Stability of multi-step fixed point iterative methods

Marcel-Adrian Şerban

Department of Mathematics, Babeş-Bolyai University

mserban@math.ubbcluj.ro

Problem 1 (Limit shadowing property problem) *Let (X, d) be a metric space and $f : X \to X$ an operator. Let $x_n \in X$, $n \in \mathbb{N}$, be such that:*

$$d(x_{n+1}, f(x_n)) \to 0 \text{ as } n \to +\infty.$$

In which conditions there exists $x \in X$ such that

$$d(x_n, f^n(x)) \to 0 \text{ as } n \to +\infty \text{ ?}$$

By definition if f is a solution of the above problem, then we say that *the operator f has the limit shadowing property with respect to Picard iteration.*

Problem 2 (The stability problem) *Let (X, d) be a metric space and $f : X \to X$ an operator. Let us consider the Picard iteration algorithm for f*

$$x_0 \in X,\ x_{n+1} = f(x_n),\ n \in \mathbb{N}.$$

By definition, the Picard iteration algorithm is stable with respect to f if it is convergent (i.e., f is WPO) and f has the limit shadowing property with respect to this algorithm. The problem is to give conditions on f which imply that the Picard iteration algorithm is stable.

Let (X, d) be a metric space and $T : X^k \to X$ an operator. Let us consider the following multi-step algorithm for f

$$x_0, x_1, \dots, x_{k-1} \in X,\ x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1}),\ n \in \mathbb{N}.$$

We define the operator $A_T : X^k \to X^k$

$$A_T(u_1, \dots, u_k) = (u_2, \dots, u_k, T(u_1, \dots, u_k)).$$

It is easy to see that for $x_0, x_1, \dots, x_{k-1} \in X$ we have

$$(x_{n+1}, x_{n+2}, \dots, x_{n+k}) = A_T^n(x_0, x_1, \dots, x_{k-1})$$

In this paper we give conditions on $T : X^k \to X$ such that the Picard iteration algorithm for the operator $A_T : X^k \to X^k$ is stable.

References

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