Bilateral Inequalities for Harmonic, Geometric and Hölder Means

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For $0 < a < b$, the *harmonic*, *geometric* and *Hölder* means are given by

$$H(a,b) = \frac{2ab}{a+b}, \quad G(a,b) = \sqrt{ab}, \quad Q(a,b) = \left(\frac{a^2+b^2}{2}\right)^{1/2}.$$

They are special cases ($p = -1, 0, 2$) of power means

$$M_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p}, & \text{for } p \neq 0 \\ \sqrt{ab}, & \text{for } p = 0. \end{cases}$$

We consider the problem of finding $\alpha,\ \beta \in \mathbb{R}$ for which

$$\alpha H(a,b) + (1-\alpha)Q(a,b) < G(a,b) < \beta H(a,b) + (1-\beta)Q(a,b).$$

Similar problems for other means have been studied in [1], [2], [3].These inequalities are equivalent to

$$\beta < \frac{Q(a,b) - G(a,b)}{Q(a,b) - H(a,b)} < \alpha,$$

and, denoting by $t = b/a$, $t > 1$, the problem reduces to find $\inf f$ and $\sup f$, where

$$f(t) = \frac{Q(1,t) - G(1,t)}{Q(1,t) - H(1,t)}.$$

We find the best bounds for α and β using the motonicity of the function f. Then we replace Q by M_p, $p \geq 2$ and address the same problem.

References

[1] Alzer, H., Qiu, S.-L., Inequalities for means in two variables, Arch. Math. (Basel) 80 (2003), 201-215

[2] Anisiu, M.-C., Anisiu, V., Bilateral inequalities for means, Revue d'Analyse Numérique et de Théorie de l'Approximation 42(2) (2013), 85-93

[3] Xia, W.-F., Chu, Y.-M., Optimal inequalities related to the logarithmic, identric, arithmetic and harmonic means, Revue d'Analyse Numérique et de Théorie de l'Approximation 39(2) (2010), 176-183.