\gamma-1)\beta,(2\gamma-1)\beta,0)=\sum_{p,n}^{*}(\alpha,\beta,\gamma)$

$$=\left\{f\in\sum\nolimits_{p}^{*}:\left|\frac{z^{2}(D^{n}f(z))^{'}+1}{(2\gamma-1)z^{2}(D^{n}f(z))^{'}+(2\gamma\alpha-1)}\right|<\beta,\right.$$

$$\left.(z\in U^{*}\,;\,0\leq\alpha<1;0<\beta\leq 1;\frac{1}{2}\leq\gamma\leq 1;\ n\in\mathbb{N}_{0})\right\}.\tag{1.10}$$

2. **Inclusion properties of the class** $\sum_{n}(A,B,\lambda)$

We begin by recalling the following result (Jack's lemma), which we shall apply in proving our first theorem.

Lemma 2.1. [8] *Let the (nonconstant) function $w(z)$ be analytic in U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in U$, then*

$$z_{0}w^{'}(z_{0})=\gamma w(z_{0}),\tag{2.1}$$

where γ is a real and $\gamma\geq 1$.

Theorem 2.2. *The following inclusion property holds true for the class* $\sum_{n}(A,B,\lambda)$

$$\sum\nolimits_{n+1}(A,B,\lambda)\subset\sum\nolimits_{n}(A,B,\lambda)\,(n\in\mathbb{N}_{0}).\tag{2.2}$$

Proof. Let $f(z) \in \sum_{n+1}(A,B,\lambda)$ and suppose that

$$z^2 F'_{\lambda,n}(z) = -\frac{(1-2\lambda)(1+Aw(z))}{1+Bw(z)}, \tag{2.3}$$

where the function $w(z)$ is either analytic or meromorphic in U, with $w(0)=0$. Then, by using (1.5) and (2.3), we have

$$z^2 F'_{\lambda,n+1}(z) = -(1-2\lambda)\left[\frac{1+Aw(z)}{1+Bw(z)} + \frac{(A-B)zw'(z)}{(1+Bw(z))^2}\right]. \tag{2.4}$$

We claim that $|w(z)| < 1$ for $z \in U$. Otherwith there exists a point $z_0 \in U$ such that $\max\limits_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1.$ Applying Jack's lemma, we have $z_0 w'(z_0) = \gamma w(z_0) (\gamma \geq 1).$ Writing $w(z_0) = e^{i\theta} (0 \leq \theta \leq 2\pi)$ and putting $z = z_0$ in (2.4), we get

$$\begin{aligned} &\left|\frac{z_0^2 F'_{\lambda,n+1}(z_0) + (1-2\lambda)}{Bz_0^2 F'_{\lambda,n+1}(z_0) + (1-2\lambda)A}\right|^2 - 1 \\ &= \frac{\left|1+\gamma+Be^{i\theta}\right|^2 - \left|1+B(1-\gamma)e^{i\theta}\right|^2}{\left|1+B(1-\gamma)e^{i\theta}\right|^2} \\ &= \frac{\gamma^2(1-B^2) + 2\gamma(1+B^2+2B\cos\theta)}{\left|1+B(1-\gamma)e^{i\theta}\right|^2} \geq 0, \end{aligned} \tag{2.5}$$

which obviously contradicts our hypothesis that $f(z) \in \sum_{n+1}(A,B,\lambda)$. Thus we must have $|w(z)| < 1$ $(z \in U),$ so from (2.3), we conclude that $f(z) \in \sum_n(A,B,\lambda),$ which evidently completes the proof of Theorem 1.

Theorem 2.3. *Let α be a complex number such that $Re(\alpha) > 0$. If $f(z) \in \sum_n(A,B,\lambda)$, then the function $G_{\lambda,n}(z)$ given by*

$$G_{\lambda,n}(z) = \frac{\alpha}{z^{\alpha+1}} \int_0^z t^\alpha F_{\lambda,n}(t)dt \tag{2.6}$$

is also in the same class $\sum_n(A,B,\lambda)$.

Proof. From (2.6), we have

$$zG'_{\lambda,n}(z) = \alpha F_{\lambda,n}(z) - (\alpha+1)G_{\lambda,n}(z). \tag{2.7}$$

Put

$$z^2 G'_{\lambda,n}(z) = -\frac{(1-2\lambda)(1+Aw(z))}{1+Bw(z)}, \tag{2.8}$$

where $w(z)$ is either analytic or meromorphic in U with $w(0) = 0$. Then , by using (2.7) and (2.8), we have

$$z^2 F'_{\lambda,n}(z) = -(1-2\lambda)\left[\frac{1+Aw(z)}{1+Bw(z)} + \frac{(A-B)zw'(z)}{\alpha(1+Bw(z))^2}\right]. \qquad (2.9)$$

The remaining part of the proof is similar to that of Theorem 1 and so is omitted.

3. Properties of the class $\sum_{p,n}^{*}(A,B,\lambda)$

Theorem 3.1. *Let* $f(z) \in \sum_p^*$ *be given by* (1.8). *Then* $f(z) \in \sum_{p,n}^{*}(A,B,\lambda)$ *if and only if*

$$\sum_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)](1+B)\,|a_k| \le (B-A)(1-2\lambda), \qquad (3.1)$$

where the parameters A, B, n *and* λ *are constrained as in* (1.7).

Proof. Let $f(z) \in \sum_{p,n}^{*}(A,B,\lambda)$ be given by (1.8). Then , from (1.8) and (1.6), we have

$$\left|\frac{z^2 F'_{\lambda,n}(z) + (1-2\lambda)}{Bz^2 F'_{\lambda,n}(z) + (1-2\lambda)A}\right|$$

$$= \left|\frac{\sum\limits_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)]\,|a_k|\, z^{k+1}}{(B-A)(1-2\lambda) - B\sum\limits_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)]\,|a_k|\, z^{k+1}}\right| < 1 \quad (z \in U^*). \quad (3.2)$$

Since $|\mathrm{Re}(z)| \le |z|\ (z \in \mathbb{C})$, we have

$$\mathrm{Re}\left\{\frac{\sum\limits_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)]\,|a_k|\, z^{k+1}}{(B-A)(1-2\lambda) - B\sum\limits_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)]\,|a_k|\, z^{k+1}}\right\} < 1. \qquad (3.3)$$

Choose values of z on the real axis so that $z^2 F'_{\lambda,n}(z)$ is real. Upon clearing the denominator in (3.3) and letting $z \to 1^-$ through real values we obtain (3.1).

In order to prove the converse, we assume that the inequality (3.1) holds true. then, if we let $z \in \partial U$, we find from (1.8) and (3.1) that

$$\left|\frac{z^2 F'_{\lambda,n}(z) + (1-2\lambda)}{Bz^2 F'_{\lambda,n}(z) + (1-2\lambda)A}\right|$$

$$\leq \frac{\sum\limits_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)]\,|a_k|}{(B-A)(1-2\lambda) - B\sum\limits_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)]\,|a_k|}$$

$$< 1 (z \in \partial U = \{z : z \in \mathbb{C} \text{ and } |z| = 1\}). \tag{3.4}$$

Hence, by the maximum modulus theorem, we have $f(z) \in \sum_{p,n}^{*}(A, B, \lambda)$.

Corollary 3.2. *If the function* $f(z)$ *defined by* (1.8) *is in the class* $\sum_{p,n}^{*}(A, B, \lambda)$*, then*

$$|a_k| \leq \frac{(B-A)(1-2\lambda)}{k(k+2)^n[1+\lambda(k-1)](1+B)} \quad (k \geq p; p \in \mathbb{N}; n \in \mathbb{N}_0), \tag{3.5}$$

with equality for the function

$$f(z) = \frac{1}{z} + \frac{(B-A)(1-2\lambda)}{k(k+2)^n[1+\lambda(k-1)](1+B)} z^k \quad (k \geq p; p \in \mathbb{N}; n \in \mathbb{N}_0). \tag{3.6}$$

Putting $A = (2\gamma\alpha - 1)\beta$ and $B = (2\gamma - 1)\beta$ $(0 \leq \alpha < 1, 0 < \beta \leq 1 \text{ and } \frac{1}{2} \leq \gamma \leq 1)$ in Theorem 2.3, we obtain:

Corollary 3.3. *A function* $f(z)$ *defined by* (1.8) *is in the class* $\sum_{p,n}^{*}(\alpha, \beta, \gamma, \lambda)$ *if and only if*

$$\sum_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)\,|a_k| \leq 2\beta\gamma(1-2\lambda)(1-\alpha). \tag{3.7}$$

Next we prove the following growth and distortion properties for the class $\sum_{p,n}^{*}(A, B, \lambda)$.

Theorem 3.4. *If a function* $f(z)$ *defined by* (1.8) *is in the class* $\sum_{p,n}^{*}(A, B, \lambda)$, *then*

$$\left\{m! - \frac{(p-1)!(B-A)(1-2\lambda)}{(p-m!)(p+2)^n[1+\lambda(p-1)](1+B)} r^{p+1}\right\} r^{-(m+1)} \leq \left|f^{(m)}(z)\right|$$

$$\leq \left\{m! + \frac{(p-1)!(B-A)}{(p-m!)(p+2)^n[1+\lambda(p-1)](1+B)} r^{p+1}\right\} r^{-(m+1)} \tag{3.8}$$

$$(0 < |z| = r < 1; p \in \mathbb{N}; m, n \in \mathbb{N}_0; m < p).$$

The result is sharp for the function $f(z)$ *given by*

$$f(z) = \frac{1}{z} + \frac{(B-A)(1-2\lambda)}{p(p+2)^n[1+\lambda(p-1)](1+B)} z^p \quad (p \in \mathbb{N};\ n \in \mathbb{N}_0). \tag{3.9}$$

Proof. In view of Theorem 2.3, we have

$$\frac{p(p+2)^n[1+\lambda(p-1)](1+B)}{p!}\sum_{k=p}^{\infty}k!\,|a_k| \le \sum_{k=p}^{\infty}k(k+2)^n[1+\lambda(k-1)](1+B)\,|a_k|$$
$$\le (B-A)(1-2\lambda),$$

which yields

$$\sum_{k=p}^{\infty}k!\,|a_k| \le \frac{p!(B-A)(1-2\lambda)}{p(p+2)^n[1+\lambda(p-1)](1+B)} \quad (p\in\mathbb{N};\ n\in\mathbb{N}_0). \tag{3.10}$$

Now, by differentiating both sides of (1.8) m times with respect to z, we have

$$f^{(m)}(z) = (-1)^m m! z^{-(m+1)} + \sum_{k=p}^{\infty}\frac{k!}{(k-m)!}\,|a_k|\,z^{k-m},$$
$$(p\in\mathbb{N}; m,n\in\mathbb{N}_0; m<p), \tag{3.11}$$

and Theorem 3.1 follows easily from (3.10) and (3.11).

Finally, it is easy to see that the bounds in (3.8) are attained for the function $f(z)$ given by (3.9).

By the same way as in the proof given by Cho et al. [5] , we have the radii of meromorphically starlikeness of order $\phi(0\le\phi<1)$ and meromorphically convexity of order $\phi(0\le\phi<1)$ for functions in the class $\sum_{p,n}^{*}(A,B,\lambda)$.

Theorem 3.5. *Let the function $f(z)$ defined by* (1.8) *be in the class* $\sum_{p,n}^{*}(A,B,\lambda)$, *then, we have*

(i) $f(z)$ is meromorphically starlike of order $\phi(0\le\phi<1)$ in the disc $|z|<r_1$, that is,

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \phi \quad (|z|<r_1; 0\le\phi<1), \tag{3.12}$$

where

$$r_1 = \inf_{k\ge p}\left\{\frac{k(k+2)^n[1+\lambda(k-1)](1+B)(1-\phi)}{(B-A)(1-2\lambda)(k+2-\phi)}\right\}^{\frac{1}{k+1}}. \tag{3.13}$$

(ii) $f(z)$ is meromorphically convex of order $\phi(0\le\phi<1)$ in the disc $|z|<r_2$, that is,

$$\operatorname{Re}\left\{-(1+\frac{zf''(z)}{f'(z)})\right\} > \phi \quad (|z|<r_2\,; 0\le\phi<1), \tag{3.14}$$

where

$$r_2 = \inf_{k \geq p} \left\{ \frac{(k+2)^n[1+\lambda(k-1)](1+B)(1-\phi)}{(B-A)(1-2\lambda)(k+2-\phi)} \right\}^{\dfrac{1}{k+1}} . \tag{3.15}$$

Each of these results is sharp for the function $f(z)$ given by (3.6).

4. **Neighborhoods and partial sums**

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [7] and Ruscheweyh [13], and (more recently) by Altintas et al. ([1], [2] and [3]) , Liu [10] and Liu and Srivastava ([11] and [12]), we begin by introducing here the $\delta-$ neighborhood of a function $f(z) \in \sum$ of the form (1.1) by means of the definition given below:

$$N_\delta(f) = \left\{ g \in \sum \ : \ g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k \text{ and} \right.$$

$$\sum_{k=0}^{\infty} \frac{k(k+2)^n[1+\lambda(k-1)](1+|B|)}{(B-A)(1-2\lambda)} |a_k - b_k| \leq \delta,$$

$$\left. (-1 \leq A < B \leq 1, 0 \leq \lambda < \frac{1}{2}, \delta > 0, p \in \mathbb{N}, n \in \mathbb{N}_0) \right\}. \tag{4.1}$$

Making use of the definition (4.1), we now prove Theorem 6 below:

Theorem 4.1. *Let the function $f(z)$ defined by* (1.1) *be in the class $\sum_n(A, B, \lambda)$. If $f(z)$ satisfies the following condition:*

$$\frac{f(z) + \epsilon z^{-1}}{1+\epsilon} \in \sum\nolimits_n (A, B, \lambda) \ \ (\epsilon \in \mathbb{C}, |\epsilon| < \delta, \delta > 0),$$

then

$$N_\delta(f) \subset \sum\nolimits_n (A, B, \lambda). \tag{4.3}$$

Proof. It is easily seen from (1.6) that $g(z) \in \sum_n(A, B, \lambda)$ if and only if for any complex number σ with $|\sigma| = 1$,

$$\frac{z^2 G'_{\lambda,n}(z) + (1-2\lambda)}{Bz^2 G'_{\lambda,n}(z) + (1-2\lambda)A} \neq \sigma \quad (z \in U), \tag{4.4}$$

which is equivalent to

$$\frac{(g * h)(z)}{z^{-1}} \neq 0 \ \ (z \in U), \tag{4.5}$$

which, for convenience,

$$h(z) = \frac{1}{z} + \sum_{k=0}^{\infty} c_k z^k$$
$$= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{k(k+2)^n[1+\lambda(k-1)](1-\sigma B)}{\sigma(B-A)(1-2\lambda)} z^k. \tag{4.6}$$

From (4.6), we have

$$|c_k| \leq \frac{k(k+2)^n[1+\lambda(k-1)](1+|B|)}{(B-A)(1-2\lambda)} \quad (0 \leq \lambda < \frac{1}{2}\,; n \in \mathbb{N}_0). \tag{4.7}$$

Now , if $f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k \in \sum$ satisfies the condition (4.2), then (4.5) yields

$$\left| \frac{(f*h)(z)}{z^{-1}} \right| \geq \delta \qquad (z \in U; \delta > 0). \tag{4.8}$$

By letting

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k \in N_\delta(f), \tag{4.9}$$

so that

$$\left| \frac{[g(z)-f(z)]*h(z)}{z^{-1}} \right| = \left| \sum_{k=0}^{\infty} (b_k - a_k) c_k z^{k+1} \right|$$
$$\leq |z| \sum_{k=0}^{\infty} \frac{k(k+2)^n[1+\lambda(k-1)](1+|B|)}{(B-A)(1-2\lambda)} |b_k - a_k|$$
$$< \delta \quad (z \in U; \delta > 0). \tag{4.10}$$

Thus we have (4.5), and hence also (4.4) for any $\sigma \in \mathbb{C}$ such that $|\sigma| = 1$, which implies that $g(z) \in \sum_n (A, B, \lambda)$. This evidently proves the assertion (4.3) of Theorem 6.

We now define the $\delta-$neighborhood of a function $f(z) \in \sum_p^*$ of the form (1.8) as follows:

$$N_\delta^+(f) = \Big\{ g \in \sum\nolimits_p^* \, : \, g(z) = \frac{1}{z} + \sum_{k=p}^{\infty} |b_k| \, z^k \quad \text{and}$$
$$\sum_{k=0}^{\infty} \frac{k(k+2)^n[1+\lambda(k-1)](1+B)}{(B-A)(1-2\lambda)} \, ||b_k| - |a_k|| \leq \delta,$$
$$(-1 \leq A < B \leq 1;\ 0 \leq \lambda < \frac{1}{2}; \delta > 0; p \in \mathbb{N}; n \in \mathbb{N}_0) \Big\}. \tag{4.11}$$

Making use of the definition (4.11), we now prove Theorem 3.4 below:

Theorem 4.2. *Let the function* $f(z)$ *defined by* (1.8) *be in the class* $\sum_{p,n}^{*}(A,B,\lambda)$, $-1 \leq A < B \leq 1, 0 < B \leq 1$, $0 \leq \lambda < \dfrac{1}{2}, p \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$, *then*

$$N_{\delta}^{+}(f) \subset \sum\nolimits_{p,n}^{*}(A,B,\lambda) \;\; (\delta = \frac{p+1}{p+2}). \tag{4.12}$$

The result is sharp .

Proof. Making use the same method as in the proof of Theorem 6, we can show that [cf. Eq. (4.6)]

$$h(z) = \frac{1}{z} + \sum_{k=p}^{\infty} c_k z^k$$

$$= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{k(k+2)^n[1+\lambda(k-1)](1-\sigma B)}{\sigma(B-A)(1-2\lambda)} z^k. \tag{4.13}$$

Thus under the hypothesis $-1 \leq A < B \leq 1\,, 0 < B \leq 1$, $0 \leq \lambda < \dfrac{1}{2}\,, p \in \mathbb{N}$ and $n \in \mathbb{N}_0$, if $f(z) \in \sum_{p,n+1}^{*}(A,B,\lambda)$ is given by (1.8), we obtain

$$\left|\frac{(f*h)(z)}{z^{-1}}\right| = \left|1 + \sum_{k=p}^{\infty} c_k \left|a_k\right| z^{k+1}\right|$$

$$\geq 1 - \frac{1}{p+2} \sum_{k=p}^{\infty} \frac{k(k+2)^{n+1}[1+\lambda(k-1)](1+B)}{(B-A)(1-2\lambda)} \left|a_k\right|,$$

which in view of Theorem 2.3, yields

$$\left|\frac{(f*h)(z)}{z^{-1}}\right| \geq 1 - \frac{1}{p+2} = \frac{p+1}{p+2} = \delta.$$

The remaing part of the proof of Theorem 3.4 is similar to that of Theorem 6, and we skip the details involved.

To show the sharpness, we consider the functions $f(z)$ and $g(z)$ given by

$$f(z) = \frac{1}{z} + \frac{(B-A)(1-2\lambda)}{p(p+2)^{n+1}[1+\lambda(p-1)](1+B)} z^p \in \sum\nolimits_{p,n+1}^{*}(A,B,\lambda) \tag{4.14}$$

and

$$g(z) = \frac{1}{z} + \left[\frac{(B-A)(1-2\lambda)}{p(p+2)^{n+1}[1+\lambda(p-1)](1+B)} + \right.$$
$$\left. \frac{(B-A)(1-2\lambda)\delta'}{p(p+2)^{n}[1+\lambda(p-1)](1+B)}\right] z^p, \tag{4.15}$$

where $\delta^{'} > \delta = \dfrac{p+1}{p+2}$. Clearly, the function $g(z)$ belongs to $N^{+}_{\delta'}(f)$. On the other hand, we find from Theorem 2.3 that g(z) is not in the class $\sum^{*}_{p,n}(A,B,\lambda)$.

Thus the proof of Theorem 3.4 is completed.

Next we prove the following result.

Theorem 4.3. *Let* $f(z) \in \sum$ *be given by* (1.1) *and define the partial sums* $s_1(z)$ *and* $s_m(z)$ *as follows:*

$$s_1(z) = \frac{1}{z} \quad \text{and} \quad s_m(z) = \frac{1}{z} + \sum_{k=0}^{m-2} a_k z^k \ \ (m \in \mathbb{N}\backslash\{1\}). \tag{4.16}$$

Suppose also that

$$\sum_{k=0}^{\infty} d_k\, |a_k| \leq 1 \quad \left(d_k = \frac{k(k+2)^n[1+\lambda(k-1)](1+|B|)}{(B-A)(1-2\lambda)}\right). \tag{4.17}$$

Then we have

$(i) f(z) \in \sum_n(A,B,\lambda)$,

$$(ii) \ \ Re\left\{\frac{f(z)}{s_m(z)}\right\} > 1 - \frac{1}{d_{m-1}} \quad (z \in U;\ m \in \mathbb{N}) \tag{4.18}$$

and

$$(iii) Re\left\{\frac{s_m(z)}{f(z)}\right\} > \frac{d_{m-1}}{1+d_{m-1}} \quad (z \in U;\ m \in \mathbb{N}). \tag{4.19}$$

The estimates in (4.18) *and* (4.19) *are sharp for each* $m \in \mathbb{N}$.

Proof. (i) It is not difficult to see that

$$z^{-1} \in \sum\nolimits_n (A,B,\lambda) \quad (n \in \mathbb{N}_0).$$

Thus, from Theorem 6 and the hypothesis (4.17) of Theorem 3.5, we have

$$N_1(z^{-1}) \subset \sum\nolimits_n (A,B,\lambda) \quad (n \in \mathbb{N}_0), \tag{4.20}$$

which shows that $f(z) \in \sum_n(A,B,\lambda)$ as asserted by Theorem 3.5.

(ii) For the coefficients d_k given by(4.17), it is not difficult to verify that

$$d_{k+1} > d_k > 1 \quad (k \in \mathbb{N}). \tag{4.21}$$

Therefore, we have

$$\sum_{k=0}^{m-2} |a_k| + d_{m-1} \sum_{k=m-1}^{\infty} |a_k| \leq \sum_{k=0}^{\infty} d_k |a_k| \leq 1, \tag{4.22}$$

where we have used the hypothesis (4.17) again.

By setting

$$h_1(z) = d_{m-1} \left\{ \frac{f(z)}{s_m(z)} - \left(1 - \frac{1}{d_{m-1}}\right) \right\} = 1 + \frac{d_{m-1} \sum\limits_{k=m-1}^{\infty} a_k z^{k+1}}{1 + \sum\limits_{k=0}^{m-2} a_k z^{k+1}}, \tag{4.23}$$

and applying (4.22), we find that

$$\left| \frac{h_1(z) - 1}{h_1(z) + 1} \right| \leq \frac{d_{m-1} \sum\limits_{k=m-1}^{\infty} |a_k|}{2 - 2 \sum\limits_{k=0}^{m-2} |a_k| - d_{m-1} \sum\limits_{k=m-1}^{\infty} |a_k|} \leq 1 \quad (z \in U), \tag{4.24}$$

which readily yields the assertion (4.18) of Theorem 3.5. If we take

$$f(z) = \frac{1}{z} - \frac{z^{m-1}}{d_{m-1}}, \tag{4.25}$$

then

$$\frac{f(z)}{s_m} = 1 - \frac{z^m}{d_{m-1}} \to 1 - \frac{1}{d_{m-1}} \quad \text{as } z \to 1^-,$$

which shows that the bound in (4.18) is the best possible for each $n \in \mathbb{N}$.

(iii) Just as in Part (ii) above, if we put

$$\begin{aligned} h_2(z) &= (1 + d_{m-1}) \left(\frac{s_m(z)}{f(z)} - \frac{d_{m-1}}{1 + d_{m-1}} \right) \\ &= 1 - \frac{(1 + d_{m-1}) \sum\limits_{k=m-1}^{\infty} a_k z^{k+1}}{1 + \sum\limits_{k=0}^{\infty} a_k z^{k+1}}, \end{aligned} \tag{4.26}$$

and make use of (4.22), we can deduce that

$$\left| \frac{h_2(z) - 1}{h_2(z) + 1} \right| \leq \frac{(1 + d_{m-1}) \sum\limits_{k=m-1}^{\infty} |a_k|}{2 - 2 \sum\limits_{k=0}^{m-2} |a_k| - (1 - d_{m-1}) \sum\limits_{k=m-1}^{\infty} |a_k|} \leq 1 \quad (z \in U),$$

which leads us immediately to the assertion (4.19) of Theorem 3.5.

The bound in (4.19) is sharp for each $m \in N$, with the extremal function $f(z)$ given by (4.25). The proof of Theorem 3.5 is thus completed.

5. **Convolution properties**

For the functions

$$f_j(z) = \frac{1}{z} + \sum_{k=p}^{\infty} |a_{k,j}|\, z^k \qquad (j = 1, 2; p \in \mathbb{N}), \tag{5.1}$$

we dnote by $(f_1 * f_2)(z)$ the Hadamard product (or convolution) of the functions $f_1(z$ and $f_2(z)$, that is,

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=p}^{\infty} |a_{k,1}|\, |a_{k,2}|\, z^k. \tag{5.2}$$

Theorem 5.1. *Let the functions* $f_j(z)\,(j = 1, 2)$ *defined by* (5.1) *be in the class* $\sum_{p,n}^{*}(\alpha, \beta, \gamma, \lambda)$. *Then* $(f_1 * f_2)(z) \in \sum_{p,n}^{*}(\delta, \beta, \gamma, \lambda)$, *where*

$$\delta = 1 - \frac{2\beta\gamma(1 - 2\lambda)(1 - \alpha)^2}{p(p + 2)^n[1 + \lambda(p - 1)](1 + 2\beta\gamma - \beta)}. \tag{5.3}$$

The result is sharp for the functions

$$f_j(z) = \frac{1}{z} + \frac{2\beta\gamma(1 - 2\lambda)(1 - \alpha)}{p(p + 2)^n[1 + \lambda(p - 1)](1 + 2\beta\gamma - \beta)} z^p \;\; (j = 1, 2; p \in \mathbb{N}; n \in \mathbb{N}_0). \tag{5.4}$$

Proof. Employing the technique used earlier by Schild and Silverman [14], we need to find the largest δ such that

$$\sum_{k=p}^{\infty} \frac{k(k + 2)^n[1 + \lambda(k - 1)](1 + 2\beta\gamma - \beta)}{2\beta\gamma(1 - 2\lambda)(1 - \delta)}\, |a_{k,1}|\, |a_{k,2}| \leq 1 \tag{5.5}$$

for $f_j(z) \in \sum_{p,n}^{*}(\alpha, \beta, \gamma, \lambda)(j = 1, 2)$. Since $f_j(z) \in \sum_{p,n}^{*}(\alpha, \beta, \gamma, \lambda)(j = 1, 2)$, we readily see that

$$\sum_{k=p}^{\infty} \frac{k(k + 2)^n[1 + \lambda(k - 1)](1 + 2\beta\gamma - \beta)}{2\beta\gamma(1 - 2\lambda)(1 - \alpha)}\, |a_{k,j}| \leq 1 \quad (j = 1, 2). \tag{5.6}$$

Therefore, by the Cauchy-Schwarz inequality , we obtain

$$\sum_{k=p}^{\infty} \frac{k(k + 2)^n[1 + \lambda(k - 1)](1 + 2\beta\gamma - \beta)}{2\beta\gamma(1 - 2\lambda)(1 - \alpha)} \sqrt{|a_{k,1}|\, |a_{k,2}|} \leq 1\,. \tag{5.7}$$

This implies that we need only to show that

$$\frac{|a_{k,1}|\,|a_{k,2}|}{(1-\delta)} \le \frac{\sqrt{|a_{k,1}|\,|a_{k,2}|}}{(1-\alpha)} \quad (k \ge p) \tag{5.8}$$

or , equivalently , that

$$\sqrt{|a_{k,1}|\,|a_{k,2}|} \le \frac{(1-\delta)}{(1-\alpha)} \quad (k \ge p). \tag{5.9}$$

Hence, by the inequality (5.7), it is sufficient to prove that

$$\frac{2\beta\gamma(1-2\lambda)(1-\alpha)}{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)} \le \frac{(1-\delta)}{(1-\alpha)} \quad (k \ge p). \tag{5.10}$$

It follows from (5.10) that

$$\delta \le 1 - \frac{2\beta\gamma(1-2\lambda)(1-\alpha)^2}{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)} \quad (k \ge p). \tag{5.11}$$

Now, defining the function $\varphi(k)$ by

$$\varphi(k) = 1 - \frac{2\beta\gamma(1-2\lambda)(1-\alpha)^2}{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)} \quad (k \ge p). \tag{5.12}$$

We see that $\varphi(k)$ is an increasing function of k. Therefore , we conclude that

$$\delta \le \varphi(p) = 1 - \frac{2\beta\gamma(1-2\lambda)(1-\alpha)^2}{p(p+2)^n[1+\lambda(p-1)](1+2\beta\gamma-\beta)}, \tag{5.13}$$

which evidently completes the proof of Theorem 4.1.

Using arguments similar to those in the proof of Theorem 4.1, we obtain the following result.

Theorem 5.2. *Let the function $f_1(z)$ defined by* (5.1) *be in the class* $\sum_{p,n}^{*}(\alpha,\beta,\gamma,\lambda)$. *Suppose also that the function $f_2(z)$ defined by (5.1) be in the class* $\sum_{p,n}^{*}(\zeta,\beta,\gamma,\lambda)$. *Then* $(f_1 * f_2)(z) \in \sum_{p,n}^{*}(\xi,\beta,\gamma,\lambda)$, *where*

$$\xi = 1 - \frac{2\beta\gamma(1-2\lambda)(1-\alpha)(1-\zeta)}{p(p+2)^n[1+\lambda(p-1)](1+2\beta\gamma-\beta)}\,. \tag{5.14}$$

The result is sharp for the functions $f_j(z)(j=1,2)$ given by

$$f_1(z) = \frac{1}{z} + \frac{2\beta\gamma(1-2\lambda)(1-\alpha)}{p(p+2)^n[1+\lambda(p-1)](1+2\beta\gamma-\beta)} z^p \quad (p \in \mathbb{N}; n \in \mathbb{N}_0), \tag{5.15}$$

and

$$f_2(z) = \frac{1}{z} + \frac{2\beta\gamma(1-2\lambda)(1-\zeta)}{p(p+2)^n[1+\lambda(p-1)](1+2\beta\gamma-\beta)} z^p \quad (p \in \mathbb{N}; n \in \mathbb{N}_0). \tag{5.16}$$

Theorem 5.3. *Let the functions $f_j(z)(j=1,2)$ defined by* (5.1) *be in the class* $\sum_{p,n}^{*}(\alpha,\beta,\gamma,\lambda)$. *Then the function $h(z)$ defined by*

$$h(z)=\frac{1}{z}+\sum_{k=p}^{\infty}(|a_{k,1}|^2+|a_{k,2}|^2)z^k \tag{5.17}$$

belongs to the class $\sum_{p,n}^{*}(\tau,\beta,\gamma,\lambda)$, *where*

$$\tau=1-\frac{4\beta\gamma(1-2\lambda)(1-\alpha)^2}{p(p+2)^n[1+\lambda(p-1)](1+2\beta\gamma-\beta)}. \tag{5.18}$$

This result is sharp for the functions $f_j(z)(j=1,2)$ given already by (5.4).

Proof. Noting that

$$\sum_{k=p}^{\infty}\frac{\{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)\}^2}{[2\beta\gamma(1-2\lambda)(1-\alpha)]^2}\,|a_{k,j}|^2$$

$$\leq(\sum_{k=p}^{\infty}\frac{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)}{2\beta\gamma(1-2\lambda)(1-\alpha)}\,|a_{k,j}|)^2\leq 1\ (j=1,2), \tag{5.19}$$

for $f_j(z)\in\sum_{p,n}^{*}(\alpha,\beta,\gamma,\lambda)(j=1,2)$, we have

$$\sum_{k=p}^{\infty}\frac{\{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)\}^2}{2\,[2\beta\gamma(1-2\lambda)(1-\alpha)]^2}(|a_{k,1}|^2+|a_{k,2}|^2)\leq 1\,. \tag{5.20}$$

Therefore, we have to find the largest τ such that

$$\frac{1}{(1-\tau)}\leq\frac{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)}{4\beta\gamma(1-2\lambda)(1-\alpha)^2}\ (k\geq p), \tag{5.21}$$

that is, that

$$\tau\leq 1-\frac{4\beta\gamma(1-2\lambda)(1-\alpha)^2}{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)}\ (k\geq p). \tag{5.22}$$

Now, defining a function $\Psi(k)$ by

$$\Psi(k)=1-\frac{4\beta\gamma(1-2\lambda)(1-\alpha)^2}{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)}\ (k\geq p). \tag{5.23}$$

We observe that $\Psi(k)$ is an increasing function of k. We thus conclude that

$$\tau\leq\Psi(p)=1-\frac{4\beta\gamma(1-2\lambda)(1-\alpha)^2}{p(p+2)^n[1+\lambda(p-1)](1+2\beta\gamma-\beta)}\ , \tag{5.24}$$

which completes the proof of Theorem 4.3.

Putting $n=\lambda=0$ in Theorem 4.3, we obtain:

Corollary 5.4. *Let the functions $f_j(z)(j = 1,2)$ defined by* (5.1) *be in the class $\sum_p^*(\alpha,\beta,\gamma)$. Then the function $h(z)$ defined by* (5.17) *belongs to the class $\sum_p^*(\tau,\beta,\gamma)$, where*

$$\tau = 1 - \frac{4\beta\gamma(1-\alpha)^2}{p(1+2\beta\gamma-\beta)}. \tag{5.25}$$

The result is sharp.

Remark 5.5. The result obtained by Cho et al. ([5] and [6]) is not correct. The correct result is given by Corollary 3.

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