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THE FUNDAMENTAL TRANSFORMATION FORMULA OF DIVIDED DIFFERENCES ON UNDIRECTED NETWORKS

DANIELA MARIAN

ABSTRACT. We establish a fundamental transformation formula of divided differences on undirected networks. We adopt the definition of network as metric space introduced by P. M. Dearing and R. L. Francis (1974).

1. Preliminary notions and result

The definition of network as metric space was introduced in [1] and was used in [2], [4], [3], etc.

We consider an undirected, connected graph $G = (W, A)$, without loops or multiple edges. To each vertex $w_i \in W$ we associate a point v_i from the Euclidean space $\mathbf{R}^q, q \in \mathbf{N}, q \geq 2$. This yields a finite subset V of $\mathbf{R}^q$, called the vertex set of the network. We also associate to each edge $(w_i, w_j) \in A$ a rectifiable arc $[v_i, v_j] \subset \mathbf{R}^q$ called edge of the network. We assume that any two edges have no interior common points. We denote by $E = \{e_1, ..., e_m\}$, $e_k = [v_{i_k}, v_{j_k}]$, $k = 1, 2, ..., m$ the set of all edges. We define the network $N = (V, E)$ by

$$N = \{x \in \mathbf{R}^q \mid \exists (w_i, w_j) \in A \text{ so that } x \in [v_i, v_j]\}.$$

It is obvious that N is a geometric image of G, which follows naturally from an embedding of G in $\mathbf{R}^q$. Suppose that for each edge $e_k = [v_{i_k}, v_{j_k}] \in E$, $k = 1, 2, ..., m$, there exist a continuous one-to-one mapping $T_{e_k} : [0,1] \to [v_{i_k}, v_{j_k}]$ so that $T_{e_k}(0) = v_{i_k}$, $T_{e_k}(1) = v_{j_k}$ and $T_{e_k}([0,1]) = [v_{i_k}, v_{j_k}]$.

Any connected and closed subset of an edge $e_k = [v_{i_k}, v_{j_k}] \in E$ bounded by two points x and y is called a closed subedge and is denoted by $[x, y]$. If one or both of x, y are missing we say than the subedge is open in x, or in y, or is open and we denote this by $(x, y]$, $[x, y)$, or (x, y), respectively. We denote by θ_{e_k} the inverse function of T_{e_k}. We consider that $e_k = [v_{i_k}, v_{j_k}]$ has the positive length l_{e_k}. Using θ_{e_k}, it is possible to compute the length of $[x, y]$ as

$$l([x, y]) = |\theta_{e_k}(x) - \theta_{e_k}(y)| \cdot l_{e_k}.$$

Particularly we have

$$l([v_i, v_j]) = l_{e_k}, l([v_i, x]) = \theta_{e_k}(x)\, l_{e_k}$$

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and

$$l([x, v_j]) = (1 - \theta_{e_k}(x)) \cdot l_{e_k}.$$

A path $L(x, y)$ linking two points x and y in N is a sequence of edges and at most two subedges at extremities, starting in x and ending in y. If $x = y$ then the path is called cycle. The length of a path (cycle) is the sum of the lengths of all its component edges and subedges and will be denoted by $l(L(x, y))$. If a path (cycle) contains only distinct vertices then we call it elementary.

A network is connected if for every pair of points $x, y \in N$ there exists a path $L(x, y) \subset N$. A connected network without cycles is called tree.

Let $D(x, y)$ be a shortest path between the points $x, y \in N$. This path is also called geodesic. We define a distance on N as follows:

Definition 1.1. [1]*For every pair of points $x, y \in N$, the distance from x to y, $d(x, y)$ in the network N is the length of a shortest path from x to y :*

$$d(x, y) = l(D(x, y)).$$

It is obvious that (N, d) is a metric space.

For $x, y \in N$, we denote

(1) $$\langle x, y \rangle = \{z \in N \mid d(x, z) + d(z, y) = d(x, y)\},$$

and $\langle x, y \rangle$ is called the metric segment between x and y.

We consider a nonnegative integer $n \geq 0$, two points $x, y \in N$, $D(x, y) \subset \langle x, y \rangle$ a shortest path from x to y, a function $f : N \to \mathbf{R}$ and the distinct points

(2) $$x_1, x_2, ..., x_{n+1}$$

included on the path $D(x, y)$.

In [3] E. Iacob denote:

$$\mathcal{P}_n(x) = \left\{ P : D(x, y) \to \mathbf{R} \mid P(t) = \sum_{k=0}^{n} c_k d^k(x, t), c_k \in \mathbf{R} \right\}.$$

The elements of $\mathcal{P}_n(x)$ are called metric polynomials. For a metric polynomial P, the maximum number k for which the coefficient c_k is different by zero is called the degree of P.

E. Iacob established that exist a single polynomial $P^* \in \mathcal{P}_n(x)$ which is equal with f on the points (2). The polynomial P^* is denoted with

$$L(\mathcal{P}_n(x); x_1, ..., x_{n+1}; f)$$

and is called interpolation metric polynomial of Lagrange type.

Theorem 1.1. [3]*The metric polynomial*

$$L(\mathcal{P}_n(x); x_1, ..., x_{n+1}; f) : D(x, y) \to \mathbf{R}$$

(3)
$$L(\mathcal{P}_n(x); x_1, ..., x_{n+1}; f)(t) =$$
$$= \sum_{i=1}^{n+1} f(x_i) \cdot \frac{(d(x,t) - d(x,x_1)) \cdot ... \cdot (d(x,t) - d(x,x_{i-1}))}{(d(x,x_i) - d(x,x_1)) \cdot ... \cdot (d(x,x_i) - d(x,x_{i-1}))} \cdot$$
$$\cdot \frac{(d(x,t) - d(x,x_{i+1})) \cdot ... \cdot (d(x,t) - d(x,x_{n+1}))}{(d(x,x_i) - d(x,x_{i+1})) \cdot ... \cdot (d(x,x_i) - d(x,x_{n+1}))}$$

belongs to the set $\mathcal{P}_n(x)$ and satisfy the conditions

(4)
$$L(\mathcal{P}_n(x); x_1, ..., x_{n+1}; f)(x_i) = f(x_i), \textit{for all } i = 1, 2, ..., n+1$$

Moreover $L(\mathcal{P}_n(x); x_1, ..., x_{n+1}; f)$ is the unique metric polynomial from $\mathcal{P}_n(x)$ which satisfy the condition (4).

In [5] the coefficient c_n of $L(\mathcal{P}_n(x); x_1, ..., x_{n+1}; f)$ corresponding of $d^n(x,t)$ is called the divided difference of the function f on the points (2) related to x. We denote

(5)
$$c_n := [x_1, x_2, ..., x_{n+1}; f]_x .$$

Theorem 1.2. [5] *The divided difference (5) has the following properties:*

(6)
$$[x_1, x_2, ..., x_{n+1}, f]_x =$$
$$= \sum_{i=1}^{n+1} f(x_i) \cdot \frac{1}{(d(x,x_i) - d(x,x_1)) \cdot ... \cdot (d(x,x_i) - d(x,x_{i-1}))} \cdot$$
$$\cdot \frac{1}{(d(x,x_i) - d(x,x_{i+1})) \cdot ... \cdot (d(x,x_i) - d(x,x_{n+1}))},$$

(7)
$$\left[x_1, x_2, ..., x_{n+1}; d^k(x,t)\right]_x = \begin{cases} 0, k = 0, 1, ..., n-1 \\ 1, k = n \end{cases}$$

and

(8)
$$\left[x_1, x_2, ..., x_{n+1}; c_0 + c_1 d(x,t) + ... + c_k d^k(x,t)\right]_x = \begin{cases} 0, k = 0, 1, ..., n-1 \\ c_k, k = n \end{cases},$$

$\forall c_1, c_2, \ldots, c_k \in \mathbf{R}$.

We denote now

(9) $\varphi(t)_x = (d(x,t) - d(x,x_1)) \cdot (d(x,t) - d(x,x_2)) \cdot ... \cdot (d(x,t) - d(x,x_{n+1}))$

and

(10)
$$\begin{aligned} \varphi'(x_i)_x &= (d(x,x_i) - d(x,x_1)) \cdot ... \cdot (d(x,x_i) - d(x,x_{i-1})) \cdot \\ &\cdot (d(x,x_i) - d(x,x_{i+1})) \cdot ... \cdot (d(x,x_i) - d(x,x_{n+1})), \\ \forall i &= 1.2, ..., n+1 \end{aligned}$$

With the notations (9) and (10) we obtain

$$L\left(\mathcal{P}_n(x);x_1,...,x_{n+1};f\right)(t)=\sum_{i=1}^{n+1}\frac{f(x_i)\cdot\varphi(t)_x}{\varphi'(x_i)_x\cdot(d(x,t)-d(x,x_i))}$$

and

$$[x_1,x_2,...,x_{n+1};f]_x=\sum_{i=1}^{n+1}\frac{f(x_i)}{\varphi'(x_i)_x}.$$

2. The fundamental transformation formula of divided differences

In [7] T. Popoviciu established a fundamental transformation formula of usual divided differences. This also can be find in [8] and [6]. In what follows we establish a analogous fundamental transformation formula of divided differences on undirected network.

We consider a network N, two fixed points $x,y\in N, D(x,y)\subset\langle x,y\rangle$, a non-negative integer $n\geq 0$, a natural number $m\geq n+1$ and m distinct points

$$x_1,x_2,...,x_m \tag{11}$$

included on the path $D(x,y)$.

We denote

$$\Delta_j^i(f)_x=[x_i,x_{i+1},...,x_{i+j};f]_x\,,\Delta_0^i(f)_x=f(x_i)\,,$$

$$\begin{aligned}\varphi_{i,j+1}(t)_x &= (d(x,t)-d(x,x_i))\cdot(d(x,t)-d(x,x_{i+1}))\cdot\\ &\quad ...\cdot(d(x,t)-d(x,x_{i+j}))\,,\\ \varphi_{i,0}(t)_x &= 1\end{aligned}$$

and

$$\begin{aligned}\varphi'_{i,j+1}(x_r)_x &= (d(x,x_r)-d(x,x_i))\cdot...\cdot(d(x,x_r)-d(x,x_{r-1}))\cdot\\ &\quad (d(x,x_r)-d(x,x_{r+1}))\cdot...(d(x,x_r)-d(x,x_{i+j}))\end{aligned}$$

where $i=1,2,...,m-j$, $j=0,1,...,m-1$ and $r=i,i+1,...,i+j$.

With this last notations we have

$$\varphi(t)_x=\varphi_{1,n+1}(t)_x\,,$$

$$\varphi'(x_i)_x=\varphi'_{1,n+1}(x_i)_x\,,\forall i=1,2,\ldots,n+1$$

and

$$\Delta_j^i(f)_x=\sum_{r=i}^{i+j}\frac{f(x_r)}{\varphi'_{i,j+1}(x_r)_x}$$

We consider now a function $f : D(x,y) \to \mathbf{R}$ and the linear combination

$$F = \sum_{i=1}^{m} \lambda_i f(x_i), \tag{12}$$

the coefficients $\lambda_i \in \mathbf{R}, i = 1, 2, ..., m$ being independent by the function f.

Theorem 2.1. *The expression (12) always can be expresses like follows:*

$$F = \sum_{i=1}^{n} \mu_i \Delta_{i-1}^{1}(f)_x + \sum_{i=1}^{m-n} \gamma_i \Delta_n^i (f)_x, \tag{13}$$

where the coefficients μ_i and γ_i do not depend by the function f. These coefficients are completely determined by the coefficients λ_i.

Proof. Indeed, if we equal the second member of (12) with the second member of (13) we have

$$\sum_{i=1}^{m} \lambda_i f(x_i) = \sum_{i=1}^{n} \mu_i \Delta_{i-1}^{1}(f)_x + \sum_{i=1}^{m-n} \gamma_i \Delta_n^i (f)_x,$$

$$\sum_{i=1}^{m} \lambda_i f(x_i) = \mu_1 \Delta_0^1(f)_x + \mu_2 \Delta_1^1(f)_x + ... + \mu_n \Delta_{n-1}^1(f)_x + \gamma_1 \Delta_n^1(f)_x + \gamma_2 \Delta_n^2(f)_x + ... + \gamma_{m-n} \Delta_n^{m-n}(f)_x,$$

$$\sum_{i=1}^{m} \lambda_i f(x_i) = \mu_1 f(x_1) + \mu_2 \sum_{i=1}^{2} \frac{f(x_i)}{\varphi'_{1,2}(x_i)_x} + ... + \mu_n \sum_{i=1}^{n} \frac{f(x_i)}{\varphi'_{1,n}(x_i)_x} +$$

$$\gamma_1 \sum_{i=1}^{n+1} \frac{f(x_i)}{\varphi'_{1,n+1}(x_i)_x} + \gamma_2 \sum_{i=2}^{n+2} \frac{f(x_i)}{\varphi'_{2,n+1}(x_i)_x} + ... + \gamma_{m-n} \sum_{i=m-n}^{m} \frac{f(x_i)}{\varphi'_{m-n,n+1}(x_i)_x}$$

We identify now the coefficients and we obtain the following linear system of m equation with the unknowns $\mu_1, \mu_2, ..., \mu_n$ and $\gamma_1, \gamma_2, ..., \gamma_{m-n}$:

$$\begin{cases} \lambda_1 = \mu_1 + \frac{\mu_2}{\varphi'_{1,2}(x_1)_x} + ... + \frac{\mu_n}{\varphi'_{1,n}(x_1)_x} + \frac{\gamma_1}{\varphi'_{1,n+1}(x_1)_x} + 0 \cdot \gamma_2 + ... + 0 \cdot \gamma_{m-n} \\ \lambda_2 = 0 \cdot \mu_1 + \frac{\mu_2}{\varphi'_{1,2}(x_2)_x} + ... + \frac{\mu_n}{\varphi'_{1,n}(x_2)_x} + \frac{\gamma_1}{\varphi'_{1,n+1}(x_2)_x} + \frac{\gamma_2}{\varphi'_{2,n+1}(x_i)_x} + ... + 0 \cdot \gamma_{m-n} \\ \quad ... \\ \lambda_m = 0 \cdot \mu_1 + 0 \cdot \mu_2 + ... + 0 \cdot \mu_n + 0 \cdot \gamma_1 + 0 \cdot \gamma_2 + ... + \frac{\gamma_{m-n}}{\varphi'_{m-n,n+1}(x_m)_x} \end{cases}$$

The determinant of this system is

$$\frac{1}{\varphi'_{1,2}(x_2)_x \cdot \varphi'_{1,3}(x_3)_x \cdot ... \cdot \varphi'_{1,n}(x_n)_x \cdot \varphi'_{1,n+1}(x_{n+1})_x \cdot \varphi'_{2,n+1}(x_{n+2})_x \cdot ... \cdot \varphi'_{m-n,n+1}(x_m)_x}$$

and it is different by zero, consequently the coefficients μ_i, γ_i are unique determined. □

For establish the coefficients $\mu_1, \mu_2, ..., \mu_n$ and $\gamma_1, \gamma_2, ..., \gamma_{m-n}$ we convenient search the function f.

First, for establish the coefficients $\mu_1, \mu_2, ..., \mu_n$ we consider the function

$$\begin{aligned} f &: D(x,y) \to \mathbf{R} \\ f(t) &= \varphi_{1,j-1}(t)_x = (d(x,t) - d(x,x_1))(d(x,t) - d(x,x_2)) \cdot ... \\ &\cdot (d(x,t) - d(x,x_{j-1})) \end{aligned}$$

where $j \in \{1, 2, ..., n\}$.

Lemma 2.1. *We have*

$$\Delta^1_{i-1}(f)_x = \begin{cases} 0, \ \text{if } i = 1, 2, ..., j-1 \\ 1, \ \text{if } i = j \\ 0, \ \text{if } i = j+1, j+2, ..., n \end{cases}$$

and

$$\Delta^i_n(f)_x = 0, \forall i = 1, 2, ..., m-n$$

Proof. 1. For $i = 1, 2, ..., j-1$ we have

$$\Delta^1_{i-1}(f)_x = \sum_{r=1}^{i} \frac{f(x_r)}{\varphi'_{1,i}(x_r)_x} =$$

$$= \sum_{r=1}^{i} \frac{(d(x,x_r) - d(x,x_1))(d(x,x_r) - d(x,x_2)) \cdot ... \cdot (d(x,x_r) - d(x,x_{j-1}))}{\varphi'_{1,i}(x_r)_x}.$$

But $i \le j-1$ hence all the terms of the sum are zero, so $\Delta^1_{i-1}(f)_x = 0$.
2. For $i = j$ we apply the relation (8) and we have

$$\begin{aligned} \Delta^1_{i-1}(f)_x &= [x_1, x_2, ..., x_i; (d(x,t) - d(x,x_1))(d(x,t) - d(x,x_2)) \cdot \\ &... \cdot (d(x,t) - d(x,x_{i-1}))] = 1 \end{aligned}$$

because f is a metric polynomial of degree $i-1$.
3. For $i = j+1, j+2, ..., n$ we also apply the relation (8) and we have

$$\begin{aligned} \Delta^1_{i-1}(f)_x &= [x_1, x_2, ..., x_i; (d(x,t) - d(x,x_1))(d(x,t) - d(x,x_2)) \cdot ... \\ &\cdot (d(x,t) - d(x,x_{j-1}))] = 0 \end{aligned}$$

because the degree of the metric polynomial f is $j-1$ and $j-1 \le i-2$ for $i = j+1, j+2, ..., n$.
4. From the relation (8) we have

$$\begin{aligned} \Delta^i_n(f)_x &= [x_i, x_{i+1}, ..., x_{i+n}; (d(x,t) - d(x,x_1))(d(x,t) - d(x,x_2)) \cdot ... \\ &\cdot (d(x,t) - d(x,x_{j-1}))] = 0 \end{aligned}$$

because the degree of the metric polynomial f is $j-1$ and $j-1 \le n-1$. □

Lemma 2.2. *For every $j = 1, 2, ..., n$*

$$\mu_j = \sum_{i=j}^{m} \lambda_i \left(d\left(x, x_i\right) - d\left(x, x_1\right)\right)\left(d\left(x, x_i\right) - d\left(x, x_2\right)\right) \cdot ... \cdot \left(d\left(x, x_i\right) - d\left(x, x_{j-1}\right)\right). \tag{14}$$

Proof. From Theorem 2.1 we have

$$\sum_{i=1}^{m} \lambda_i f\left(x_i\right) = \sum_{i=1}^{n} \mu_i \Delta_{i-1}^{1}\left(f\right)_x + \sum_{i=1}^{m-n} \gamma_i \Delta_n^{i}\left(f\right)_x.$$

We apply now Lemma 2.1 and we obtain

$$\mu_j = \sum_{i=1}^{m} \lambda_i f(x_i) = \sum_{i=1}^{m} \lambda_i \varphi_{1,j-1}(x_i)_x =$$

$$= \sum_{i=j}^{m} \lambda_i \left(d\left(x, x_i\right) - d\left(x, x_1\right)\right)\left(d\left(x, x_i\right) - d\left(x, x_2\right)\right) \cdot ... \cdot \left(d\left(x, x_i\right) - d\left(x, x_{j-1}\right)\right).$$

□

For establish now the coefficients $\gamma_1, \gamma_2, ..., \gamma_{m-n}$ we consider the function

$$f_j^* : D(x, y) \to \mathbf{R},$$

$$f_j^*(t) = \begin{cases} 0, \text{ if } x = x_1, x_2, \ldots, x_{j+n-1} \\ \varphi_{j+1,n-1}(t)_x, \text{ if } x = x_{j+n}, x_{j+n+1}, \ldots, x_m \end{cases}$$

where $j = 1, 2, \ldots, m - n$.

Lemma 2.3. *We have*

$$\Delta_{i-1}^{1}\left(f_j^*\right)_x = 0 \text{ for every } i = 1, 2, \ldots n$$

and

$$\Delta_n^{i}\left(f_j^*\right)_x = \begin{cases} 0, \text{ if } i = 1, 2, \ldots, j-1 \\ \frac{1}{d(x, x_{j+n}) - d(x, x_j)}, \text{ if } i = j \\ 0, \text{ if } i = j+1, j+2, \ldots, m-n \end{cases}$$

Proof. 1. For every $i = 1, 2, \ldots n$ we have $i \leq n < j + n$ hence $f_j^*(x_1) = f_j^*(x_2) = \ldots = f_j^*(x_i) = 0$. Consequently

$$\Delta_{i-1}^{1}\left(f_j^*\right)_x = \left[x_1, x_2, ..., x_i; f_j^*\right]_x = \sum_{r=1}^{i} \frac{f_j^*(x_r)}{\varphi_{1,i}^{'}(x_r)_x} = 0.$$

2. For every $i = 1, 2, \ldots, j-1$ we have $i + n \leq j + n - 1$ hence

$$f_j^*(x_i) = f_j^*(x_{i+1}) = \ldots = f_j^*(x_{i+n}) = 0.$$

Consequently

$$\Delta_n^{i}\left(f_j^*\right)_x = \sum_{r=i}^{n+i} \frac{f_j^*(x_r)}{\varphi_{i,n+1}^{'}(x_r)_x} = 0.$$

3. For $i=j$ we have

$$\Delta_n^j\left(f_j^*\right)_x=\sum_{r=j}^{n+j}\frac{f_j^*\left(x_r\right)}{\varphi_{j,n+1}'\left(x_r\right)_x}.$$

But

$$f_j^*\left(x_j\right)=f_j^*\left(x_{j+1}\right)=\ldots=f_j^*\left(x_{j+n-1}\right)=0 \text{ and } f_j^*\left(x_{j+n}\right)=\varphi_{j+1,n-1}\left(x_{j+n}\right)_x.$$

Hence $\Delta_n^j\left(f_j^*\right)_x=$

$$\frac{f_j^*\left(x_{j+n}\right)}{\left(d\left(x,x_{j+n}\right)-d\left(x,x_j\right)\right)\left(d\left(x,x_{j+n}\right)-d\left(x,x_{j+1}\right)\right)\cdot\ldots\cdot\left(d\left(x,x_{j+n}\right)-d\left(x,x_{j+n-1}\right)\right)}$$

$$=\frac{\left(d\left(x,x_{j+n}\right)-d\left(x,x_{j+1}\right)\right)\left(d\left(x,x_{j+n}\right)-d\left(x,x_{j+2}\right)\right)\cdot\ldots\cdot\left(d\left(x,x_{j+n}\right)-d\left(x,x_{j+n-1}\right)\right)}{\left(d\left(x,x_{j+n}\right)-d\left(x,x_j\right)\right)\left(d\left(x,x_{j+n}\right)-d\left(x,x_{j+1}\right)\right)\cdot\ldots\cdot\left(d\left(x,x_{j+n}\right)-d\left(x,x_{j+n-1}\right)\right)}$$

$$=\frac{1}{d(x,x_{j+n})-d(x,x_j)}.$$

4. Finally we compute $\Delta_n^i\left(f_j^*\right)_x=\left[x_i,x_{i+1},\ldots,x_{i+n};f_j^*\right]_x$ for $i=j+1,j+2,\ldots,m-n$. We see that $i+n\geq j+n+1$. We have

$$f_j^*(x_r)=0 \text{ for every number } r \text{ so that } i\leq r<j+n-1$$

and

$$f_j^*\left(x_r\right)=\varphi_{j+1,n-1}(x_r)_x \text{ for every number } r \text{ so that } j+n\leq r\leq i+n.$$

But

$$\varphi_{j+1,n-1}(x_r)_x=\left(d\left(x,x_r\right)-d\left(x,x_{j+1}\right)\right)\left(d\left(x,x_r\right)-d\left(x,x_{j+2}\right)\right)\cdot\ldots$$
$$\cdot\left(d\left(x,x_r\right)-d\left(x,x_{j+n-1}\right)\right)=0$$

for every $r=j+1,j+2,\ldots,j+n-1$. Consequently

$$f_j^*\left(x_r\right)=\varphi_{j+1,n-1}\left(x_r\right)_x \text{ for every } r=i,i+1,\ldots i+n$$

and we obtain $\Delta_n^i\left(f_j^*\right)_x=0$ because the degree of the polynomial f_j^* is $n-1$ and the divided difference is considered on $n+1$ points. □

Lemma 2.4. *For every* $j=1,2,\ldots,m-n$

$$\gamma_j=\left(d(x,x_{j+n})-d(x,x_j)\right)\cdot \tag{15}$$
$$\cdot\sum_{i=j+n}^{m}\lambda_i\left(d\left(x,x_i\right)-d\left(x,x_{j+1}\right)\right)\left(d\left(x,x_i\right)-d\left(x,x_{j+2}\right)\right)\cdot$$
$$\ldots\cdot\left(d\left(x,x_i\right)-d\left(x,x_{j+n-1}\right)\right).$$

Proof. From Theorem 2.1 we have

$$\sum_{i=1}^{m}\lambda_i f(x_i)=\sum_{i=1}^{n}\mu_i\Delta_{i-1}^{1}(f)_x+\sum_{i=1}^{m-n}\gamma_i\Delta_n^{i}(f)_x\,.$$

We apply Lemma 2.3 and we have

$$\gamma_j = (d(x,x_{j+n})-d(x,x_j))\cdot\sum_{i=1}^{m}\lambda_i\, f_j^*(x_i)=$$

$$= (d(x,x_{j+n})-d(x,x_j))\cdot\sum_{i=j+n}^{m}\lambda_i\,\varphi_{j+1,n-1}(x_i)_x=(d(x,x_{j+n})-d(x,x_j))\cdot$$

$$\cdot\sum_{i=j+n}^{m}\lambda_i\left(d\left(x,x_i\right)-d\left(x,x_{j+1}\right)\right)\left(d\left(x,x_i\right)-d\left(x,x_{j+2}\right)\right)\cdot$$

$$\ldots\cdot\left(d\left(x,x_i\right)-d\left(x,x_{j+n-1}\right)\right).$$

□

From Lemma 2.2 and Lemma 2.4 we obtain the following fundamental transformation formula of divided differences.

Theorem 2.2. *For every natural number n we have:*

$$\sum_{i=1}^{m}\lambda_i f(x_i) = \sum_{j=1}^{n}\left[\sum_{i=j}^{m}\lambda_i\varphi_{1,j-1}(x_i)_x\right]\Delta_{j-1}^{1}(f)_x+$$

$$\sum_{j=1}^{m-n}\left[(d(x,x_{j+n})-d(x,x_j))\cdot\sum_{i=j+n}^{m}\lambda_i\,\varphi_{j+1,n-1}(x_i)_x\right]\Delta_n^{j}(f)_x\,.$$

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Str. Albac Nr.11 Ap.10, Cluj-Napoca
E-mail address: dcirdan@math2.ubbcluj.ro