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# INFEASIBLE PRIMAL-DUAL ALGORITHM FOR MINIMIZING CONVEX QUADRATIC PROBLEMS

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Abstract. Problems with a convex quadratic objective function and linear constraints are important in their own right, and they also arise as sub problems in Methods for general constrained optimization, such as sequential quadratic Programming and augmented Lagrangian methods.

In this paper, we propose and implement an infeasible primal-dual algorithm in order to minimize a convex quadratic function subject to bounded and linear equality constraints.

Preliminary experimentations are particularly encouraging.

**Key words :** feasible interior points methods, convex quadratic Programming, infeasible interior points methods.

## 1. Introduction

Interior point's methods are recognized to be efficient for solving many optimization problems. However, finding a strictly feasible initial point (phase1) is difficult.

In theory, we can overcome this difficulty by introducing some artificial variables and by transforming the problem in a new one into a space of superior dimension. This transformation requires using parameters in general (unknown) to big values as in the approach of "Big M" in linear programming. Inconveniences of this approach are known:

(1) One does not know if the size of these large parameters values risk to destabilize the algorithm.
(2) The reformulation can impair the structure of the original problem because we add lines and columns. Because of these failings, this approach is not very welcomed, or is carefully used. Considerable research efforts are dedicated to initializing interior point's methods.

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Several approaches are proposed in which the variant phase1 and phase2 and where no large parameter is used. The artificial variables are introduced through lines and supplementary columns. The relative process to this approach spread between 1986 and 1991. All these works main objective consisted in elaborating algorithms that do not necessarily start inside the feasible domain (of the original problem) along with theoretical properties, as the polynomial complexity. Researchers did not globally aim at the numeric aspect. So, in these methods, the initial point is not necessary but transforming the problem is unavoidable.

Efforts of research are oriented towards numeric performances. About this, a set of practical variants is proposed with the comparative numeric tests from 1989 to 1992. All these algorithms do not require feasible initial point transformation but start from any positive point and tempt to achieve feasibility and optimality in a simultaneous manner.

These algorithms are called infeasible interior point's methods; the most part of these algorithms is that of Newton type. This last class is the object of our study. We concentrated our efforts on methods favoured by a rich theory and also by a lot of numeric subtleties. Indeed, the latter starts phase 2 directly.

Regarding this, several researchers like Zhang, and all authors of principalsŠ relative development of these methods regarding the linear programming, think that these algorithms are more effective. These subjects seem to be very logical. On one hand, phase 1 is eliminated and on the other hand, phase 2 iteration does not defer too much from the feasible case. The preliminary study that we did stimulates of the numeric behaviour of the convex quadratic programming development.

## 2. General presentation of the convex quadratic program

A convex quadratic program with constraints means optimization problem, in which the objective is a convex quadratic form. The constraints are linear.

Without loss of generalities, we can write it as follwos:

$$(QP)\begin{cases} min\ c^t x + \frac{1}{2}x^t Qx \\ Ax = b \\ x \succeq 0 \end{cases}$$

where $Q$ is a $n \times n$ matrix assumed to be positive semidefinite, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ and $A$ is a $m \times n$ matrix of full rank.

The dual of $(QP)$ is:

$$(QD)\begin{cases} max\ b^t y - \frac{1}{2}x^t Qx \\ A^t y + z - Qx = c \\ z \geq 0, y \in \mathbb{R}^m \end{cases}$$

where $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$.

We impose the following assumptions:

- $(\mathbf{H1})$ : $S_{int} = \{x \in \mathbb{R}^n \ / \ Ax = b, x \rangle 0\} \neq \Phi$ interior feasible solutions of (P) is non-empty;
- $(\mathbf{H2})$ :$T_{int} = \{(y, z) \in \mathbb{R}^m \times \mathbb{R}^n \ / \ A^t y + z - Qx = c, z \rangle 0\} \neq \Phi$ interior feasible solutions of (D) is non-empty.

These assumptions are often used to develop the interior pointŠs algorithms.

2.1. **Principle.** Most of the new interior pointŠs methods are motivated by the the logarithmic barrier function technique of Frisch (1955) to problem $(QP)$ application.

Indeed, to the problem $(QP)$, one associates the problem gate non linear next one:

$$(QP_\mu) \begin{cases} min \ c^t x + \frac{1}{2} x^t Q x - \mu \sum\limits_{i=1}^{n} ln \ x_i = f_\mu(x) \\ Ax = b \\ x \rangle 0 \end{cases}$$

ă The principle of these methods is to solve the system of Karuch-Kuhn-Tucker (KKT) partner to the problem $(QP_\mu)$ by the method of damped Newton, while leaving from any positive point which is not necessarily feasible. The resolution of $(QP_\mu)$ is equivalent at that of $(QP)$ with that if $x^*(\mu)$ is an optimal solution of $(QP_\mu)$ then $x^* = \lim\limits_{\mu \to 0} x^*(\mu)$ is an optimal solution of $(QP)$. To achieve feasibility and optimality we introduce a merit function defined by:

$$\boldsymbol{\phi}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{x}^t \mathbf{z} + \mathbf{r}(\mathbf{x}, \mathbf{y}, \mathbf{z})$$

where $r(x, y, z) = \|Ax - b\| + \|-Qx + A^t y + z - c\|$.

It is clear that $r$ measure feasibility and $x^t z$ (duality gap ) control the optimality. The idea is to make the value of this function towards zero during iterations.

2.2. **Resolution of** $(P_\mu)$**.** $x$ is an optimal solution of $(P_\mu)$ if an only if there is $y \in \mathbb{R}^m$ such that:

$$(\mathbf{1}) \begin{cases} c - \mu X^{-1} e - A^t y + Qx = 0 \\ Ax = b \\ x > 0 \end{cases}$$

where: $X^{-1} = diag(1/x_i)$. We apply the method of damped Newton to solve the system of nonlinear equations $(\mathbf{1})$ from an infeasible starting point (which is not necessarly feasible) $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$, $(x, z) \rangle 0$ and $\mu = x^t z / n \rangle 0$, we gets the following system:

$$\begin{cases} X\Delta z + Z\Delta x = -XZe + \sigma\mu e \qquad 0 < \sigma < 1, \qquad Z = diag(z_i) \\ A\Delta x \ = b - Ax \\ -Q\Delta x + A^t\Delta y + \Delta z = c - A^t y + Qx - z \end{cases}$$

where the solution is: $(\Delta x, \Delta y, \Delta z)$, the new iterate is then: $(\hat{x}, \hat{y}, \hat{z}) = (x, y, z) + \alpha(\Delta x, \Delta y, \Delta z)$

With $\alpha \rangle 0$ is the displacement step chosen such a way that $(\hat{x}, \hat{z}) \rangle 0$ and $\phi^k$ decreases. If the test of stop is not satisfied one replaces $\mu$ by $\mu_1(\mu_1 \prec \mu)$ and reiterate.

Our infeasible interior point algorithm is described as follows:

**BASIC ALGORITHM**

**Beginning :**

**Initialisation:**

Start with $(x,\ z) \rangle 0$, $y \in \mathbb{R}^m$ (arbitrary) and calculate $\phi$. Either $\varepsilon \ \rangle 0$ a parameter of precision.

$\mathbf{K = 0}$

when $\phi \rangle \varepsilon$ do:

**step 0:**

Calculate $\mu = (1/n)\,(x)^t\, z$ and choose $\sigma \in (0, 1)$

**step 1:**

Solve the following linear system:

$$\begin{bmatrix} Z & 0 & X \\ A & 0 & 0 \\ -Q & A^t & I \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \mu\sigma e - XZe \\ b - Ax \\ c - A^t y + Qx - z \end{bmatrix}$$

**step 2:**

finds a step of displacement $\alpha \rangle 0$ such as:

$x = x + \alpha\Delta x \rangle 0$, $z = z + \alpha\Delta z \rangle 0$ and $\phi$ decreasing.

**step 3:**

$y = y + \alpha\Delta y$

$\mathbf{K = k + 1}$

**End.**

2.3. **Convergence of the Algorithm.** The convergence of the algorithm is studied in [19, 20, 21] for linear and complementarity programming, we extend these results for quadratic convex programming. Under hypotheses (**H1**) and (**H2**), the convergence of the algorithm is based on the following lemma:

**Lemma 1.** *Let $\left\{\left(x^k, y^k, z^k\right)\right\}$ be the sequence of iterates generated by the algorithm then we have:*

**1)** $A(x^k + \alpha^k\Delta x^k) - b = (1 - \alpha^k)(Ax^k - b) = v^{k+1}(Ax^0 - b)$

**2)** $A^t(\,y^k+\alpha^k\Delta y^k)-Q(x^k+\alpha^k\Delta x^k)+(z^k+\alpha^k\Delta z)-c=v^{k+1}(A^ty0+z0-Qx0-c)$

**3)** $(\mathbf{x}^k+\alpha^k\Delta x^k)^t(z^k+\alpha^k\Delta z^k)=(x^k)^tz^k(1-\alpha^k+\alpha^k\sigma^k)+(\alpha^k)^2(\Delta x^k)^t\Delta z^k,$

where $v^{k+1}=(1-\alpha^k)v^k=(1-\alpha^j)\succeq 0, v^0=1.$

**Proposition 2.** *The sequence* $\left\{\phi^k\right\}$ *generated by the algorithm satisfies:*

$$\phi^{k+1}=\left(1-\delta^k\right)\phi^k$$

Where $\delta^k=\delta\left(\alpha^k\right)=\frac{\left[\alpha^k\left(1-\sigma^k\right)\left(x^k\right)^tz^k+\alpha^kv^kr0-\left(\alpha^k\right)^2\left(\Delta x^k\right)^t\Delta z^k\right]}{\left[(x^k)^tz^k+v^kr0\right]}.$

**Corollary 3.** *It is easy to prove that the sequence* $\left\{\phi^k\right\}$ *converge linearly if* $0\,\langle\,\alpha^k\leq 1$ *to which case we have* $0\,\langle\,\delta^k\,\langle\,1$ *and if* $\delta^k$ *offers toward* $1$ *the convergence becomes super linear.*

**Proposition 4.** *Let us suppose the initial point is given by* $\left(x^0,y^0,z^0\right)=\zeta\left(e,0,e\right)$ $(\,\zeta\,\rangle\,0)$ *then: the algorithm converges on at most* $O\left(n^2\left|log\left(\varepsilon\right)\right|\right)$ *iterations (* $\varepsilon$ *a parameter of precision ).*

2.4. **Determination of the displacement step.** The displacement step choice is based on the decreased monotonous of the merit function and on the strict positivity of $(x,z)$. To get the global convergence, two supplementary hypotheses are necessary to know:

**C1**) $h\left(\alpha\right)=\left[min(X\left(\alpha\right)z\left(\alpha\right))-\gamma\left(x\left(\alpha\right)\right)^tz\left(\alpha\right)/n\right]\succeq 0\quad \alpha\in(0,1]$

ă **C2**) $g\left(\alpha\right)=\left[\left(x\left(\alpha\right)\right)^tz\left(\alpha\right)-v\left(\alpha\right)\left(x^0\right)^tz^0\right]\succeq 0\quad \alpha\in(0,1]$

ă where $0\,\langle\,\gamma\,\langle\,1$ satisfied $\gamma\leq\min\left(X^0z^0\right)/(\left(x^0\right)^tz^0/n)$ et $X\left(\alpha\right)=\text{diag}\left(x^{k+1}\right)$, $x\left(\alpha\right)=x^{k+1}$, $z\left(\alpha\right)=z^{k+1}$, $v\left(\alpha\right)=v^{k+1}$ ă The **C1** condition is essential for interior pointŠs methods. Its role is to prevent iterates to approach prematurely the border (before the optimality), while the **C2** condition gives the priority to the feasibility on complementarity (the feasibility is achieved at the latest at the same time than complementarity:

$(x^k)^tz^k/\left(x^0\right)^tz^0\succeq((r^k/r^0)=v^k).$

Let us determine then $\alpha^k$ while taking account the two previous conditions and the maximization of $\delta\left(\alpha\right)$ in $(0,1]$ that is to say:

$$\alpha^k=\text{arg max}\left\{\delta\left(\alpha\right):h\left(\beta\right)\succeq 0,g\left(\beta\right)\succeq 0\text{ for any }\beta\leq\alpha\right\}\qquad(1)$$

The solution of (1) is given below by the lemma:

**Lemma 5.** *If the* **C1** *condition is verified to every iteration,then the problem (1.3) admits a unique solution:*

$$\alpha^k = \begin{cases} min\ \left(1, \alpha_1^k, \alpha_2^k\right) & si\ \left(\Delta x^k\right)^t \Delta z^k \leq 0 \\ min\ \left(1, \alpha_1^k, \alpha_3^k\right) & si\ \left(\Delta x^k\right)^t \Delta z^k\ \rangle\ 0 \end{cases}$$

*where:*

$$\alpha_1^k = min\ \{\alpha\ \rangle\ 0 : h(\alpha) = 0\}$$

ă $\alpha_2^k = \begin{cases} 1 & si\ \left(\Delta x^k\right)^t \Delta z^k = 0 \\ min\ \{\alpha\ \rangle\ 0 : g(\alpha) = 0\} & si\ \left(\Delta x^k\right)^t \Delta z^k\ \langle\ 0 \end{cases}$ ă *and*

$$\alpha_3^k = \left[(1-\sigma)\left(x^k\right)^t z^k + v^k r^0\right] / 2\ \left(\Delta x^k\right)^t \Delta z^k. \text{ [18]}$$

**Remark 1.** *Let us note that the previous choice of step constitutes a condition sufficient only for the convergence, which gives us a certain liberty in practice. Indeed, a less expensive choice is possible, it is about choosing $\alpha$ so that $(x, z)\ \rangle\ 0$ (strictly positive) while taking account of the decrease of the merit function.*

ă *In the implementation our suitable choice of the largest step size is given by :* ă $\alpha_x = \beta\acute{\alpha_x}$ *and* $\alpha_z = \beta\acute{\alpha_z}$ $(0\langle\beta\langle 1)$ ă *where* ă

$$\acute{\alpha_x} = \begin{cases} min\left(-x_i/\Delta x_i\right) & si\ \Delta x_i\ \langle\ 0 \\ 1 & si\ \Delta x_i \succeq 0 \end{cases} \quad ă\ \acute{\alpha_z} = \begin{cases} min\left(-z_i/\Delta z_i\right) & si\ \Delta z_i\ \langle\ 0 \\ 1 & si\ \Delta z_i \succeq 0 \end{cases}$$

ă Whose new iterate is: $x = x + \alpha_x \Delta x$, $z = z + \alpha_z \Delta z$ and $y = y + \alpha_z \Delta x$.

All time, it is important to signal that performances of interior points methods feasible or no, depend greatly on the choice of the displacement step.

2.5. **Calculation of the displacement direction.** In the algorithms, the cost of an iteration is dominated by the calculation of the displacement direction $\Delta w = (\Delta x, \Delta y, \Delta z)$ that is by solving the following of the linear system:

$$\begin{bmatrix} Z & 0 & X \\ A & 0 & 0 \\ -Q & A^t & I \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \mu\sigma e - XZe \\ b - Ax \\ c - A^t y + Qx - z \end{bmatrix} \tag{2}$$

Several procedures can be used to solve (2) like Gauss elimination. The obtained results are valid but limited with small dimensions. Moreover, the strategy is not optimal. The manipulated matrix is of dimension $(2\times n + m, 2\times n + m)$. To calculate the displacement direction, we introduce a more economic alternative which consists in reducing the implemented matrix size. For that, with simple calculations, one obtains the system of equations according to:

$$\begin{cases} A(QX+Z)^{-1}XA^t\Delta y = b - Ax + A(QX+Z)^{-1} \\ \qquad [Xze - \mu\sigma e - X(c - A^t y + Qx - z)] \\ \Delta x = (QX+Z)^{-1}[XA^t\,\Delta y + \mu\sigma e - Xze - X(c - A^t y + Qx - z)] \\ \Delta z = (X)^{-1}(\mu\sigma e - Xze - Z\Delta x) \end{cases}$$

Only matrix $A(QX+Z)^{-1}XA^t$of size $(m \times m)$ will be necessary for to resolve the system in question.

The matrix: $A(QX+Z)^{-1}XA^t$ is positive semi definite and:

**i)** $A(QX+Z)^{-1}XA^t = \left(A(QX+Z)^{-1}XA^t\right)^t$

**ii)** $x \neq 0, \langle\ \left(AA(QX+Z)^{-1}XA^t\right)\ \left(A(QX+Z)^{-1}XA^t\right)^t x, x\ \rangle =$
$\langle\ \left(A(QX+Z)^{-1}XA^t\right)x, A(QX+Z)^{-1}XA^t x\ \rangle =$
$\left\|\left(A(QX+Z)^{-1}XA^t\right)x\right\|^2\rangle\ 0$

Therefore, the Cholesky factorization (which is a particular case of the Gauss method) is frequently used in solving the system.

2.6. **Numerical experiments.** This paragraph is dedicated to preliminarily numerical results presentation in order to test our algorithm, implemented on a Pentium II and in TURBO-PASCAL programming.

Examples are stated under the following canonical form:

$$(QP)\begin{cases} min\ c^t x + \frac{1}{2}x^t Qx \\ Ax = b \\ x \succeq 0 \end{cases}$$

The dual of $(QP)$ is:

$$(QD)\begin{cases} max\ b^t y - \frac{1}{2}x^t Qx \\ A^t y + z - Qx = c \\ z \geq 0, y \in \mathbb{R}^m \end{cases}$$

ă **Example 1:** ă

$$\begin{cases} \underset{(x,t)}{min}\ f(x,t) = 6.5x + 0.5x^2 - t_1 - 2t_2 - 3t_3 - 2t_4 - t_5 \\ Az \leq\ b \\ z = (x,t)^t \\ x \succeq 0,\ t_1 \succeq 0, t_2 \succeq 0 \\ 0 \leq\ t_i\ \leq 1 \quad i = 3,4 \\ 0 \leq\ t_5 \leq\ 2 \end{cases}$$

$b = (26, -11, 24, 12, 3)$

$$A = \begin{pmatrix} 1 & 2 & 8 & 1 & 3 & 5 \\ -8 & -4 & -2 & 2 & 4 & -1 \\ 2 & 0.5 & 0.2 & -3 & -1 & -4 \\ 0.2 & 2 & 0.1 & -4 & 2 & 2 \\ -0.1 & -0.5 & 2 & 5 & -5 & 3 \end{pmatrix}$$

ă The optimal solution of $(QP)$ is:

$z^* = (x^*, t^*)^t = (0, 7.987342, 0.253165, 2, 2, 0)^t$

The optimal solution of $(QD)$ is:

$y^* = (-0.246835, 0, 0, -0.253165, 0)^t$

objective function: $-18.493671$ ă **Example 2:**

$$\begin{cases} \underset{x}{min}\ f(x) = 0.5 \sum_{i=1}^{6} \beta_i (x_i - \alpha_i)^2 \\ Ax \preceq b \\ x \succeq 0 \end{cases}$$

$b = (-5, 2, -1, -3, 5)^t$ ă

$$A = \begin{pmatrix} -3 & 7 & 0 & -5 & 1 & 1 \\ 7 & 0 & -5 & 1 & 1 & 0 \\ 0 & -5 & 1 & 1 & 0 & 2 \\ -5 & 1 & 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 2 & -1 & -1 \end{pmatrix}$$

ă **case 1**: $\beta_i = 1$, $\alpha_i = 2$ for $i : 1, ..., 6$ ă The optimal solution of primal problem is: ă $x^* = (1.010453, 0.749129, 1.303136, 1.442509, 0, 0)^t$ ă The optimal solution of dual problem is: ă $y^* = (-0.661232, -0.646117, -1.217533, -0.709915, 0)^t$ ă Objective function: 2.680600 ă **case 2**: $\beta_i = 1$, $\alpha_i = 2$ for $i : 1, ..., 6$ ă The optimal solution of primal problem is:

$x^* = (1.715685, 0.965687, 2.397060, 1.431373, 0.544120, 0)^t$

The optimal solution of dual problem is:

$y^* = (-7.862744, -3.183211, -13.518993, -2, -11.590071)^t$

Objective function: $-11.467500$

**case 3**: $\beta_i = 1$, $\alpha_i = 2$ for $i : 1, ..., 6$

The optimal solution for primal problem is:

$x^* = (1.715685, 0.965687, 2.397060, 1.431373, 0.544120, 0)^t$

The optimal solution for dual problem is:

$y^* = (-15.725487, -6.366421, -27.037984, -4, -23.180141)^t$

Objective function: $-22.935000$

**case 4**: $\beta_i = 2$, $\alpha_i = 0$ for $i : 1, ..., 6$

The optimal solution of primal problem is:

$x^* = (2.05, 1.30, 3.40, 2.10, 2.55, 0)^t$

The optimal solution for dual problem is:
$y^* = (-13, -4.80, -22.40, -2.60, -19.40)^t$
Objective function: $-11.400000$

**Example 3:**

$$\begin{cases} \min_x f(x) = \sum_{i=1}^{9} x_i x_{i+1} + \sum_{i=1}^{8} x_i x_{i+2} + x_1 x_9 + x_1 x_{10} + x_2 x_{10} + x_1 x_5 + x_4 x_7 \\ \sum_{i=1}^{10} x_i = 1 \\ x \succeq 0 \end{cases}$$

ă The optimal solution of primal problem is:
$x^* = (0, 0.249335, 0.25, 0, 0.000665, 0.017431, 0, 0.25, 0.232568, 0)^t$
The optimal solution of dual problem is:
$y^* = 0.25$
Objective function: $0.125010$

## 3. Conclusion

In this paper, we presented an implementing method for convex quadratic programming which stimulates greatly the development of the numeric behaviour of infeasible methods for problems of optimization. One can conclude that these methods constitute a valid solution as to the algorithm initialization problem. This one deserves some supplementary efforts essentially when choosing the step displacement. This, until now, is the object of numerous researches aiming to reduce the iteration cost and, by the same time, improve the numeric behaviour distinctly. This one deals not only with the linear and convex quadratic programming but is extended to non linear programming.

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