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FRICTIONAL CONTACT PROBLEMS WITH NORMAL COMPLIANCE AND COULOMB'S LAW FOR NONLINEAR ELASTIC BODIES

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Abstract. The subject of this work is the study of a problem modeling the frictional contact between a non linear elastic body and a rigid foundation at the presence of rapel forces. First, we present variational formulation for this problem, after we indicate sufficient conditions in order to have the existence, the uniqueness and the Lipschitz continuous dependence of solution with respect to the data. Finally, we prove the dependence of the solution by the parameter θ. The proofs are based on results of topological degree theory as well as on convexity, monotonicity and fixed point arguments see [1].

1. **Introduction**

In this paper we consider perturbed quasivariational inequalities of the form

$$u \in V, \quad \langle Au, v-u\rangle_V + \langle Bu, v-u\rangle_{\mathbb{V}} + j(u,v) - j(u,u) \geq \langle f, v-u\rangle_V \quad \forall v \in V$$

where V denotes a real Hilbert space and $A : V \to V$ is a strongly monotone and Lipschitz continuous operator on V.

$$(h_1)\textbf{:} \begin{cases} a)\exists m > 0 \text{ such that } \langle Au - Av, u-v\rangle_V \geq m\,|u-v|_V^2 \quad \forall u, v \in V \\ b)\exists M > 0 \text{ such that } |Au - Av|_V \leq M\,|u-v|_V \quad \forall u, v \in V \end{cases}$$

Let $B : V \to V$, satisfies:

(h_2)**:** There exists $C \geq 0$ such that $\langle Bv, v\rangle_{\mathbb{V}} \geq -C\,|v|_V^2 \quad \forall v \in V$

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$$(h_3)\textbf{:}\ \begin{cases} \text{For every sequence } \{\eta_n\} \subset V \text{ such that } \eta_n \to \eta \in V\ , \\ \quad \text{then there exist a subsequence } \{\eta_{n'}\} \subset V \\ \qquad\qquad B\eta_{n'} \to B\eta \text{ strongly in } V. \end{cases}$$

(h_4)**:** $\langle Bu - Bv, v - u\rangle_{\mathbb{V}} < (m - \alpha)\,|u - v|_V^2 \quad \forall u, v \in V, u \neq v.$

(h_5)**:** $\exists \beta, 0 \leq \beta \leq (m - \alpha),\ \langle Bu - Bv, v - u\rangle_{\mathbb{V}} \leq \beta\,|u - v|_V^2 \quad \forall u, v \in V.$

The functional $j : V \times V \to \mathbb{R}$ satisfies

(h_6)**:** $j(\eta, .) : V \to \mathbb{R}$ is a convex functional on V, for all $\eta \in V$,

It is well known that there exists the directional derivative $j_2^{'}$ given by

(h_7)**:** $j_2^{'}(\eta, u; v) = \lim_{\lambda \to 0}\,[j(\eta, u + \lambda v) - j(\eta, v)] \quad \forall \eta, u, v \in V,$

We consider now the following assumptions:

$$(J_1)\textbf{:}\ \begin{cases} \text{For every sequence } \{u_n\} \subset V \text{ with } |u_n|_V \to \infty \\ \text{and every sequence } \{t_n\} \subset [0,1] \text{ one has} \\ \liminf_{n\to\infty} \left[\frac{1}{|u_n|_V^2} j_2^{'}(t_n u_n, u_n; -u_n)\right] < m - C \end{cases}$$

$$(J_2)\textbf{:}\ \begin{cases} \text{For every sequence } \{u_n\} \subset V \text{ with } |u_n|_V \to \infty \\ \text{and every bounded sequence } \{\eta_n\} \subset V \text{ one has} \\ \liminf_{n\to\infty} \left[\frac{1}{|u_n|_V^2} j_2^{'}(\eta_n, u_n; -u_n)\right] < m. \end{cases}$$

$$(J_3)\textbf{:}\ \begin{cases} \text{For every sequence } \{u_n\} \subset V \text{ and } \{\eta_n\} \subset V \text{ such that} \\ u_n \to u \in V,\ \eta_n \to \eta \in V \text{ and for every } v \in V \text{ then one has} \\ \limsup_{n\to\infty} [j(\eta_n, v_n) - j(\eta_n, u_n)] \leq j(\eta, v) - j(\eta, u). \end{cases}$$

(J_4)**:** $j(u, v) - j(u, u) + j(v, u) - j(v, v) < m\,|u - v|_V^2 \quad \forall u, v \in V, u \neq v$

(J_5)**:** $\quad j(u, v) - j(u, u) + j(v, u) - j(v, v) \leq \alpha\,|u - v|_V^2\ ,$

$\forall u, v \in V$,for some $\alpha \in \mathbb{R}$ with $\alpha < m$.

Theorem 1. *We consider the following problem :*

$$\langle Au, v - u\rangle_V + \langle Bu, v - u\rangle_{\mathbb{V}} + j(u, v) - j(u, u) \geq \langle f, v - u\rangle_V \quad \forall v \in V$$

Let $(h_1), (h_2)$ *and* (h_6) *hold.*

(1) Under the assumptions $(J_1), (J_2), (J_3), (J_5)$ and (h_3), the problem has at least a solution.

(2) Under the assumptions $(J_1), (J_2), (J_3), (J_5), (h_3)$ and (h_4),the problem has a unique solution. .

(3) Under the assumptions $(J_1), (J_2), (J_3), (J_5)$, (h_3) and (h_5), the problem has a unique solution $u = u(f)$ which depends Lipschitz continuously on $f \in V$ with the Lipschitz constant $(m - \alpha - \beta)^{-1}$, *i.e.*

$$|u(f_1) - u(f_2)|_V \leq \frac{1}{(m - \alpha - \beta)} |f_1 - f_2|_V \quad \forall f_1, f_2 \in V$$

Proof. It is based on results of topological degree theory as well as on convexity, monotonicity, compactness and fixed point arguments see [1].

Remark 1. *The coercivity conditions* $(J_1), (J_2)$ *and* (h_1) (a) *are needed in order to use the weakly sequential compactness property of the closed, bounded convex sets of* V*, see [1].*

2. **The elastic contact problem**

2.1. **Formulation of the mechanical problem and assumptions.** Let us consider an elastic, homogeneous isotrop body whose material particles occupy a bounded domain $\Omega \subset \mathbb{R}^n (n = 1, 2, 3)$ and whose boundary Γ, assumed to be sufficiently smooth is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that meas $\Gamma_1 > 0$.

We denote by u the displacement vector, σ represents the stress field and $\varepsilon(u)$ is the small strain tensor such that that $\varepsilon = (\varepsilon_{ij}) : H_1 \to \mathcal{H}$

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial u_j} + \frac{\partial u_j}{\partial u_i}\right)$$

where the spaces H_1 and $\mathcal{H}$ are defined below. The elastic constitutive law of the material is assumed to be

$$\sigma = F(\varepsilon(u), \theta)$$

In which F is a given nonlinear function, and θ is a parameter.

We assume that the body is clamped on Γ_1 and thus the displacement field vanishes there, that the surface tractions h act on Γ_2 and that the body rests on a rigid foundation on the part Γ_3 of the boundary and that the normal stress σ_ν satisfies the normal compliance condition:

$$\sigma_\nu = -p_\nu(u_\nu)$$

where $\nu = (\nu_i)$ represents the outward unit normal vector on $\Gamma_j, (j = 1, 2, 3)$, u_ν represents the normal displacement $(u_\nu = u.\nu)$, p_ν is a prescribed nonnegative function and u_ν when it is positive, represents the penetration of the body in the foundation. The associated friction law on Γ_3 is chosen as

$$\begin{cases} |\sigma_\tau| \le p_\tau(u_\nu) \\ |\sigma_\tau| < p_\tau(u_\nu) \Rightarrow u_\tau = 0 \\ |\sigma_\tau| = p_\tau(u_\nu) \Rightarrow \sigma_\tau = -\lambda u_\tau, \lambda \ge 0 \end{cases}$$

here τ is the tangent unit vector in the positive sense on Γ_j $(j = 1, 2, 3)$, p_τ is a nonnegative function, the so-called friction bound, u_τ denotes the tangential displacement $(u_\tau = u - u_\nu \nu)$ and σ_τ represents the tangential force on the contact boundary.

For example, we can consider

(1): $\quad p_\nu(r) = c_\nu(r_+)^{m_\nu}, p_\tau(r) = c_\tau r_+$

where $m_\nu \in]0, 1], c_\nu$ and c_τ are positives constants and $r_+ = \max\{0, r\}$.

Also, the friction law can be used with

(2): $\quad p_\nu = \mu p_\nu$ or $p_\tau = \mu p_\nu(1 - \alpha p_\nu)_+$

where $\mu > 0$ is a coefficient of friction and α is a small positive coefficient related to the wear and hardness of the surface.

2.2. **Position of the problem.** The mechanical problem may be formulated as follows:

Problem (P): Find a displacement field $u : \Omega \to \mathbb{R}^n$ and a stress field $\sigma : \Omega \to S_n$ such that :

(3): $\quad Div\ \sigma + f_0 = 0 \quad in\ \Omega$

(4): $\quad \sigma = F(x, \varepsilon(u), \theta) \quad in\ \Omega$

(5): $\quad u = 0 \quad on \quad \Gamma_1$

(6): $\quad \gamma(\sigma\nu + \Phi(x, u)) = h\ on\ \Gamma_2$

and on Γ_3,

$$\textbf{(7):}\quad \begin{cases} \sigma_\nu = -p_\nu(u_\nu) \\ |\sigma_\tau| \leq p_\tau(u_\nu) \\ |\sigma_\tau| < p_\tau(u_\nu) \Rightarrow u_\tau = 0 \\ |\sigma_\tau| = p_\tau(u_\nu) \Rightarrow \sigma_\tau = -\lambda u_\tau, \textit{pour un certain } \lambda \geq 0 \end{cases}$$

(6) is called rapel forces and it means that the surface tractions are proportional to the displacement. It's the case of building and matlats, ...).

To provide the variational analysis of the problem (P) we need additional notations. Let

$H = \left(\mathbb{L}^2(\Omega)\right)^n, \;\; H_1 = \left(H^1(\Omega)\right)^n.$

$\mathcal{H} = \left(\mathbb{L}^2(\Omega)\right)^{n\times n}, \;\; \mathcal{H}_1 = \left(H^1(\Omega)\right)^{n\times n}.$

The spaces H, H_1 and $\mathcal{H}$ are real Hilbert spaces endowed with the canonical inner products denoted by $\langle .,. \rangle_H$, $\langle .,. \rangle_{H_1}$ and $\langle .,. \rangle_{\mathcal{H}}$, respectively. The associate norms on H, H_1 and $\mathcal{H}$ are denoted by $|.|_H$, $|.|_{H_1}$ and $|.|_{\mathcal{H}}$, respectively.

In the study of the mechanical problem (P) we assume that the elasticity operator $F : \Omega \times S_n \times \mathbb{R}^M \to S_n$ satisfies

$$\textbf{(H}_1\textbf{):}\quad \begin{cases} (\mathbf{a}) \quad \exists m_F > 0 \text{ such that } \forall \varepsilon_1, \varepsilon_2 \in S_n, \forall \theta \in \mathbb{R}^M \\ \left(F(x,\varepsilon_1,\theta) - F(x,\varepsilon_2,\theta)\right).(\varepsilon_1 - \varepsilon_2) \geq m_F \left|\varepsilon_1 - \varepsilon_2\right|^2 \;\; a.e.in\ \Omega,. \\ (\mathbf{b}) \quad \exists L_1, L_2 > 0 \text{ such that } \forall \varepsilon_1, \varepsilon_2 \in S_2, \forall \theta_1, \theta_2 \in \mathbb{R}^M \\ \left|F(x,\varepsilon_1,\theta_1) - F(x,\varepsilon_2,\theta_2)\right| \leq L_1 \left|\varepsilon_1 - \varepsilon_2\right| + L_2 \left|\theta_1 - \theta_2\right| \;\; a.e.in\ \Omega, \\ (\mathbf{c}) x \to F(x,\varepsilon,\theta) \text{ is measurable function with respect to the} \\ \text{Lebesgue measure } a.e.in\ \Omega, \forall \varepsilon \in S_n, \forall \theta \in \mathbb{R}^M \\ (\mathbf{d}) \quad F(x, 0_n, 0_M) = 0_n. \end{cases}$$

We assume that the forces and the tractions have the regularity

$$\textbf{(H}_2\textbf{):}\qquad f_0 \in H = \mathbb{L}^2(\Omega)^n, \;\; h \in \mathbb{L}^2(\Gamma_2)^n,$$

also,

$$\textbf{(H}_3\textbf{):}\qquad \theta \in \mathbb{L}^2(\Omega)^M$$

The function Φ is defined by:

$$\Phi : \Gamma_2 \times \mathbb{R}^n \to \mathbb{R}^n$$

such that

$$\textbf{(H}_4\textbf{):} \begin{cases} (\mathbf{a})\ \exists m_\Phi > 0 \text{ such that} \\ (\Phi(x,u_1) - \Phi(x,u_2)) \,.\, (u_1 - u_2) \geq m_\Phi \left|u_1 - u_2\right|^2 \\ \text{ a.e. in } \ \Gamma_2, \forall u_1, u_2 \in \mathbb{R}^n \\ (\mathbf{b})\ \exists L_\Phi > 0 \text{ such that} \\ \left|\Phi(x,u_1) - \Phi(x,u_2)\right| \leq L_\Phi \left|u_1 - u_2\right| \ a.e.in\, \Gamma_2, \forall u_1, u_2 \in \mathbb{R}^n \\ (\mathbf{c})\ x \mapsto \Phi(x,u) \text{ is measurable function with respect to the} \\ \quad \text{Lebesgue measure } a.e.in\ \Gamma_2, \forall u \in \mathbb{R}^n. \\ (\mathbf{d})\ \Phi(x, 0_n) = 0_n \end{cases}$$

We also assume that the normal compliance functions satisfy the following hypothesis for $r = \nu, \tau$:

$$\textbf{(H}_5\textbf{):} \begin{cases} (\mathbf{a})\ p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+ \text{ such that} \\ \qquad p_r(.,r) \text{ is Lebesgue measurable on } \Gamma_3,\ \forall r \in \mathbb{R} \\ (\mathbf{b}) \text{ The mapping } p_\tau(.,r)\ = 0 \text{ for } r \leq 0; \\ (\mathbf{c}) \text{ There exists an } L_r > 0 \text{ such that} \\ \left|p_r(x,r_1) - p_r(x,r_2)\right| \leq L_r \left|r_1 - r_2\right|, \forall r_1, r_2 \in \mathbb{R},\ a.e.on\, \Gamma_3, \end{cases}$$

$(H_5^{'})$**:** $(p_\nu(x,r_1) - p_\nu(x,r_2))\,.\,(r_1 - r_2) \geq 0, \forall r_1, r_2 \in \mathbb{R},\ a.e.on\, \Gamma_3,$

Remark 2. *Certainly the functions defined in* (1) *satisfy the conditions* (H_5) *and* $(H_5^{'})$. *Also, if* p_ν *defined in* (2) *is Lipschitz then the conditions* (H_5) *is satisfied.*

Using the hypothesis $(H_5)(b)$ and (c) it follows that:

(8): $\quad \left|p_r(x,t)\right| \leq L_\tau \left|t\right|, \forall t \in \mathbb{R}, a.e.on\, \Gamma_3.$

Remark 3. *Using* (H_1) *we find that for all* $\tau \in \mathcal{H}$ *the function* $x \to F(x, \tau(x), \theta(x))$ *belongs to* $\mathcal{H}$ *and hence we may consider* $F(.,\theta)$ *as an operator defined on* $\mathcal{H}$ *with range in* $\mathcal{H}$ *by:* $F(.,\theta) : \mathcal{H} \to \mathcal{H}$

$$F(\varepsilon, \theta)(x) = F(x, \varepsilon(x), \theta(x))\ a.e.in\ \Omega\ \forall \varepsilon \in \mathcal{H}$$

Moreover, $F(.,\theta)$ *is a strongly monotone Lipschitz continuous operator:*

(9): $\exists L_1 > 0 : |F(\varepsilon_1, \theta) - F(\varepsilon_2, \theta)|_{\mathcal{H}} \leq L_1 |\varepsilon_1 - \varepsilon_2|_{\mathcal{H}}$.

(10): $\langle F(\varepsilon_1, \theta) - F(\varepsilon_2, \theta), \varepsilon_1 - \varepsilon_2 \rangle_{\mathcal{H}} \geq m_F |\varepsilon_1 - \varepsilon_2|^2_{\mathcal{H}}$

The inequality (9) is a particular case of

(11): $\exists L_1, L_2 > 0 : |F(\varepsilon_1, \theta_1) - F(\varepsilon_2, \theta_2)|_{\mathcal{H}} \leq L_1 |\varepsilon_1 - \varepsilon_2|_{\mathcal{H}} + L_2 |\theta_1 - \theta_2|_{\mathbb{L}^2(\Omega)^M}$.

Therefore $F(., \theta)$ is invertible and its inverse $F^{-1}(., \theta) : \mathcal{H} \to \mathcal{H}$ is also a strongly Lipschitz continuous operator.

Remark 4. *The assumptions* (H_4) *allows us to consider the operator denoted by* $\Phi : H \to \mathbb{L}^2(\Gamma_2)^n$

$$\Phi(v)(x) = \Phi(x, v(x)) \text{ a.e. in } \Gamma_2 \ \forall v \in H$$

Moreover, Φ *is a strongly monotone Lipschitz continuous operator and therefore* Φ *is invertible and its inverse* $\Phi^{-1} : \mathbb{L}^2(\Gamma_2)^n \to H$ *is also a strongly Lipschitz continuous operator.*

We denote by V the closed subspace of H_1 given by

(12): $V = \{ v \in H_1 / \gamma v = 0 \ sur \ \Gamma_1 \}$

Since meas $\Gamma_1 > 0$, Korn's inequality holds:

$$|\varepsilon(v)|_{\mathcal{H}} \geq C |v|_{H_1} \ \ \forall v \in V$$

C denotes a strictly positive generic constant which may depend on Ω, Γ_1, Γ_2,Γ_3 and F.

We endow V with the inner product defined by

(13): $\langle u, v \rangle_V = \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}} \quad \forall u, v \in V$

and let $|.|_V$ the associated norm. It follows from the Korn's inequality that $|.|_V$ and $|.|_{H_1}$are equivalent norms on V. Therefore, $(V, |.|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, Korn's inequality and (13) we have a constant C_0 depending on Ω, Γ_1 et Γ_3 such that:

(14): $|v|_{\mathbb{L}^2(\Gamma_3)^n} \leq C_0 |v|_V \ , \forall v \in V.$

The functional $v \rightarrow \langle f, v\rangle_H + \langle h, \gamma v\rangle_{\mathbb{L}^2(\Gamma_2)^n}, \forall v \in V$ is linear and continue on V; it results, by using the Riesz Fréchet theorem, the existence of an element $f \in V$ such that

(15): $\langle f, v\rangle_V = \langle f_0, v\rangle_H + \langle h, \gamma v\rangle_{\mathbb{L}^2(\Gamma_2)^n} \forall v \in V.$

For all fixed w in V and for all fixed θ in $\mathbb{L}^2(\Omega)^M$, the functional defined on V by: $v \rightarrow \langle F\varepsilon(w), \theta), \varepsilon(v)\rangle_{\mathcal{H}} + \langle \Phi(w), v\rangle_{\mathbb{L}^2(\Gamma_2)^n}$ is a continuous linear functional on V. Then using Riesz-Fréchet's theorem, there exists an element $A_\theta w \in V$ such that:

(16): $\langle A_\theta w, v\rangle_V = \langle F\varepsilon(w), \theta), \varepsilon(v)\rangle_{\mathcal{H}} + \langle \Phi(w), v\rangle_{\mathbb{L}^2(\Gamma_2)^n} \forall v \in V.$

Let $B : V \rightarrow V$ defined by

(17): $\langle Bu, v\rangle_V = \int\limits_{\Gamma_3} p_\nu(u_\nu - g)v_\nu ds, \forall u, v \in V.$

and let $j : V \times V \rightarrow \mathbb{R}$ be the functional

(18): $j(u, v) = \int\limits_{\Gamma_3} p_\tau(u_\nu - g) \left|v_\tau\right| ds, \forall u, v \in V.$

Using the conditions $(H_5)(b), (c)$ it follows that for all $v \in V$ the functions

(19): $x \longmapsto p_r(x, v(x)),\ (r = \nu, \tau),$

belong to $\mathbb{L}^2(\Gamma_3)$ and hence the integrals in (17) and (18) are well defined.

2.3. **Variational Formulation.**

Theorem 2. *If* $(u, \sigma) \in H_1 \times \mathcal{H}_1$ *are sufficiently smooth functions satisfying* $(3) - (7)$ *then*

(20): $$u \in V : \langle \sigma, \varepsilon(v) - \varepsilon(u)\rangle_{\mathcal{H}} + \langle \Phi(x, u), v - u\rangle_{\mathbb{L}^2(\Gamma_2)^n} + \langle Bu, v - u\rangle_{\mathbb{V}}$$
$$+ j(u, v) - j(u, u) \succeq \langle q, v - u\rangle_V, \forall v \in V.$$

Proof. Let $u, v \in U_{ad}$, by using the Green formula we obtain:

$$\begin{aligned} \langle f_0, v - u\rangle_H &= -\langle Div\sigma, \varepsilon(v) - \varepsilon(u)\rangle_{\mathcal{H}} \\ &= \langle \sigma, \varepsilon(v) - \varepsilon(u)\rangle_{\mathcal{H}} - \langle \sigma\nu, \gamma(v - u)\rangle_{H'_\Gamma \times H_\Gamma} \end{aligned}$$

but

$$\begin{aligned}\langle \sigma\nu, \gamma(v-u)\rangle_{H'_\Gamma \times H_\Gamma} &= \int\limits_{\Gamma} \sigma\nu(v-u)ds = \sum_{j=1}^{3} \int\limits_{\Gamma_j} \sigma\nu^j(v-u)ds \\ &= \int\limits_{\Gamma_2} h(v-u)ds - \int\limits_{\Gamma_2} \Phi(u)(v-u)ds + \int\limits_{\Gamma_3} \sigma\nu^1(v-u)ds\end{aligned}$$

Let $(u_\nu, u_\tau), (v_\nu, v_\tau)$ and $(\sigma_\nu, \sigma_\tau)$ the components of the vectors u, v and $\sigma\nu$ in the orthonorm system (ν, τ). From (5) and (7) it results that we obtain on Γ_3

$$\begin{aligned}\sigma\nu(v-u) &= \sigma_\nu(v_\nu - u_\nu) + \sigma_\tau(v_\tau - u_\tau) \\ &= -p_\nu(u_\nu)(v_\nu - u_\nu) + \sigma_\tau(v_\tau - u_\tau)\end{aligned}$$

Then

$$\begin{aligned}\langle f_0, v-u\rangle_H + \langle h, \gamma(v-u)\rangle_{\mathbb{L}^2(\Gamma_2)^n} &= \langle \sigma, \varepsilon(v-u)\rangle_{\mathcal{H}} + \langle \Phi(u), v-u\rangle_{\mathbb{L}^2(\Gamma_2)^n} + \\ &+ \int\limits_{\Gamma_3} p_\nu(u_\nu)(v_\nu - u_\nu)ds - \int\limits_{\Gamma_3} \sigma_\tau(v_\nu - u_\nu)ds\end{aligned}$$

So, by (15) we obtain:

$$\begin{aligned}\langle f, v-u\rangle_H &= \langle \sigma, \varepsilon(v) - \varepsilon(u)\rangle_{\mathcal{H}} + \langle \Phi(u), v-u\rangle_{\mathbb{L}^2(\Gamma_2)^n} + \\ &+ \langle Bu, v-u\rangle_{\mathbb{V}} - \int\limits_{\Gamma_3} \sigma_\tau(v_\nu - u_\nu)ds\end{aligned}$$

Using (7) it results

$$\begin{aligned}-\sigma_\tau(v_\nu - u_\nu) &= -\sigma_\tau v_\nu + \sigma_\tau u_\nu, \\ -\sigma_\tau v_\nu &\le p_\nu(u_\nu)\,|v_\nu| \\ \sigma_\tau u_\nu &= -p_\nu(u_\nu)\,|u_\nu|\end{aligned}$$

It follows that

$$\begin{aligned}\langle f, v-u\rangle_H &\le \langle \sigma, \varepsilon(v) - \varepsilon(u)\rangle_{\mathcal{H}} + \langle \Phi(u), v-u\rangle_{\mathbb{L}^2(\Gamma_2)^n} + \\ &+ \langle Bu, v-u\rangle_{\mathbb{V}} + j(u,v) - j(u,u)\end{aligned}$$

it results

$$\langle A_\theta u, v-u\rangle_V + \langle Bu, v-u\rangle_{\mathbb{V}} + j(u,v) - j(u,u) \geq \langle f, v-u\rangle_H$$

and using (4), yields to the following variational formulation of the problem (P):

Find a displacement field $u : \Omega \to H_1$, such that

(21): $u \in V,\ \langle F(\varepsilon(u),\theta), \varepsilon(v)-\varepsilon(u)\rangle_{\mathcal{H}} + \langle \Phi(u), v-u\rangle_{\mathbb{L}^2(\Gamma_2)^n} +$

$$\langle Bu, v-u\rangle_{\mathbb{V}} + j(u,v) - j(u,u) \geq \langle f, v-u\rangle_V, \forall v \in V.$$

Remark 5. *If u is a solution (21) then (u,σ) satisfy the mechanical problem (P), where σ is given by (4).*

2.4. **Existence and uniqueness results.** Let $L_0 > 0$ a constant such that:

$$L_0 = \frac{m_F + m_\Phi}{C_0^2}$$

Theorem 3. *Assume that $(H_1)-(H_5)$ and $L_\tau < L_0$ hold. Then*

1) the variational problem (P_V) has at least a solution $u \in V$

2) in addition to $(H_5^{'})$ the problem (P_V) has a unique solution which depends Lipschitz continuously on f.

Proof. The proof fellows from the abstract result provided by theorem1. It will be carried out in several steps. We are going to prove that if the hypothesis $(H_1)-(H_5)$ hold then the conditions $(h_1)-(h_6)$, $(j_2),(j_3)$ and (j_5) will be satisfied.

Lemma 4. *We suppose that $(H_1)-(H_5)$ hold, the we obtain that the conditions (h_1) and (h_6) are satisfied.*

Proof. **1)** We see that (16) et $(H_5)(b),(c)$ give $(h_1)(a)$ with $m = m_F + m_\Phi$ and $(h_1)(b)$ with $M = L_1 + L_\Phi$.

2) Moreover, from (18) we deduce that $j(u,.)$ is convex $\forall u \in V$.

Using (8) and (18) we obtain that $\forall u, v_1, v_2 \in V$

$$|j(u,v_1) - j(u,v_2)| \leq L_\tau\, |u_\nu|_{\mathbb{L}^2(\Gamma_2)}\, |v_1 - v_2|_{\mathbb{L}^2(\Gamma_2)^d} \leq C(u)\, |v_1 - v_2|_{\mathbb{V}}$$

Then $j(u,.)$ is continuous on V for all $u \in V$.

Lemma 5. *Under assumptions* $(H_4),(H_5)$*, and* $L_\tau < L_0$*, The functional* J *satisfies the conditions* $(J_2),(J_3)$ *and* (J_5)*.*

Proof. **1)** Using (18) it results that ,$\forall\ \eta, u \in V, \forall \lambda \in]0,1[$:

$$\frac{1}{\lambda}\left[j(\eta, u - \lambda u) - j(\eta, u)\right] = \int\limits_{\Gamma_3} p_\tau(u_\nu)(-\left|u_\tau\right|)ds \leq 0$$

Then $j_2^{'}(\eta, u; -u) \leq 0 \quad \forall\ \eta, u \in V$. It follows that for every sequence $\{u_n\}$ and $\{\eta_n\}$ in V, we have

$$\lim\inf_{n\to\infty} \frac{1}{|u_n|_V^2}\left[j_2^{'}(\eta_n, u_n; -u_n)\right] \leq 0 < m$$

and we deduce that J satisfies (J_2).

2)Let now $\{u_n\} \subset V$, $\{\eta_n\} \subset V$ be two sequences such that $u_n \rightharpoonup u$ and $\eta_n \rightharpoonup \eta$ weakly in V.

Using the compactness property of the trace map of $H^1(\Omega)$ in $\mathbb{L}^2(\Gamma)$ it follows that

(30): $u_n \to u$ in $\mathbb{L}^2(\Gamma_3)$ strongly for a subsequence,

and

(31): $\eta_n \to \eta$ in $\mathbb{L}^2(\Gamma_3)$ strongly for a subsequence,

Using $(H_5)(c)$ and (31) we have

(32): $p_\tau(., \eta_{n\nu} - g) \to p_\tau(., \eta_\nu - g)$ in $\mathbb{L}^2(\Gamma_3)$ strongly for a subsequence,

Therefore we deduce that

(33): $j(\eta_n, v) \to j(\eta, v)\ \forall v \in V$.

Also, (30) gives

(34): $|u_{n\tau}| \to |u_\tau|$ in $\mathbb{L}^2(\Gamma_3)$ strongly for a subsequence,

So, by $(18),(32)$ and (34) we obtain

(35): $j(\eta_n, u_n) \to j(\eta, v)$ for a subsequence.

Using (33) and (35) we have for all $v \in V$,

$$\lim\sup_{n\to\infty} \left[j(\eta_n, v) - j(\eta_n, u_n)\right] = j(\eta, v) - j(\eta, u)$$

The (J_3) is satisfied.

3) Let $u, v \in V$. Using $(H_5)(c)$ and (18) we obtain:

$$\begin{aligned} j(u,v) - j(u,u) + j(v,u) - j(v,v) &\leq \int\limits_{\Gamma_3} |p_\tau(u_\nu) - p_\tau(v_\nu)|\, |v_\tau - u_\tau|\, ds \\ &\leq L_\tau\, |u - v|^2_{\mathbb{L}^2(\Gamma_3)} \end{aligned}$$

Using now (14) in the previous inequality we deduce

$$j(u,v) - j(u,u) + j(v,u) - j(v,v) \leq L_\tau C_0^2\, |u - v|^2_{\mathbb{L}^2(\Gamma_3)}$$

Then (J_5) is satisfied with $\alpha = L_\tau C_0^2,\ L_\tau < L_0$.

Lemma 6. *Under assumptions (H_4) and (H_5) we deduce that (J_1), (h_2) and (h_3) are satisfied, and under assumptions $(H_4), (H_5)$ and $(H_5^{'})$ we obtain the condition (h_5).*

Proof. **1)** Using (17) we obtain $\langle Bu, v\rangle_V = \int\limits_{\Gamma_3} p_\nu(u_\nu) v_\nu ds \quad \forall v \in V$.

Let $v_\nu \geq 0$, since $p_\nu \geq 0$, it results

$$\langle Bv, v\rangle_V \geq 0$$

then (h_2) is satisfied with $C = 0$.

2) By using (18) we have for all $\eta, u \in V$, $j_2^{'}(\eta, u; -u) \leq 0$. which results (J_1) with $C = 0$.

3)Let now $\eta_n \to \eta$ weakly in V. Using the compactness property of the trace map of $H^1(\Omega)$ in $\mathbb{L}^2(\Gamma)$ it follows that $\eta_n \to \eta$ in $\mathbb{L}^2(\Gamma_3)$ strongly for a subsequence,

It results from (17)

$$\langle Bu_1 - Bu_2, v\rangle_V = \langle p_\nu(u_{1\nu}) - p_\nu(u_{2\nu}), v_\nu\rangle_{\mathbb{L}^2(\Gamma_3)}\, \forall u_1, u_2, v \in V$$

Taking $v = Bu_1 - Bu_2$ in the previous equality, we have

$$\begin{aligned} |v|^2_V &\leq |p_\nu(u_{1\nu}) - p_\nu(u_{2\nu})|_{\mathbb{L}^2(\Gamma_3)} . |v|_{\mathbb{L}^2(\Gamma_3)} \\ &\leq C\, |p_\nu(u_{1\nu}) - p_\nu(u_{2\nu})|_{\mathbb{L}^2(\Gamma_3)} . |v|_{\mathbb{V}} . \end{aligned}$$

Then

(36): $|Bu_1 - Bu_2| \leq C\, |p_\nu(u_{1\nu}) - p_\nu(u_{2\nu})|_{\mathbb{L}^2(\Gamma_3)}$.

So, by (H_5) we obtain

(37): $p_\nu(\eta_{n\nu} - g) \to p_\nu(\eta_\nu - g)$ in $\mathbb{L}^2(\Gamma_3)$ strongly for a subsequence.

Finally, (36) and (37) give $B\eta_n \to B\eta$ in V strongly for a subsequence.

4)Using (17) and $(H_5^{'})$ it follows that (h_5) is satisfied for $\beta = 0 : \forall u_1, u_2 \in V$:

(38): $\langle B, u_1 - Bu_2, u_2 - u_1\rangle_V = \langle p_\nu(u_{1\nu}) - p_\nu(u_{2\nu}), u_{2\nu} - u_{1\nu}\rangle_{\mathbb{L}^2(\Gamma_3)} \leq 0.$

Proof of theorem 3.

The proof is based on the application of the theorem1. It follows by using the lemma 4, lemma 5 and lemma 6.

3. **The dependence of the solution on the parameter**

Theorem 7. *under the assumptions* $(H_1)-(H_5)$, *let* $(u_i, \sigma_i), (i = 1, 2)$ *the variational solution of the problem (P) associée to the parameter* θ_i *such that* $\theta_i \in \mathbb{L}^2(\Omega)^M$ *is satisfied. Then there exists a positive constant* $C > 0$ *which is depend to* Ω, Γ_1 *and* Γ *such that:*

$$|u_1 - u_2|_{H_1} + |\sigma_1 - \sigma_2|_{H_1} \leq C\, |\theta_1 - \theta_2|_{\mathbb{L}^2(\Omega)^M}$$

Proof. Let $(u_i, \sigma_i), (i = 1, 2)$, the variational solutions of the problem (P).

$$\langle \sigma_i, \varepsilon(v - u_i)\rangle_{\mathcal{H}} + \langle \Phi(u_1), v - u_i\rangle_{\mathbb{L}^2(\Gamma_2)^n} + \langle Bu_i, v - u_i\rangle_{\mathbb{V}} +$$

$$+j(u_i, v) - j(u_i, u_i) \geq \langle f, v - u_i\rangle_H$$

Where $v = u_2$ for $i = 1$, and $v = u_1$ for $i = 2$.

$$\langle \sigma_1, \varepsilon(u_2 - u_1)\rangle_{\mathcal{H}} + \langle \Phi(u_1), u_2 - u_1\rangle_{\mathbb{L}^2(\Gamma_2)^n} + \langle Bu_1, u_2 - u_1\rangle_{\mathbb{V}} +$$

$$+j(u_1, u_2) - j(u_1, u_1) + \succeq \langle f, u_2 - u_1\rangle_V$$

and

$$\langle \sigma_2, \varepsilon(u_1 - u_2)\rangle_{\mathcal{H}} + \langle \Phi(u_2), u_1 - u_2\rangle_{\mathbb{L}^2(\Gamma_2)^n} + \langle Bu_2, u_1 - u_2\rangle_{\mathbb{V}} +$$

$$+j(u_2, u_1) - j(u_2, u_2) + \succeq \langle f, u_1 - u_2\rangle_V$$

it follows that

$$\langle \sigma_1 - \sigma_2, \varepsilon(u_2 - u_1)\rangle_{\mathcal{H}} + \langle \Phi(u_1) - \Phi(u_2), u_2 - u_1\rangle_{\mathbb{L}^2(\Gamma_2)^n} +$$

$$+ \langle Bu_1 - Bu_2, u_2 - u_1\rangle_{\mathbb{V}} + j(u_1, u_2) - j(u_1, u_1) + j(u_2, u_1) - j(u_2, u_2) \succeq 0$$

Then, using (j_5), we deduce that

$$j(u_1,u_2)-j(u_1,u_1)+j(u_2,u_1)-j(u_2,u_2)\le \alpha\left|u_1-u_2\right|_V^2 \;,\; \alpha<m$$

by (33), we obtain that:

$$\langle \sigma_1-\sigma_2,\varepsilon(u_2-u_1)\rangle_{\mathcal{H}}+\langle \Phi(u_1)-\Phi(u_2),u_2-u_1\rangle_{\mathbb{L}^2(\Gamma_2)^n}+\alpha\left|u_1-u_2\right|_V^2\ge 0$$

it follows that

$$\langle \sigma_1-F(\varepsilon(u_2),\theta_1),\varepsilon(u_2-u_1)\rangle_{\mathcal{H}}+\langle F(\varepsilon(u_2),\theta_1)-\sigma_2,\varepsilon(u_2-u_1)\rangle_{\mathcal{H}}+$$

$$\langle \Phi(u_1)-\Phi(u_2),u_2-u_1\rangle_{\mathbb{L}^2(\Gamma_2)^n}+\alpha\left|u_1-u_2\right|_V^2\ge 0$$

Then

$$\left|\langle F(\varepsilon(u_2),\theta_1)-\sigma_1,\varepsilon(u_2-u_1)\rangle_{\mathcal{H}}+\langle \Phi(u_2)-\Phi(u_1),u_2-u_1\rangle_{\mathbb{L}^2(\Gamma_2)^n}\right|$$

$$\le\left|\langle F(\varepsilon(u_2),\theta_1)-\sigma_2,\varepsilon(u_2-u_1)\rangle_{\mathcal{H}}\right|+\alpha\left|u_1-u_2\right|_V^2$$

Using the Cauchy-Schwartz inequality and $(H_1)(b)$on the right member of the previous inequality, and $(H_1)(a),(H_4)(a)$ and Korn's inequality on the left member, we obtain that

$$m\left|u_1-u_2\right|_V^2\le cL_2\left|\theta_1-\theta_2\right|_{\mathbb{L}^2(\Omega)^M}\left|u_1-u_2\right|_V+\alpha\left|u_1-u_2\right|_V^2$$

Then

$$(m-\alpha)\left|u_1-u_2\right|_V\le K\left|\theta_1-\theta_2\right|_{\mathbb{L}^2(\Omega)^M}\;,\;\text{ where } K \text{ is a constant} >0$$

Since $(m-\alpha)>0$, then there exists a constant $C>0$ such that

(39): $\left|u_1-u_2\right|_{H_1}\le C\left|\theta_1-\theta_2\right|_{\mathbb{L}^2(\Omega)^M}$

Other way, we have:

$$\left|\sigma_1-\sigma_2\right|_{\mathcal{H}_1}=\left|\sigma_1-\sigma_2\right|_{\mathcal{H}}=\left|F(\varepsilon(u_1),\theta_1)-F(\varepsilon(u_2),\theta_2)\right|_{\mathcal{H}}$$

Using $(H_1)(b)$ and (34) we obtain that

(40): $\left|\sigma_1-\sigma_2\right|_{\mathcal{H}_1}\le C\left|\theta_1-\theta_2\right|_{\mathbb{L}^2(\Omega)^M}$

The wanted inequality is now a consequence of (39) and (40)

This theorem prove well the dependence of the solution on the parameter θ and this result is very important from the mechanical point of view because it prove that small perturbations on the parameter θ gives small perturbations on the solution (u, σ) of the problem without frisher.

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